# FRACTAL MEASURES IN POLISH GROUPS AND BANACH SPACES: CARDINAL INVARIANTS

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STRUCTURES IN BANACH SPACES
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# HAUSDORFF MEASURE

$\forall n \ \mathrm{diam}\, E_n \leq \delta$

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \{ \sum (\mathrm{diam}\, E_n)^s : \{E_n\} \ \delta\text{-cover of } E \}$$

$$\mathcal{H}^\phi(E) = \lim_{\delta \to 0} \inf \{ \sum \phi(\mathrm{diam}\, E_n) : \{E_n\} \ \delta\text{-cover of } E \}$$

$\phi$ ... gauge

Hausdorff function

- Brownian motion $\phi(r) = r \log \log \frac{1}{r}$
- microscopic sets $\phi(r) = \frac{1}{\log \frac{1}{r}}$
- Strong measure zero (Borel 1919)

<u>THM</u> (Besicovitch 1933) $X$ has strong measure zero $\Leftrightarrow \forall \phi \ \mathcal{H}^\phi(X) = 0$

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## CARDINAL INVARIANTS

$\mathcal{J}$ ... ideal of sets

- Uniformity $\operatorname{non} \mathcal{J} = \min\{|A| : A \notin \mathcal{J}\}$
- covering $\operatorname{cov} \mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}, \bigcup \mathcal{A} = X\}$
- additivity $\operatorname{add} \mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}, \bigcup \mathcal{A} \notin \mathcal{J}\}$
- cofinality $\operatorname{cof} \mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a base of } \mathcal{J}\}$

# CICHOŃ DIAGRAM

$$\begin{tikzcd}
\mathrm{cov}\,\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathrm{non}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cof}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cof}\,\mathcal{N} \arrow[dd, dash] \\
 & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & \\
\mathrm{add}\,\mathcal{N} \arrow[r, dash] & \mathrm{add}\,\mathcal{M} \arrow[r, dash] & \mathrm{cov}\,\mathcal{M} \arrow[r, dash] & \mathrm{non}\,\mathcal{N}
\end{tikzcd}$$

$\mathcal{N}$ Leb. null sets in $\mathbb{R}$

$\mathcal{M}$ meager sets in $\mathbb{R}$

$\mathfrak{b}$ bounding

$\mathfrak{d}$ dominating

→ $\mathcal{N}(\mathcal{H}^\varphi)$ — null sets of $\mathcal{H}^\varphi$

→ $\mathrm{non}\,\mathcal{N}(\mathcal{H}^\varphi) = \mathrm{non}\,\mathcal{H}^\varphi$

# CICHOŃ DIAGRAM

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\mathrm{cov}\,\mathcal{N} \arrow[r, no head] \arrow[dd, no head] & \mathrm{non}\,\mathcal{M} \arrow[r, no head] \arrow[d, no head] & \mathrm{cof}\,\mathcal{M} \arrow[r, no head] \arrow[d, no head] & \mathrm{cof}\,\mathcal{N} \arrow[dd, no head] \\
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$\mathfrak{b}$ bounding

$\mathfrak{d}$ dominating

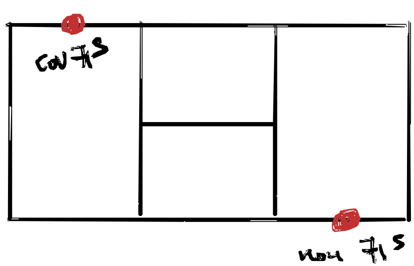
→ $\mathcal{N}(\mathcal{H}^\phi)$ -- null sets of $\mathcal{H}^\phi$ → $\mathrm{non}\,\mathcal{N}(\mathcal{H}^\phi) = \mathrm{non}\,\mathcal{H}^\phi$

THEOREM (Fremlin 2003) Let $X$ be a separable metric space.

