Closed subideals of bounded operators

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Structures in Banach Spaces
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1. Aim of talk/notation

- Aim of talk: to describe examples and properties of non-trivial closed subideals and closed n-subideals of the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X.
- The talk is based on
  - ⋆ H.-O. Tylli & H. Wirzenius: *Exotic closed subideals of algebras of bounded operators*, Proc. Amer. Math. Soc. 152 (2024).
  - ⋆ H.-O. Tylli & H. Wirzenius: *Structure of closed subideals of* $\mathcal{L}(X)$, in preparation.
- Notation: Let X, Y be ∞-dim. (real or complex) Banach spaces, and denote

  $\mathcal{L}(X, Y) = \{$ bounded linear operators $X \to Y\}$.

  $\mathcal{S}(X, Y) = \{$ strictly singular operators $X \to Y\}$.

  $\mathcal{K}(X, Y) = \{$ compact operators $X \to Y\}$.

  $\mathcal{A}(X, Y) = \{$ approximable operators $X \to Y\} := \overline{\mathcal{F}(X, Y)}$, where $\mathcal{F}(X, Y) = \{$ bounded finite rank operators $X \to Y\}$ and closure in operator norm.

- $\mathcal{F}(X, Y) \subsetneq \mathcal{A}(X, Y) \subset \mathcal{K}(X, Y) \subset \mathcal{S}(X, Y) \subset \mathcal{L}(X, Y)$.
- For $X = Y$ write $\mathcal{L}(X) := \mathcal{L}(X, X)$ etc.

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- Observation : Suppose $\mathcal{I}, \mathcal{J} \subset \mathcal{L}(X)$ are closed ideals, and there is a Banach algebra isomorphism $\theta : \mathcal{I} \to \mathcal{J}$. Then

$$\mathcal{I} = \mathcal{J}. \tag{1}$$

  In other words: distinct closed ideals of $\mathcal{L}(X)$ are never isomorphic as Banach algebras.

  Reason: θ above is of the form

$$\theta(S) = USU^{-1} \quad (S \in \mathcal{I})$$

  for some linear isomorphism $U \in \mathcal{L}(X)$ by (Chernoff '73).

- (Tylli & W. '22) There is a Banach space X (failing the approximation property) with an uncountable family $\mathfrak{F}$ of distinct closed ideals of $\mathcal{K}(X)$, where none of the ideals $\mathcal{I} \in \mathfrak{F}$ are ideals of $\mathcal{L}(X)$.
- Question (Schechtman): Are the ideals in $\mathfrak{F}$ pairwise non-isomorphic as Banach algebras?
- (Tylli & W. '24) $\mathcal{I}$ and $\mathcal{J}$ are isomorphic as Banach algebras for all $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$.
  - ⋆ Conclusion: the above observation (1) fails for closed subideals of $\mathcal{L}(X)$.

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Definition

Let $\mathcal{J}, \mathcal{I} \subset \mathcal{L}(X)$ be closed subalgebras. Then $\mathcal{J}$ is called a closed subideal (or a closed $\mathcal{I}$-subideal) of $\mathcal{L}(X)$ if

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where $\mathcal{J}$ is an ideal of $\mathcal{I}$, and $\mathcal{I}$ is an ideal of $\mathcal{L}(X)$. $\mathcal{J}$ is a non-trivial subideal if $\mathcal{J}$ is not an ideal of $\mathcal{L}(X)$.

- (Fong & Radjavi '83) Examples of non-trivial, but non-closed, $\mathcal{K}(\ell^2)$-subideals of $\mathcal{L}(\ell^2)$. (Terminology due to Patnaik & Weiss '13.)
- Remark: If the closed ideal $\mathcal{I} \subset \mathcal{L}(X)$ has an approximate identity, then there are no non-trivial closed $\mathcal{I}$-subideals of $\mathcal{L}(X)$.

  (A net $(U_\alpha) \subset \mathcal{I}$ is an approximate identity if $S = \lim_\alpha U_\alpha S = \lim_\alpha S U_\alpha$ for all $S \in \mathcal{I}$.)

  - ⋆ Consequence: If H is a Hilbert space, then there are no non-trivial closed subideals of $\mathcal{L}(H)$. Reason: every closed ideal $\mathcal{I} \subset \mathcal{L}(H)$ has an approximate identity.

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- **Question:** Non-trivial closed subideals of $\mathcal{L}(X)$ for classical Banach spaces X?
- If $X = \ell^p$, $1 \leq p < \infty$ or $X = c_0$, then $\mathcal{L}(X)$ does not have any non-trivial closed subideals. Reason:
  - (i) In these cases $\mathcal{K}(X)$ is the unique non-trivial closed ideal of $\mathcal{L}(X)$, and X has the approximation property, so $\mathcal{A}(X) = \mathcal{K}(X)$.
  - (ii) General fact for any X: If $\mathcal{I}$ is a closed subideal of $\mathcal{L}(X)$, then $\mathcal{A}(X) \subset \mathcal{I}$.
- How about non-trivial closed subideals of $\mathcal{L}(X)$ for $X = L^p(0,1)$, where $1 \leq p < \infty$, $p \neq 2$ or $X = C(0,1)$? In these cases, $\mathcal{L}(X)$ has plenty of closed ideals.

