Spaces of vector valued Lipschitz functions and the Daugavet property

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Structures in Banach Spaces

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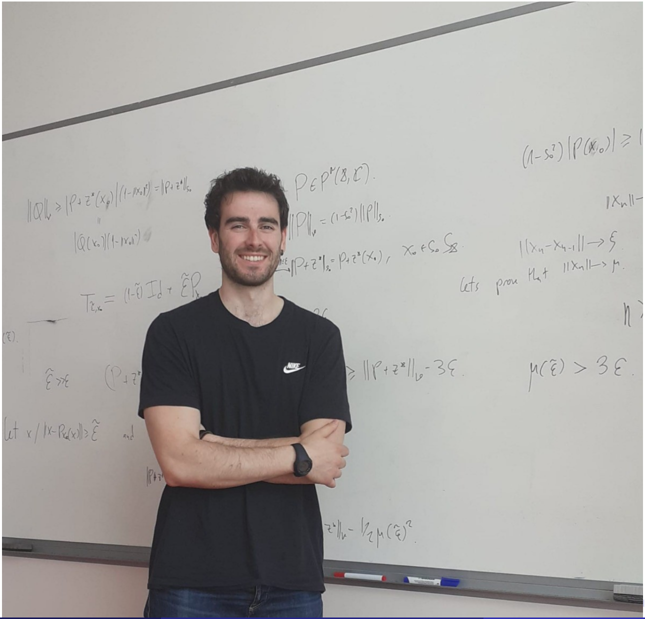
Collaborator



Spaces of Lipschitz functions and Lipschitz free space

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$$\mathrm{Lip}_0(M, X) := \{f : M \longrightarrow X \text{ such that } f \text{ is Lipschitz}, f(0) = 0\},$$

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$$\mathcal{F}(M)^* = \mathrm{Lip}_0(M).$$

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The above space is known as the *Lipschitz-free space over M*.

Linearisation properties of $\mathcal{F}(M)$

Given a metric space M, a Banach space X and a Lispchitz map $f : M \longrightarrow X$ such that $f(0) = 0$, there exists a bounded operator $\hat{f} : \mathcal{F}(M) \longrightarrow X$ defined by

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This operator $\hat{f}$ satisfies that $\|\hat{f}\| = \|f\|_L$ and that the following diagram is commutative

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta"'] & X \\ \mathcal{F}(M) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

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From here it follows that the mapping

$$\begin{array}{ccc}
\mathrm{Lip}_0(M,X) & \longrightarrow & L(\mathcal{F}(M),X), \\
f & \longmapsto & \hat{f}
\end{array}$$

is an onto linear isometry, so $\mathrm{Lip}_0(M,X) = L(\mathcal{F}(M),X)$.

McShane extension theorem

Theorem

Let M metric with origin 0, $0 \in N \subseteq M$ *and* $f : N \longrightarrow \mathbb{R}$ *Lipschitz. Then there exists an extension* $F : M \longrightarrow \mathbb{R}$ *of* f *such that* $\|F\|_{\mathrm{Lip}_0(M)} = \|f\|_{\mathrm{Lip}_0(N)}$.

Geometric properties on $\text{Lip}_0(M)$ and $\mathcal{F}(M)$

A lot of properties have been analysed in $\text{Lip}_0(M)$ as well as in its predual $\mathcal{F}(M)$ (approximation properties, the property of being an L_1 or L_∞ space, octahedrality, Daugavet property etc.).

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3. *For every $x^* \in S_{X^*}$, $\varepsilon > 0$ and w^*-open $\emptyset \neq W \subseteq B_{X^*}$, $\exists\, y^* \in W$ with*
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Theorem (V. Kadets, Y. Ivakhno and D. Werner (2007); García-Lirola, Procházka, R.Z. (2018))

Let M be a complete metric space. Then $\mathcal{F}(M)$ has the Daugavet property if, and only if, M is length.

M length implies $\mathcal{F}(M)$ Daugavet (sketch)

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1. Is $\mathrm{Lip}_0(M, X)$ a dual space?
2. **McShane extension theorem is false for vector-valued functions!**

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- If $\varphi(m_i) \approx h(m_i)$, $\|\varphi\| \approx 1$ and $\frac{r}{r_0} \approx 0$ enough, then $g := \frac{\varphi+\psi}{\|\varphi+\psi\|}$ does the trick.

Perturbating vector-valued Lipschitz functions

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We need to find, given h, a function $\tilde{h}$ such that $\tilde{h}(m_i) \approx h(m_i)$, h constant (flat) at some $B(x_0, r)$ (r small) and $\|h\| \approx 1$.

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Then, given $x_0 \in X$, $R, \varepsilon > 0$ there exists $\delta > 0$ and $\psi : X \longrightarrow X$ such that $\psi(x) = x$ if $x \notin B(x_0, R)$, $\psi(z) = x_0, z \in B(x_0, \delta)$ and $\|\psi\| \leq 1 + \varepsilon$.

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This solved an open question by García-Lirola, Procházka and R.Z. 2018.

Spaces of Lipschitz functions

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Let M be a complete metric space. Then $\mathcal{F}(M)$ has the Daugavet property if, and only if, M is length and if, and only if, $\mathrm{Lip}_0(M)$ has the Daugavet property.

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- Finally, if $0 < d(x_n, y_n) \lll s_n \lll r_n$, taking $g_n := \frac{\psi_n + \varphi_n}{\|\psi_n + \varphi_n\|}$ we get $g_n \to g$ weakly and $\|f - g_n\| \geq 2 - \eta$.

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We say that M has the LPIP if $\forall m_1, \ldots, m_n, x_0 \in M$ and $\varepsilon > 0$ there exists a Lipschitz map $\varphi : M \longrightarrow M$ satisfying:

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3. there exists $\eta > 0$ such that $\varphi(z) = x_0$ at $B(x_0, \eta)$.

R. Smith kindly provided us Talimdjioski result. This allowed us to prove the Daugavet on $\mathrm{Lip}_0(M, X)$ if M is length.

Old memories ...

In September, 2023 in the conference "Æasy to define, Hard to analyse : First conference on Lipschitz free spaces", I presented the following sufficient condition for the Daugavet property on $\mathcal{F}(M)\widehat{\otimes}_\pi X$:

Definition (Local perturbation of the identity (LPIP))

We say that M has the LPIP if $\forall m_1, \ldots, m_n, x_0 \in M$ and $\varepsilon > 0$ there exists a Lipschitz map $\varphi : M \longrightarrow M$ satisfying:

1. $\|\varphi\| \leq 1 + \varepsilon$,
2. $d(\varphi(m_i), m_i) < \varepsilon$ for all $1 \leq i \leq n$ and,
3. there exists $\eta > 0$ such that $\varphi(z) = x_0$ at $B(x_0, \eta)$.

R. Smith kindly provided us Talimdjioski result. This allowed us to prove the Daugavet on $\mathrm{Lip}_0(M, X)$ if M is length. This altruist gesture shows that science progresses when it is based on cooperation and not on competition.

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