$\mathrm{non}\,\mathcal{H}^s \geq \mathrm{cov}\,\mathcal{M}$, $\mathrm{cov}\,\mathcal{H}^s \leq \mathrm{non}\,\mathcal{M}$

If $X$ is analytic, $\mathcal{H}^s(X) > 0$: $\mathrm{cov}\,\mathcal{M} \leq \mathrm{non}\,\mathcal{H}^s \leq \mathrm{non}\,\mathcal{N}$

$\mathrm{cov}\,\mathcal{N} \leq \mathrm{cov}\,\mathcal{H}^s \leq \mathrm{non}\,\mathcal{M}$

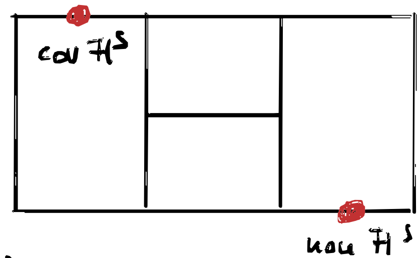



INGREDIENTS:

→ $\phi(r) = r^s$ doubling: $\phi(2r) \leq C\phi(r)$

→ $\mathcal{H}^s$ semifinite: $\forall B$ Borel s.t. $\mathcal{H}^s(B) = \infty$ $\exists B' \subseteq B$ $0 < \mathcal{H}^s(B') < \infty$

# CONSISTENCIES




THEOREM. The following are relatively consistent:

→ $\mathrm{non}\,\mathcal{H}^{1/2} < \mathrm{non}\,\mathcal{N}$ (Shelah, Steprāns 2005, Goto 2021)

→ $\forall s>0$ $\mathrm{non}\,\mathcal{H}^{s} > \mathrm{cov}\,\mathcal{M}$

$X = 2^\omega$

→ $\mathrm{cov}\,\mathcal{H}^{1/2} > \mathrm{cov}\,\mathcal{N}$ (Elekes, Steprāns 2019)

→ $\mathrm{cov}\,\mathcal{H}^{1/2} < \mathrm{non}\,\mathcal{M}$

→ Easily extends to finite dimensional Banach spaces.

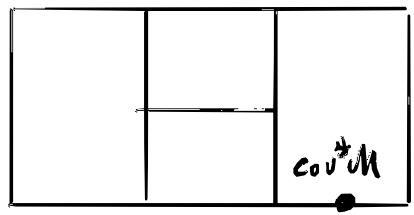
→ So neither $\mathrm{non}\,\mathcal{H}^s$ nor $\mathrm{cov}\,\mathcal{H}^s$ are determined in ZFC.

→ Do the estimates and inconsistencies extend to a wider class, in particular to infinite dimensional Banach spaces?

→ Nonseparable spaces dismissed: $\mathrm{non}\,\mathcal{H}^\phi = \omega_1$, $\mathrm{cov}\,\mathcal{H}^\phi = w(X)$

$$\mathrm{cov}^*\mathcal{M} = \min\{|A| : A \subseteq 2^\omega,\ \exists M \in \mathcal{M}\ A+M = 2^\omega\}$$

"transitive covering"




THEOREM Let $X$ be a metric space, $\phi$ a gauge.

If $X$ is separable, then non $\mathcal{H}^\phi \geq \mathrm{cov}\,\mathcal{M}$

If $X$ is $\sigma$-totally bounded, then non $\mathcal{H}^\phi \geq \mathrm{cov}^*\mathcal{M}$

$\{z_n : n \in \omega\}$ dense in $X$

$x \in X \longmapsto \hat{x} \in \omega^\omega$

$$\hat{x}(n) = \min\{j : d(x, z_j) \leq 2^{-n}\}$$

→ one-to-one

→ inverse is Lipschitz

→ $E \subseteq X$ $\sigma$-totally bdd $\Longleftrightarrow$ $\hat{E} \subseteq \omega^\omega$ bounded

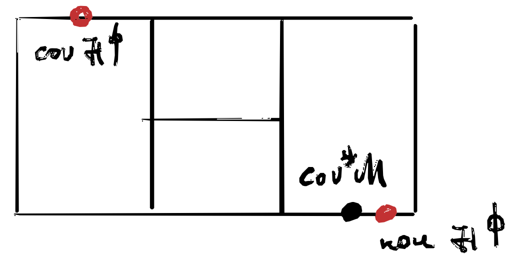
$\mathrm{cov}\,\mathcal{M}$ -- the smallest size of $F \subseteq \omega^\omega$