Theorem (Tylli & W. 2024)

Let $X = L^p(0,1)$ for $1 < p < \infty$ and $p \neq 2$. Then there are large families (even of size $2^{\mathfrak{c}}$) of non-trivial closed $\mathcal{S}(X)$-subideals $\mathcal{J} \subset \mathcal{S}(X) \subset \mathcal{L}(X)$ that, respectively

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Let $X = L^p(0,1)$ for $1 < p < \infty$ and $p \neq 2$. Then there are large families (even of size $2^{\mathfrak{c}}$) of non-trivial closed $\mathcal{S}(X)$-subideals $\mathcal{J} \subset \mathcal{S}(X) \subset \mathcal{L}(X)$ that, respectively

(i) are pairwise non-isomorphic as Banach algebras, or

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- **Question:** Non-trivial closed subideals of $\mathcal{L}(X)$ for classical Banach spaces X?
- If $X = \ell^p$, $1 \le p < \infty$ or $X = c_0$, then $\mathcal{L}(X)$ does not have any non-trivial closed subideals. Reason:
  - (i) In these cases $\mathcal{K}(X)$ is the unique non-trivial closed ideal of $\mathcal{L}(X)$, and X has the approximation property, so $\mathcal{A}(X) = \mathcal{K}(X)$.
  - (ii) General fact for any X: If $\mathcal{I}$ is a closed subideal of $\mathcal{L}(X)$, then $\mathcal{A}(X) \subset \mathcal{I}$.
- How about non-trivial closed subideals of $\mathcal{L}(X)$ for $X = L^p(0,1)$, where $1 \le p < \infty$, $p \neq 2$ or $X = C(0,1)$? In these cases, $\mathcal{L}(X)$ has plenty of closed ideals.

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  How about the converse: Are there closed $(n+1)$-subideals of $\mathcal{L}(X)$ that are not closed n-subideals of $\mathcal{L}(X)$?
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Example

Let $n \geq 2$ and $1 \leq p_1 < p_2 < \ldots < p_n$, and define

$$X = \ell^{p_1} \oplus \cdots \oplus \ell^{p_n}.$$

Then there is a closed n-subideal of $\mathcal{L}(X)$ which is not an $(n-1)$-subideal of $\mathcal{L}(X)$.

Sketch of proof:

Step 1. $\mathcal{S}(X)/\mathcal{K}(X)$ is n-nilpotent:

$$\mathcal{S}(X) = \begin{bmatrix} \mathcal{S}(\ell^{p_1}) & \mathcal{S}(\ell^{p_2}, \ell^{p_1}) & \cdots & \mathcal{S}(\ell^{p_n}, \ell^{p_1}) \\ \mathcal{S}(\ell^{p_1}, \ell^{p_2}) & \mathcal{S}(\ell^{p_2}) & \cdots & \mathcal{S}(\ell^{p_n}, \ell^{p_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{S}(\ell^{p_1}, \ell^{p_n}) & \mathcal{S}(\ell^{p_2}, \ell^{p_n}) & \cdots & \mathcal{S}(\ell^{p_n}) \end{bmatrix} = \begin{bmatrix} \mathcal{K}(\ell^{p_1}) & \mathcal{K}(\ell^{p_2}, \ell^{p_1}) & \cdots & \mathcal{K}(\ell^{p_n}, \ell^{p_1}) \\ \mathcal{S}(\ell^{p_1}, \ell^{p_2}) & \mathcal{K}(\ell^{p_2}) & \cdots & \mathcal{K}(\ell^{p_n}, \ell^{p_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{S}(\ell^{p_1}, \ell^{p_n}) & \mathcal{S}(\ell^{p_2}, \ell^{p_n}) & \cdots & \mathcal{K}(\ell^{p_n}) \end{bmatrix}$$

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Let $i_k : \ell^{p_k} \to \ell^{p_{k+1}}$ be the inclusion map. Then

$$T = \sum_{k=1}^{n-1} J_{k+1} i_k P_k \in \mathcal{S}(X) \quad (T : (x_1, \ldots, x_n) \mapsto (0, i_1 x_1, \ldots, i_{n-1} x_{n-1})), \text{ and}$$

$$T^{n-1} = J_n i_{n-1} \circ \cdots \circ i_1 P_1 \notin \mathcal{K}(X) \quad (T^{n-1} : (x_1, \ldots, x_n) \mapsto (0, \ldots, 0, i_{n-1} \cdots i_1 x_1)).$$

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- Examples inside the compact operators?
- Recall: there is a Banach space Z with a large family $\mathfrak{F}$ of non-trivial closed $\mathcal{K}(Z)$-subideals of $\mathcal{L}(Z)$.
- Question: for given $n \geq 3$, is there a space X for which $\mathcal{L}(X)$ contains a closed n-subideal $\mathcal{M}$ of $\mathcal{L}(X)$, where $\mathcal{J}_1 = \mathcal{K}(X)$ in (3), which is not an $(n-1)$-subideal of $\mathcal{L}(X)$?

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Let $n \geq 2$ and $2(n-1) < p \leq 2n$. Let $\mathcal{QN}_p$ denote the quasi p-nuclear operators. Then there is a closed subspace $X \subset c_0$ for which $\mathcal{L}(X)$ contains a closed n-subideal $\mathcal{M}$ of $\mathcal{L}(X)$, where $\mathcal{J}_1 = \overline{\mathcal{QN}_p(X)}$ in (3), which is not an $(n-1)$-subideal of $\mathcal{L}(X)$.

- $T \in \mathcal{L}(X, Y)$ is quasi p-nuclear if there is $(x_k^*) \subset \ell^p(X^*)$ such that $||Tx|| \leq \left(\sum_{k=1}^{\infty} |x_k^*(x)|^p\right)^{1/p}$ for all $x \in X$. (Persson & Pietsch '69)
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Thank you for your attention!

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10. Final remarks

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