Axiomatic Set Theory I

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CHAPTER 1

Ordinal and Cardinal Arithmetic

1. The Axiomatic System of Zermelo-Fraenkel

1.1. ZFC. In the following, we will formulate the axiomatic system of Zermelo-Fraenkel. For this we work in the language of set theory, which has only one non-logical symbol, the binary relation, membership! The language of set theory is denoted $\mathcal{L}_\in$. The Axioms (*universal closure of the following statements*):

- Axiom 1 (Extensionality)

$$\forall z(z \in x \leftrightarrow z \in y) \to x = y$$

- Axiom 2 (Foundation)

$$\exists y(y \in x) \to \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))$$

- Axiom 3 (Comprehension Scheme) For each formula φ without y free:

$$\exists y \forall x(x \in y \leftrightarrow x \in v \wedge \varphi(x))$$

- Axiom 4 (Pairing)

$$\exists z(x \in z \wedge y \in z)$$

- Axiom 5 (Union)

$$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathcal{F} \to x \in A)$$

- Axiom 6 (Replacement Scheme) For each formula φ in which B is not a free variable

$$\forall x \in A \exists! y \varphi(x,y) \to \exists B \forall x \in A \exists y \in B \varphi(x,y)$$

Remark 1.1. To formulate the last three axioms, we need some defined notions, namely the notions of a subset, emptyset, successor of a set, intersection and singleton:

(1) $x \subseteq y$ iff $\forall z(z \in x \to z \in y)$
(2) $x = \varnothing$ iff $\forall z(z \notin x)$
(3) $y = S(x)$ iff $\forall z(z \in y \leftrightarrow z \in x \vee z = x)$
(4) $y = v \cap w$ iff $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$
(5) $\mathrm{Sing}(y)$ iff $\exists y \in x \forall z \in x(z = y)$.

Note that $S(x) = x \cup \{x\}$, $\mathrm{Sing}(y) = \{y\}$ and the ordered pair (x,y) is the set $\{\{x\},\{x,y\}\}$. We continue with the axioms.

- Axiom 7 (Infinity)

$$\exists x(\varnothing \in x \wedge \forall y \in x(S(y) \in x))$$

- Axiom 8 (Power Set)
$$\exists y \forall z (z \subseteq x \to z \in y)$$
- Axiom 9 (Axiom of Choice)
$$\varnothing \notin F \wedge \forall x \in F \forall y \in F (x \neq y \to x \cap y = \varnothing) \to \exists C \forall x \in F (\mathrm{Sing}(C \cap x))$$

We refer to the above system of Axioms as ZFC. Note that ZFC is an infinite set of Axioms, because Axioms 3 (Comprehension) and 6 (Replacement) are in fact axiom schemes (one axiom for each formula). Moreover ZFC is not finitely axiomatizable.

1.2. Relations and Functions.

DEFINITION 1.2. Binary relation A set R is said to be a binary relation iff R is a set of ordered pairs, i.e. for each $u \in R$ there are x, y such that $u = (x, y) = \{\{x\}, \{x, y\}\}$.

REMARK 1.3. Recall the following notions associated to a binary relation R:

(1) R is a pre-order on A if R is reflexive and transitive on A.
(2) R partially orders A non-strictly if R is a pre-order on A and satisfies $\neg \exists x, y \in A (xRy \wedge yRx \wedge x \neq y)$.
(3) R is a total-order on A if R is irreflexive, transitive and satisfies trichotomy, i.e. for any $a, b \in A$ either aRb, or bRa or $a = b$.

DEFINITION 1.4. A binary relation R is a function if for every x there is at most one y such that $(x, y) \in R$. If there is y such that xRy then $R(x)$ denotes that unique y.

DEFINITION 1.5. For any set A, $\mathrm{id}_A = \{(x, x) : x \in A\}$ is the identity function of A.

PROOF. (Justification of existence) Note that we can justify the existence of id_A as follows:
$$\mathrm{id}_A = \{(x, x) \in \mathcal{P}(\mathcal{P}(A)) : x \in A\}.$$
□

REMARK 1.6. Note $(x, x) = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$ and whenever $x \in A$ and $x \in B$, then
$$(x, y) = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)).$$

DEFINITION 1.7. $A \times B = \{(x, y) : x \in A \wedge y \in B\}$

PROOF. (Justification of existence) The existence of $A \times B$ follows from the Axioms of Power Set and Comprehension, since $A \times B = \{(x, y) = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) : x \in A \wedge y \in B\}$. □

REMARK 1.8. To claim that $A \times B$ is a set, alternatively one can use the Axioms of Replacement and Union. By Replacement for each $y \in B$, $A \times \{y\} = \{(x, y) : x \in A\}$ is a set. Again by Replacement $S = \{A \times \{y\} : y \in B\}$ is a set. Now, by the Union Axiom $\bigcup S$ is a set. Thus, we can define $A \times B = \bigcup S$.