$\forall g \in \omega^\omega\ \exists f \in F\ \forall^\infty n\ \ f(n) \neq g(n)$

THEOREM If $X$ is a finite dimensional Banach space, then for any gauge $\phi$

$\mathrm{cov}^*\mathcal{M} \leq \mathrm{non}\,\mathcal{H}^\phi \leq \mathrm{non}\,\mathcal{N}$

$\mathrm{cov}\,\mathcal{N} \leq \mathrm{cov}\,\mathcal{H}^\phi \leq \mathrm{non}\,\mathcal{M}$




THEOREM If $X$ is a $\sigma$-compact Polish group, then for any gauge $\phi$

$\mathrm{cov}^*\mathcal{M} \leq \mathrm{non}\,\mathcal{H}^\phi$

$\mathrm{non}\,\mathcal{M} \geq \mathrm{cov}\,\mathcal{H}^\phi$

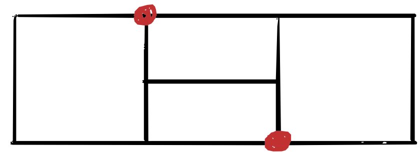
and there is $\phi$ such that $\mathrm{non}\,\mathcal{H}^\phi = \mathrm{cov}^*\mathcal{M}$

CORO (Bartoszyński–Judah 1995, Hrušák–Wohofsky–Z 2016)

If $X$ is analytic, $\sigma$-totally bounded, uncountable, then $\mathrm{non}\,\mathrm{Smz}(X) = \mathrm{cov}^*\mathcal{M}$.

→ admits a two-sided invariant metric

THEOREM If $X$ is a TSI Polish group, NOT $\sigma$-compact,
(in particular, if $X$ is an infinite dimensional separable Banach space)
and $\phi$ a gauge, then



→ $\operatorname{non} \mathcal{H}^\phi = \operatorname{cov} \mathcal{M}$

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LEMMA (Hrušák–Wohofsky–Z 2016) $X$ contains a uniform copy of $\omega^\omega$

LEMMA For $X = \omega^\omega$ and any gauge $\phi$ $\operatorname{non} \mathcal{H}^\phi \leq \operatorname{cov} \mathcal{M}$, $\operatorname{cov} \mathcal{H}^\phi \geq \operatorname{non} \mathcal{M}$.

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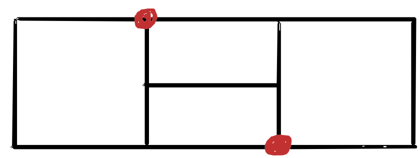

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GALVIN–MYCIELSKI–SOLOVAY THEOREM

THEOREM (G-M-S 1976, Fremlin 2008, Kysiak 2000, Hrušák–Wohofsky–Z 2016)
If $X$ is a locally compact TSI Polish group, then

$E \subseteq X$ has strong measure zero $\iff$ $\forall M$ meager $E + M \neq X$

# MORE MEASURES

$\overline{\mathcal{H}}_0^{\phi}(E) = \lim_{\delta \to 0} \inf\{\Sigma\, \phi(\operatorname{diam} E_n) : \{E_n\}$ finite $\delta$-cover of $E\}$

$\overline{\mathcal{H}}^{\phi}(E) = \inf\{\Sigma\, \overline{\mathcal{H}}_0^{\phi}(E_n) : \{E_n\}$ covers $E\}$ Munroe's "Method I"

# MORE MEASURES

$\to$ $\overline{\mathcal{H}}^{\phi}_{0}(E) = \lim_{\delta \to 0} \inf\{\sum \phi(\operatorname{diam} E_n) : \{E_n\}$ finite $\delta$-cover of $E\}$

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THEOREM ($\geq 2022$) If $X$ is a locally compact TSI Polish group, then $E$ is meager-additive $\iff$ $\forall \phi$ $\overline{\mathcal{H}}^{\phi}(E) = 0$

$\hookrightarrow$ $\forall M \in \mathcal{M}$ $E + M \in \mathcal{M}$

## MORE MEASURES

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THEOREM (Z 2022) If $X$ is a locally compact TSI Polish group, then $E$ is meager-additive $\iff \forall \phi \; \mathcal{H}^{\phi}(E) = 0$

