Projective Tensor Products: The Pursuit of Norm-Attainment

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Joint work with
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Bilinear mappings

Let X, Y, Z be vector spaces.

Definition

$A : X \times Y \to Z$ *is* **bilinear** *if*

1. $A(x_1 + x_2, y) = A(x_1, y) + A(x_2, y), \quad \forall x_1, x_2 \in X, \ \forall y \in Y.$
2. $A(x, y_1 + y_2) = A(x, y_1) + A(x, y_2), \quad \forall x \in X, \ \forall y_1, y_2 \in Y.$
3. $A(\alpha x, y) = \alpha A(x, y) = A(x, \alpha y), \quad \forall x \in X, \ \forall y \in Y, \ \forall \alpha \in \mathbb{K}.$

- We denote $B(X \times Y, Z) = \{A : X \times Y \to Z : A \text{ bilinear } \}$.
- If $Z = \mathbb{K}$ we denote $B(X \times Y) = B(X \times Y, \mathbb{K})$.

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We would like to **linearize** bilinear maps.

Tensor products

Consider the linear functional $x \otimes y : B(X \times Y) \to \mathbb{K}$ such that

- $(x \otimes y)(A) = A(x, y), \quad \forall A \in B(X \times Y)$.

Definition

The **tensor product** $X \otimes Y$ *is the subspace of the algebraic dual* $B(X \times Y)^{\#}$ *spanned by the set*

$$\{x \otimes y : x \in X,\ y \in Y\}.$$

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A **tensor** $u \in X \otimes Y$ has the following form

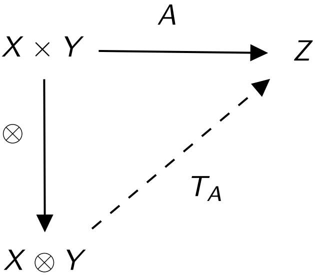
$$u = \sum_{i=1}^{n} x_i \otimes y_i, \text{ where } x_i \in X,\ y_i \in Y. \quad \textbf{(NOT UNIQUE)}$$

The **action** of $u \in X \otimes Y \subseteq B(X \times Y)^{\#}$ is defined as

$$u(A) = \sum_{i=1}^{n} A(x_i, y_i), \quad \forall A \in B(X \times Y)$$

and this value is independent of the representation of u.

Linearization $B(X \times Y, Z)$



Given $A \in B(X \times Y, Z)$, define

$$T_A\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} A(x_i, y_i).$$

Conversely, if $T : X \otimes Y \to Z$ is linear, define $A \in B(X \times Y, Z)$ by

$$A(x, y) = T(x \otimes y).$$

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Definition

Given $X \otimes Y$, we define the **projective norm** *as follows*

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}, \quad \forall u \in X \otimes Y.$$

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However, $(X \otimes Y, \|\cdot\|_\pi)$ is complete if and only if X or Y are finite-dimensional.

Definition

The **projective tensor product** $X\widehat{\otimes}_\pi Y$ *is the completion of* $(X \otimes Y, \|\cdot\|_\pi)$.

Dual space of $X\widehat{\otimes}_\pi Y$

Definition

A bilinear map $A : X \times Y \to Z$ is said to be **bounded** *if there exists $C > 0$ such that*

$$\|A(x,y)\| \leq C\|x\|\|y\|, \quad \forall (x,y) \in X \times Y.$$

- $\mathcal{B}(X \times Y, Z)$ denotes the Banach space of bounded bilinear mappings with the norm $\|A\| = \sup\{\|A(x,y)\| : \|x\| \leq 1, \|y\| \leq 1\}$.