DEFINITION 1.9. (Domain and Range) For every set R define

(1) $\mathrm{dom}(R) = \{x : \exists y((x,y) \in R)\}$,
(2) $\mathrm{ran}(R) = \{y : \exists x((x,y) \in R)\}$.

PROOF. (Justification of existence: Using Union and Comprehension) If $\{\{x\},\{x,y\}\} \in R$, then $\{x\},\{x,y\}$ belong to $\bigcup R$ and so $x,y \in \bigcup\bigcup R$. Thus, $\mathrm{dom}(R) = \{x \in \bigcup\bigcup R : \exists y((x,y) \in R)\}$, and $\mathrm{ran}(R) = \{y \in \bigcup\bigcup R : \exists x((x,y) \in R)\}$. □

Note that alternatively, one can use Replacement.

DEFINITION 1.10. (Restriction) $R \upharpoonright A = \{(x,y) \in R : x \in A\}$

PROOF. (Justification of existence) By the Axiom of Comprehension. □

REMARK 1.11. The notions of a function, injection, bijection, surjection, can be defined in a similar way.

LEMMA 1.12. Assume $\forall x \in A \exists! y \varphi(x,y)$ and assume the Axiom of Replacement. Then there is a function f such that $\mathrm{dom}(f) = A$ and such that $\forall x \in A$, $f(x)$ is the unique y such that $\varphi(x,y)$.

DEFINITION 1.13. (A set of functions) Given sets A, B let

$$B^A = {}^A B = \{f \mid f : A \to B\}.$$

PROOF. (Justification of existence: Power set and Comprehension) If f is a function from A to B, then $f \subseteq A \times B$. Therefore ${}^A B \subseteq \mathcal{P}(A \times B)$. □

DEFINITION 1.14. Let A be a set and let R be a relation on A. Then, we say that

(1) R totally orders A strictly if R is transitive, irreflexive, satisfies trichotomy on A.
(2) R well-orders A iff R totally orders A and R is well-founded on A, i.e. every $B \subseteq A$ has an R-minimal element.

LEMMA 1.15. If R is a well-order on a set A and $X \subseteq A$, then R is a well-order on X.

PROOF. Clearly R is a total order on X. Moreover, every subset of X has an R-minimal element. □

2. Ordinal Arithmetic

2.1. Ordinals.

DEFINITION 2.1. A set z is an ordinal if z is transitive, i.e. $\forall x(x \in z \to x \subseteq z)$ and the membership relation $\in$ is a well-order on z.

EXAMPLE 2.2.

- $\varnothing$,
- $\{\varnothing\}$,
- $\{\varnothing,\{\varnothing\}\}$,
- $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$

- $\cdots$

REMARK 2.3. Every natural number is an ordinal.

NOTATION. $\mathbb{ON}$ denotes the collection of all ordinals. Greek letters are used to denote ordinals.

LEMMA 2.4. Suppose α is an ordinal, $z \in \alpha$. Then z is also an ordinal.

PROOF. By transitivity of α, $z \subseteq \alpha$. Thus $\in$ is well-founded on z. We need to check if z is transitive. Let $x \in z$ and $y \in x$. Then $x \in \alpha$. But α is transitive and so $x \subseteq \alpha$. Thus $y \in \alpha$. Therefore x, y, z are elements of α. But $\in$ is transitive on α and so we have $y \in x \wedge x \in z \to y \in z$. Thus $y \in z$. That is $x \subseteq z$, i.e. z is transitive. ☐

LEMMA 2.5. Let α, β be ordinals. Then $\alpha \cap \beta$ is an ordinal.

PROOF. Since $\alpha \cap \beta \subseteq \alpha$, the $\in$ is well-founded on $\alpha \cap \beta$. We need to show that $\alpha \cap \beta$ is transitive. Let $x \in \alpha \cap \beta$ and $y \in x$. Then $x \subseteq \alpha \cap \beta$ and so $y \in \alpha \cap \beta$. Thus $x \subseteq \alpha \cap \beta$, i.e. $\alpha \cap \beta$ is a transitive set. ☐

LEMMA 2.6. Let α, β be ordinals. Then $\alpha \subseteq \beta$ if and only if $\alpha \in \beta \vee \alpha = \beta$.

PROOF. ($\Leftarrow$) If $\alpha \in \beta$, then by transitivity of β, we have $\alpha \subseteq \beta$. Therefore $\alpha \in \beta \vee \alpha = \beta$ implies that $\alpha \subseteq \beta$.