↳ $\forall M \in \mathcal{M} \; E + M \in \mathcal{M}$

## PACKING MEASURE

→ $\mathcal{P}_0^{\phi}(E) = \lim_{\delta \to 0} \sup\{\sum \phi(r_n) : \{B(x_n, r_n)\}$ is a $\delta$-packing of $E\}$

$x_n \in \overline{E}$

$x_m \notin B(x_n, r_n)$

→ $\mathcal{P}^{\phi}(E)$ ... Method I

## MORE MEASURES

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$x_n \in E$

$x_m \notin B(x_n, r_n)$

→ $\mathcal{P}^{\phi}(E)$ ... Method I

THEOREM If $X$ is a finite dimensional Banach space, then $E$ is null-additive $\iff \forall \phi\ \mathcal{P}^{\phi}(E) = 0$

↳ $\forall N \in \mathcal{N}\ E + N \in \mathcal{N}$

# HEWITT – STROMBERG MEASURE

$d(x,y) > \delta$

$$N_E(\delta) = \sup\{ |D| : D \subseteq E \text{ is } \delta\text{-separated} \}$$

$$\nu_0^\phi(E) = \liminf_{\delta \to 0} \phi(\delta) \cdot N_E(\delta)$$

$\nu^\phi(E)$ ... Method I

$$\underrightarrow{\nu}^\phi(E) = \inf\{ \sup \nu_0^\phi(E_n) : E_n \nearrow E \}$$

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Repický 2001 : perfect measure zero

Nowik – Weiss 2002 : $(T')$- sets

THEOREM

$E \subseteq \mathbb{R}$ has perfect measure zero $\iff \forall \phi \quad \underrightarrow{\nu}^\phi(E) = 0$

$E \subseteq 2^\omega$ is a $(T')$- set $\iff \forall \phi \quad \underrightarrow{\nu}^\phi(E) = 0$

# SLALOMS

$G \in \omega^\omega$, $G(n) \to \infty$: $f : \omega \to [\omega]^{<\omega}$ is a $g$-slalom if $\forall n \; |f(n)| \leq G(n)$

Let $\mathcal{S}$ be the set of all $g$-slaloms.

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$\kappa$ is the minimal size of a set $F \subseteq \omega^\omega$ such that $\forall S \in \mathcal{S}$

→ $\mathrm{add}\,\mathcal{N}$: $\exists f \in F \ \exists^\infty n \in \omega$

→ $\mathrm{cov}\,\mathcal{M}$: $\exists f \in F \ \forall^\infty n \in \omega$

$f(n) \notin S(n)$

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$$\left.\begin{array}{lll}
\mathrm{add}\,\mathcal{N} & : & \exists f \in F \ \ \exists^\infty n \in \omega \\
\mathrm{ep}_1 & : & \forall I \in [\omega]^\omega \ \exists f \in F \ \ \exists^\infty n \in I \\
\mathrm{ep}_\omega & : & \forall \langle I_k : k \in \omega \rangle \subseteq [\omega]^\omega \ \exists f \in F \ \forall k \in \omega \ \ \exists^\infty n \in I_k \\
\mathrm{cov}\,\mathcal{M} & : & \exists f \in F \ \ \forall^\infty n \in \omega
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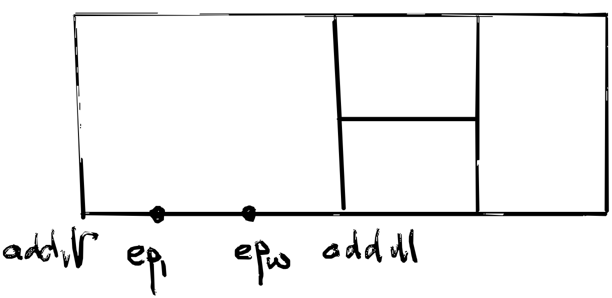


THEOREM

$$\mathrm{add}\,\mathcal{N} \le \mathrm{ep}_1 \le \mathrm{ep}_\omega \le \mathrm{add}\,\mathcal{M}$$