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It can be proved that

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X \widehat{\otimes}_\pi Y, Z)$$

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We also have that

$$B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y).$$

Norm-attainment

Given $u \in X \widehat{\otimes}_\pi Y$ we can obtain the following expression for its projective norm

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Definition

Given $u \in X\widehat{\otimes}_\pi Y$ it is said to **attain its projective norm** *if*

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i \quad \text{and} \quad \|u\|_\pi = \sum_{i=1}^{\infty}\|x_i\|\|y_i\|.$$

The set of those u which attain their projective norm is denoted by

$$\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y).$$

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Known results

S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)

We have that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ in the following cases

- If X and Y are finite-dimensional.
- If $X = \ell_1(I)$ for some set I and Y is any Banach space.
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A. Pelczynski, N. Tomczak-Jaegermann (1988)

Given n, m there are spaces X, Y with $\dim(X) = n$ and $\dim(Y) = m$ and $u \in X\widehat{\otimes}_\pi Y$ such that all the optimal representations of u have nm terms (resp. $2nm$ for $\mathbb{K} = \mathbb{C}$).

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We have that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ if X is a finite-dimensional polyhedral Banach space and Y is any dual Banach space.

Known results

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If every tensor on $X\widehat{\otimes}_\pi Y$ attains its projective norm, then every operator $T\colon X \to Y^*$ can be approximated by norm-attaining ones.

Thus, there are tensors not attaining their projective norm in $L_1(\mathbb{T})\widehat{\otimes}_\pi \ell_2^2$, $L_1[0,1]\widehat{\otimes}_\pi L_1[0,1]$, $\ell_p\widehat{\otimes}_\pi G$ $(1 < p < +\infty)$,...

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A. Rueda Zoca (2023)

$\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi \ell_2) \subseteq c_0 \otimes \ell_2$.

New results

Definition

$Z \subseteq Y$ is said to be **1-complemented** *if there exists a (linear) projection $P : Y \to Z$ with $\|P\| = 1$.*

Lemma (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let X, Y be Banach spaces such that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$. Assume $Z \subset Y$ is a 1-complemented subspace. Then, $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Z) = X \widehat{\otimes}_\pi Z$.

This allow us to get rid of the **asumption of duality**, because every dual Banach space Y is 1-complemented in its bidual Y^{**}.

New Results

L. C. García-Lirola, J. G.-V., A. Rueda Zoca (2025)

Let Z be a Banach space such that $Z^* = \ell_1(I)$ isometrically. Let X be a subspace of Z and Y be any Banach space. If either X^* or Y^* has the AP, then $\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y^*) = X^* \widehat{\otimes}_\pi Y^*$.

Key elements of the proof

- The injective tensor product $X \widehat{\otimes}_\varepsilon Y$ is the completion of $X \otimes Y$ endowed with the norm

$$\|u\|_\varepsilon = \left\{ \left| \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) \right| : \varphi \in X^*, \ \psi \in Y^*, \ \|\varphi\|, \|\psi\| \leq 1 \right\},$$

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Lemma

Let X, Y be normed spaces and $T : X \to Y$ a **linear isometry**. *Then, the operator $T^* : Y^* \to X^*$ is a quotient operator. In fact,* **given** $x^* \in X^*$, **there exists** $y^* \in Y^*$ *such that $T^*(y^*) = x^*$ and $\|y^*\| = \|x^*\|$.*

Proof of the theorem.

Take $u \in X^* \widehat{\otimes}_\pi Y^*$.
By the previous lemma, there is $z \in \ell_1(I) \widehat{\otimes}_\pi Y^*$ with

$$\|z\| = \|u\| \quad \text{and} \quad (i \otimes Id)^*(z) = u.$$

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L. C. García-Lirola, J. G.-V., A. Rueda Zoca (2025)

Let Z be a Banach space such that $Z^* = \ell_1(I)$ isometrically. Let X be a subspace of Z and Y be any Banach space which is 1-complemented in Y^{**}. If either X^* or Y^{**} has the AP, then $\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y) = X^*\widehat{\otimes}_\pi Y$.

New results

Noticing that $C(K)^* = \ell_1(I)$ for K scattered and compact, we obtain:

Corollary (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

Let X be a subspace of $C(K)$, where K is a compact Hausdorff scattered space, and Y be any Banach space which is 1-complemented in Y^{**}. If X^* has the AP then $\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y) = X^* \widehat{\otimes}_\pi Y$.