($\Rightarrow$) If $\alpha = \beta$, then clearly we are done. So, suppose $\alpha \neq \beta$. Thus $X = \beta \backslash \alpha \neq \varnothing$ and so there is $\xi = \min \beta \backslash \alpha$. Then

$$\xi \in \beta \text{ and } \xi \notin \alpha.$$

We will show that $\xi = \alpha$. First we will show that $\xi \subseteq \alpha$. Let $\mu \in \xi$. Then by transitivity of β, we have $\xi \subseteq \beta$ and so $\mu \in \beta$. If $\mu \notin \alpha$, we get a contradiction to the minimality of ξ. Thus $\mu \in \alpha$ and so $\xi \subseteq \alpha$. Now, suppose $\xi \subseteq \alpha$, but $\xi \neq \alpha$! Then take any pick $\mu \in \alpha \backslash \xi$. Then $\mu \in \beta$ (because $\alpha \subseteq \beta$ by hypothesis) and $\xi \in \beta$, since $\xi = \min \beta \backslash \alpha$. Thus, by the trichotomy of $\in$ on β we get $\mu = \xi \vee \mu \in \xi \vee \xi \in \mu$.

(1) However $\mu \in \alpha$, but $\xi \notin \alpha$. Thus $\mu \neq \xi$.
(2) By the choice of μ, $\mu \notin \xi$.
(3) Thus $\xi \in \mu$.

Since $\mu \in \alpha$ and α is transitive, $\xi \in \alpha$, which is a contradiction to the choice of ξ! Therefore $\xi = \alpha$. ☐ ☐

THEOREM 2.7. *(The collection of all ordinals "behaves" like an ordinal)*

(1) *(Transitivity) For all α, β and γ ordinals, if $\alpha \in \beta \wedge \beta \in \gamma$ then $\alpha \in \gamma$.*
(2) *(Irreflexivity) for every ordinal α, $\neg(\alpha \in \alpha)$.*
(3) *(Trichotomy) for all α, β ordinals: $\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta$.*
(4) *(Well-foundedness) If $X \neq \varnothing$ is a set of ordinals, then X has an $\in$-least element.*

PROOF. (1) Since γ is a transitive set, $\beta \subseteq \gamma$ and so $\alpha \in \gamma$.

(2) Suppose $\alpha \in \alpha$. That is α is an element of α. But $\in$ is irreflexive on α and so $\neg(\alpha \in \alpha)$. This is a contradiction. Therefore $\alpha \notin \alpha$.

(3) Let $\delta = \alpha \cap \beta$. Then $\delta \subseteq \alpha$, $\delta \subseteq \beta$. But then by a previous Lemma we have:

$$\delta \in \alpha \vee \delta = \alpha \text{ and } \delta \in \beta \vee \delta = \beta.$$

- If $\delta = \alpha$, then $\alpha \subseteq \beta$ and so $\alpha \in \beta \vee \alpha = \beta$.
- If $\delta = \beta$, then $\beta \subseteq \alpha$ and so $\beta \in \alpha \vee \beta = \alpha$.
- Thus suppose $\delta \neq \alpha$, $\delta \neq \beta$. Therefore $\delta \in \alpha$ and $\delta \in \beta$, i.e. $\delta \in \alpha \cap \beta = \delta$, which is a contradiction to (2).

(4) Let $X \neq \varnothing$ and X be a set of ordinals. Let $\alpha \in X$. If $\alpha = \min X$, then we are done. Otherwise $X_0 = \{\xi : \xi \in X \wedge \xi \in \alpha\} \neq \varnothing$. Then $\mu = \min X_0$ exists, because $X_0 \subseteq \alpha$. Thus $\mu = \min X \cap \alpha$. Note that $\mu = \min X$. Consider any $\delta \in X$ and suppose $\delta \in \mu$. Then $\delta \in \alpha$ (since $\mu \subseteq \alpha$), which is a contradiction to $\mu = \min X \cap \alpha$. □

REMARK 2.8. The above theorem shows that the collection of all ordinals, "behaves" like an ordinal. However, one may ask: Is the collection of all ordinals a set? Is there a set containing all ordinals?

THEOREM 2.9. *(Bourali-Forty Paradox) There is no set containing all ordinals.*

PROOF. Suppose not and let X be a set containing all ordinals. Then let

$$Y = \{y \in X : y \text{ is an ordinal}\}.$$

By the Axiom of Comprehension Y is a set. By the previous theorem $\in$ is well-founded on Y and Y is a transitive set. Thus, Y is an ordinal. But then $Y \in Y$, contradiction to (2) of the previous theorem. Thus, there is no such X. □ □

NOTATION. We will use the following notation:

(1) With $\mathbb{ON}$ we denote the class of all ordinals.
(2) Let α, β be ordinals. Then $\alpha < \beta$ denotes $\alpha \in \beta$ and $\alpha \leq \beta$ denotes $\alpha \in \beta \vee \alpha = \beta$.

