LARGE CARDINALS AND L-LIKE UNIVERSES

Extending ZFC:

1. $V = L$: Every set is constructible

GCH
Definable wellordering
$\diamondsuit$, $\square$, Morass

Consistency strength (ZFC + $V = L$) =
Consistency strength (ZFC)

For many interesting φ:

Consistency strength (ZFC + φ) $>$
Consistency strength (ZFC)

2. Large cardinals: inaccessible, measurable, etc.

Question 1: Can we have the advantages of both $V = L$ and large cardinals?

$(*)$ V *is an* L-like *model with large cardinals*

2 approaches:

Inner model approach: A universe with large cardinals has an *inner model* which is L-like and has large cardinals

Outer model approach: A universe with large cardinals has an *outer model* which L-like and has large cardinals

1st approach uses fine structure theory and iterated ultrapowers
2nd approach uses forcing (easier!)

Question 2: Why large cardinals?

Practical reason: many interesting statements are equiconsistent with large cardinals

Theoretical reason: *Inner model hypothesis*

Large cardinals

κ is *inaccessible* iff:
$\kappa > \aleph_0$
κ regular
$\lambda < \kappa \rightarrow 2^\lambda < \kappa$

κ is *measurable* iff:
$\kappa > \aleph_0$
$\exists$ nonprincipal, κ-complete ultrafilter on κ

Embeddings:

$V =$ universe of all sets, M an inner model

$j : V \rightarrow M$ is an *embedding* iff:
j is not the identity
j preserves formulas with parameters

Critical point of j is the least κ, $j(\kappa) \neq \kappa$

$j : V \to M$ is *α-strong* iff $V_\alpha \subseteq M$

κ is *α-strong* iff κ is the critical point of an α-strong $j : V \to M$

Strong = α-strong for all α

Kunen: No $j : V \to M$ is strong

However: κ could be strong

κ is *superstrong* iff κ is the critical point of a $j(\kappa)$-strong $j : V \to M$

κ is *Woodin* iff for each $f : \kappa \to \kappa$, κ is the critical point of a $j(f)(\kappa)$-strong $j : V \to M$

Later: Hyperstrong, n-superstrong, ...

Inner model approach

κ inaccessible $\rightarrow$ κ inaccessible in L

L is totally L-like!

κ measurable $\rightarrow$ κ is measurable in $L[U]$

U is an ultrafilter on κ

$L[U]$ is L-like: GCH, definable wellordering, $\diamondsuit$, $\square$ and (gap 1) morass

κ strong $\rightarrow$ κ strong in $L[E]$

E is a sequence of generalised ultrafilters (extenders)

$L[E]$ is L-like

Success up to Woodin limits of Woodin cardinals

Obstacle: *iterability problem*

Outer model approach

For inaccessibles:

L-coding (Jensen): V has an outer model $V[G]$ such that
ZFC holds in $V[G]$
$V[G] = L[R]$ for some real R
κ inaccessible in $V \rightarrow \kappa$ inaccessible in $V[G]$

$L[R]$ is very L-like!

Similar $L[U]$ and $L[E]$ coding theorems give L-like outer models with measurable, strong cardinals

Coding method is limited:
1. Need to have an L-like *inner* model!
2. Coding problems after a strong cardinal

Forcing

Example 1: Make GCH true in an outer model

Begin with an arbitrary universe V.

Force $f : \aleph_1 \to 2^{\aleph_0}$ onto, without adding reals. Then CH is true in the extension V_1.

$\aleph_2$ of V_1 is $(2^{\aleph_0})^+$ of V.

Force $g : \aleph_2 \to 2^{\aleph_1}$ onto, without adding subsets of $\aleph_1$. Then GCH holds at $\aleph_0$ and $\aleph_1$ in the extension V_2.

Continue to get GCH everywhere.

Does this preserve large cardinals properties?

Using an “extender ultrapower”:

Theorem 1. (GCH and superstrength) If κ is superstrong then there is an outer model in which κ is still superstrong and the GCH holds.

Can go further:

κ is *hyperstrong* iff κ is the critical point of a $j(\kappa)+1$-strong $j : V \to M$

Using a “hyperextender ultrapower”:

Theorem 2. (GCH and hyperstrength) If κ is hyperstrong then there is an outer model in which κ is still hyperstrong and the GCH holds.

κ is *n-superstrong* iff κ is the critical point of a $j^n(\kappa)$-strong $j : V \to M$, where $j^n = j \circ j \circ \cdots \circ j$ (n times).