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Example (L. C. García-Lirola, J. G.-V., A. Rueda Zoca, 2025)

The hypothesis of K being **scattered cannot be removed**:
Let $\mathbb{T}$ be the unit circle of $\mathbb{R}^2$. Then,

$$\mathrm{NA}_\pi \left(C(\mathbb{T})^* \widehat{\otimes}_\pi \mathbb{R}^2 \right) \neq C(\mathbb{T})^* \widehat{\otimes}_\pi \mathbb{R}^2.$$

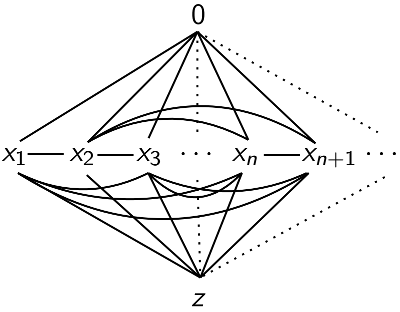
New Results (Lipschitz-free spaces)

L. C. García-Lirola, J. G.-V., A. Rueda Zoca (2025)

Let M be a complete metric space. Then $\mathrm{NA}_\pi(\mathcal{F}(M)\widehat{\otimes}_\pi Y) = \mathcal{F}(M)\widehat{\otimes}_\pi Y$ for any Banach space Y which is 1-complemented in Y^{**} if M satisfies one of the following conditions:

a) M is countable and proper.

b) M is uniformly discrete, countable, and there is a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous, and $V = \{d(x,y) \,:\, (x,y) \in M^2\} \subseteq \mathbb{R}_0^+$ is a compact set.



Example

Define $M = \{0\} \cup \{x_n : n \in \mathbb{N}\} \cup \{z\}$ whose distances are defined as $d(x_n, x_m) = d(x_n, 0) = d(x_n, z) = 1$ for every $n, m \in \mathbb{N}, n \neq m$ and $d(0, z) = 2$.

Define τ such that $(x_n)_n$ converges in τ to x_1 and the points $M \setminus \{x_1\}$ are all discrete.

Then, M **satisfies (b) and $\mathcal{F}(M)$ is not isometrically isomorphic to ℓ_1.**

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Denseness of norm-attaining tensors

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$$\|f\|_1 = \sum_{n=1}^{N} \mu(E_n)\|y_n\| = \sum_{n=1}^{N} \|\chi_{E_n}\|_1 \cdot \|y_n\|$$

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S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)

There exist X, Y such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is **NOT** dense in $X\widehat{\otimes}_\pi Y$.

Denseness of norm-attaining tensors

S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca (2022)

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Definition

A space X is said to have the **metric π-property** *if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$, we can find a finite-dimensional 1-complemented subspace $M \subset X$ and points $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon \ \forall i$.*

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- Equivalently, there are finite-rank projections $P_\alpha \colon X \to X$ with $\|P_\alpha\| = 1$ and $P_\alpha x \to x$ for every $x \in X$.

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Let X be a space with the metric π-property. $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)} = X\widehat{\otimes}_\pi Y$ if

- Y has the metric π-property or Y is uniformly convex (S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca, 2022).
- X is polyhedral and Y is a dual space (S. Dantas, L.C. García-Lirola, M. Jung, A. Rueda Zoca, 2023).

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These results are implied by the following theorem:

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Suppose that for every $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$, there exists a finite dimensional 1-complemented subspace $M \subseteq X$ and $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for each $i = 1, \ldots, n$. Assume that $\overline{\mathrm{NA}_\pi(M\widehat{\otimes}_\pi Y)} = M\widehat{\otimes}_\pi Y$. Then,

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As a consequence, $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)} = X\widehat{\otimes}_\pi Y$ in the following cases:

a) $X^* = L_1(\mu)$ and Y is 1-complemented in Y^{**}.

b) X has the metric π-property and Y is a dual space with the RNP.

Some related questions

Is every extreme point of $B_{X\widehat{\otimes}_\pi Y}$ of the form $x \otimes y$ with $x \in B_X$, $y \in B_Y$?

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Are there **reflexive** Banach spaces X, Y such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$?

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Thank you for your attention!