THEOREM If $X$ is of dimension $n < \infty$, then for any gauge $\phi$ such that $\lim_{r \to 0} \frac{\phi(r)}{r^n} = \infty$

$$\begin{tikzcd}
\mathrm{non}\,\mathcal{P}^\phi \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\,\underrightarrow{\mathcal{V}}^\phi \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\,\mathcal{V}^\phi \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\,\overline{\mathcal{H}}^\phi \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\,\mathcal{E} \\
\mathrm{add}\,\mathcal{N} \arrow[r, dash] & \mathfrak{ep}_1 \arrow[r, dash] & \mathfrak{ep}_\omega \arrow[r, dash] & \mathrm{add}^*\mathcal{M} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{cov}\,\mathcal{E} \arrow[r, dash] & \mathrm{cov}\,\overline{\mathcal{H}}^\phi \arrow[r, dash] & \mathrm{cov}\,\mathcal{V}^\phi \arrow[r, dash] & \mathrm{cov}\,\underrightarrow{\mathcal{V}}^\phi \arrow[r, dash] & \mathrm{cov}\,\mathcal{P}^\phi \arrow[r, dash] & \mathrm{cof}\,\mathcal{M}
\end{tikzcd}$$

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$$
\begin{tikzcd}
\mathrm{non}\, \mathcal{P}^{\phi} \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\, \underrightarrow{\nu}^{\phi} \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\, \nu^{\phi} \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\, \overline{\mathcal{H}}^{\phi} \arrow[r, dash] \arrow[d, dash] & \mathrm{non}\, \mathcal{E} \\
\mathrm{add}\, \mathcal{N} \arrow[r, dash] & ep_1 \arrow[r, dash] & ep_{\omega} \arrow[r, dash] & \mathrm{add}^* \mathcal{M} &
\end{tikzcd}
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\end{tikzcd}
$$

THEOREM It is relatively consistent that (for $X = \mathbb{R}$)

$$\mathrm{non}\, \mathcal{P}^1 = \omega_2 \quad \& \quad \mathrm{cov}\, \mathcal{P}^1 = \omega_1$$

$$\forall s < 1 \quad \mathrm{non}\, \mathcal{P}^s = \omega_1 \quad \& \quad \mathrm{cov}\, \mathcal{P}^s = \omega_2$$

and likewise for the other measures.

THEOREM If $X$ is a TSI Polish group, NOT $\sigma$-compact,
(in particular, if $X$ is an infinite dimensional separable Banach space)
and $\phi$ a gauge, then

→ $\operatorname{non} \mathcal{P}^\phi = \operatorname{add} \mathcal{N}$, $\operatorname{non} \underrightarrow{\mathcal{P}}^\phi = \mathfrak{ep}_1$, $\operatorname{non} \mathcal{P}^\phi = \mathfrak{ep}_\omega$, $\operatorname{non} \overline{\mathcal{H}}^\phi = \operatorname{add}^* \mathcal{M}$

→ $\operatorname{cov} \mathcal{P}^\phi = \operatorname{cov} \underrightarrow{\mathcal{P}}^\phi = \operatorname{cov} \mathcal{P}^\phi = \operatorname{cov} \overline{\mathcal{H}}^\phi = \operatorname{cof} \mathcal{M}$

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→ $\mathrm{cov}\,\mathcal{P}^\phi = \mathrm{cov}\,\underrightarrow{\nu}^\phi = \mathrm{cov}\,\nu^\phi = \mathrm{cov}\,\overline{\mathcal{H}}^\phi = \mathrm{cof}\,\mathcal{M}$

$$\mathrm{add}\,\mathcal{N} \le \mathrm{ep}_1 \le \mathrm{ep}_\omega \le \mathrm{add}\,\mathcal{M}$$

THEOREM (Hrušák-Z) $\mathfrak{p} \le \mathrm{ep}_1 \le \mathrm{ep}_\omega \le \overline{\mathrm{cov}}\,\mathcal{ED}_{\mathrm{fin}}$

COROLLARY (Hrušák-Z) Each of the following is relatively consistent:

→ $\mathrm{add}\,\mathcal{N} < \mathrm{ep}_1$

→ $\mathrm{ep}_\omega < \mathrm{add}\,\mathcal{M}$

QUESTION $\mathrm{ep}_1 = \mathrm{ep}_\omega$ ?