Combining the proofs of Theorems 1 and 2:

Theorem 3. (GCH and n-superstrength) If κ is n-superstrong then there is an outer model in which κ is still n-superstrong and the GCH holds.

κ is *ω-superstrong* iff κ is the critical point of a $j : V \to M$ which is $\sup_n j^n(\kappa)$-strong

Preserve ω-superstrength and force GCH?

Kunen: No j with critical point κ is $\sup_n j^n(\kappa) + 1$-strong.

Example 2: Add a definable wellordering

This is rather easy.

Theorem 4. If κ is ω-superstrong then there is an outer model in which κ is still ω-superstrong and there is a definable wellordering.

Interesting Example 3: Make $\Box$ true in an outer model

$\Box$: There is $\langle C_\alpha \mid \alpha \text{ singular} \rangle$ such that
C_α is cofinal in α for each α
C_α has ordertype less than α for each α
$\bar{\alpha} \in \text{Lim } C_\alpha \rightarrow C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$

Theorem 5. ($\Box$ and superstrength) If κ is superstrong then there is an outer model in which κ is still superstrong and $\Box$ holds.

Proof does not work for hyperstrong, for a good reason:

κ is *subcompact* iff for each $B \subseteq \kappa^+$ there are $\mu < \kappa$, $A \subseteq \mu^+$ and $j : (\mu^+, A) \to (\kappa^+, B)$ with critical point μ.

Jensen: If there is a subcompact cardinal then $\Box$ fails.

Theorem 6. If κ is hyperstrong then κ is subcompact.

Other examples: $\diamondsuit$, gap 1 morass behave like GCH (proofs are harder).

Higher gap morasses?

The inner model hypothesis

Weak Inner Model Hypothesis (Weak IMH):
If a first-order sentence without parameters holds in an inner model of some outer model of V (i.e., in a model compatible with V) then it already holds in an inner model of V.

(Formalise using countable transitive models of a fixed height.)

The Weak IMH is a generalisation of

Paramter-free Lévy absoluteness: If a Σ_1 sentence is true in an outer model of V then it is true in V.

A *persistently* Σ^1_1 *formula* is one of the form:

$$\exists M(M \text{ is a transitive class and } M \vDash \psi),$$

where ψ is first-order.

Theorem 7. The following are equivalent:
(a) (Parameter-free persistent Σ^1_1 *absoluteness). If a parameter-free persistent* Σ^1_1 *sentence is true in an outer model of* V *then it is true in* V.
(b) Weak Inner Model Hypothesis.

What does the Weak IMH say about V?

Theorem 8. (a) The Weak IMH implies that for some real R, *ZFC fails in* $L_\alpha[R]$ *for all ordinals* α. *In particular, there are no inaccessible cardinals and the reals are not closed under* $\#$.
(b) The Weak IMH implies that $0^\#$, $0^{\#\#}$, ... *exist.*

Absolute parameters and the IMH

Can we introduce parameters into the inner model hypothesis?

Proposition 9. The inner model hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.

With arbitrary ordinal parameters: $\aleph_1$ can be countable in an outer model.

With arbitrary real parameters:
Weak IMH $\rightarrow \exists R(\omega_1 = \omega_1$ of $L[R])$. But ω_1 of $L[R]$ can be countable in an outer model.

Absolute parameters:

p is *absolute between* V_0 *and* V_1 *via the formula* ψ iff ψ is a first-order formula without parameters which defines p in both V_0 and V_1.

IMH with arbitrary absolute parameters: Suppose that p is absolute between V and V^*, where V^* is an outer model of V, and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

Theorem 10. The inner model hypothesis with arbitrary absolute parameters is inconsistent.

Proof uses a weak form of $\square_{\beth_\omega}$ and fat stationary subsets of $\beth_\omega^+$.

Inner model hypothesis (IMH): Suppose that the ordinal α is absolute between V and V^*, where V^* is an outer model of V, and φ is a first-order sentence with parameter α which holds in an inner model of V^*. Then φ holds in an inner model of V.

Theorem 11. The IMH implies the existence of an inner model with a strong cardinal.

If core model theory can be extended from strong cardinals to Woodin cardinals without large cardinal assumptions, then the IMH implies the existence of an inner model with a Woodin cardinal.

Q. Is the (weak or strong) inner model hypothesis consistent relative to large cardinals? If so, what is its consistency strength?