LEMMA 2.10. Let α, β be ordinals. Then

$$\alpha \cap \beta = \min\{\alpha, \beta\} \text{ and } \alpha \cup \beta = \max\{\alpha, \beta\}.$$

LEMMA 2.11. If $A \neq \varnothing$ is a set of ordinals, then

(1) $\bigcap A = \min A$,
(2) $\bigcup A \in \mathbb{ON}$
(3) If $\forall \alpha \in A \exists \beta \in A(\alpha < \beta)$, then $\bigcup A$ is the smallest ordinal that exceeds all ordinals in A. Thus, we denote $\bigcup A$ also $\sup A$.

PROOF. (2) We need to show that $\bigcup A$ is a transitive set and $\in$ is well-founded on $\bigcup A$. Let $\alpha \in \bigcup A$. Thus there is $\beta \in A$ such that $\alpha \in \beta$. But β is transitive and so $\alpha \subseteq \beta$. Therefore $\alpha \subseteq \bigcup A$. To show well-foundedness of $\in$, let $X \subseteq \bigcup A$. Thus $\forall x \in X$ there is $\alpha_x \in A$ such that $x \in \alpha_x$. Now $\{\alpha_x : x \in X\}$ is a set of ordinals and so by well-foundedness of the membership relation on $\mathbb{ON}$, there is $\alpha_0 = \min\{\alpha_x : x \in X\}$. Then either $\alpha_0 = \min X$ or $\alpha_0 \cap X \neq \varnothing$, in which case $\min(\alpha_0 \cap X)$ is as desired.

(3) Let $\delta = \bigcup A$. Then $\delta = \{\alpha : \exists \beta \in A(\alpha \in \beta)\}$. Since for every $\alpha \in A$ there is $\beta \in A$ such that $\alpha < \beta$, we get that every $\alpha \in A$ is an element of δ. Also, if $\alpha < \delta$, then $\alpha \in \delta$ and so there is $\beta \in A$ such that $\alpha \in \beta$. But, then $\beta \notin \alpha$ and so α does not exceed all elements of A. □

LEMMA 2.12. Let α be an ordinal. Then

(1) $S(\alpha) = \alpha \cup \{\alpha\}$ is an ordinal,
(2) $\alpha < S(\alpha)$ and
(3) for all ordinals γ, $\gamma < S(\alpha)$ iff $\gamma \leq \alpha$.

PROOF. The membership relation is well-founded on $S(\alpha)$ and clearly $S(\alpha)$ is a transitive set. The rest is straightforward. □

DEFINITION 2.13. (Successor and Limit Ordinals) An ordinal β is

(1) a successor iff there is an ordinal α such that $\beta = S(\alpha) = \alpha \cup \{\alpha\}$,
(2) a limit ordinal iff $\beta \neq 0$ and β is not a successor ordinal,
(3) a finite ordinal or a natural number if and only if $\forall \alpha \leq \beta(\alpha = 0 \vee \alpha \text{ is a successor})$.

REMARK 2.14. If n is a natural number, then $S(n)$ is a natural number and every element of n is a natural number.

THEOREM 2.15. *(Principle of ordinary induction) If* $\varnothing \in X$ *and for all* $y \in X(S(y) \in X)$*, then every natural number is in* X*.*

PROOF. Suppose not and let $n \in \mathbb{N}\backslash X$. Consider $Y = S(n)\backslash X$. Then $n \in Y$ and so $Y \neq \varnothing$. Let $k = \min Y$. Thus $k \leq n$. Therefore $k = \varnothing$ or k is a successor. However $\varnothing \notin Y$, because $\varnothing \in X$ and so $k = S(i)$ for some i. By minimality of k, we must have $i \in X$. But then also $k = S(i) \in X$, which is a contradiction. □

REMARK 2.16. Recall the Axiom of Infinity: $\exists x(\varnothing \in X \wedge \forall y \in x(S(y) \in x))$. Thus if X is a set which contains all natural numbers, then $\{n \in X : n \text{ is a natural number}\}$ is a set.

LEMMA 2.17. Let X be a set of ordinals, which is an initial segment of $\mathbb{ON}$. That is

$$\forall \beta \in X \forall \alpha < \beta(\alpha \in X)).$$

Then X is an ordinal itself.

PROOF. Note that $\in$ is a well-order on X. Since X is an initial segment of the ordinals, X is also a transitive set. Thus X is an ordinal. □

REMARK 2.18. So in particular, every transitive set of ordinals is an ordinal.

DEFINITION 2.19. Let ω denote the set of all natural numbers.

REMARK 2.20. Note that ω is an initial segment of $\mathbb{ON}$ and so ω is an ordinal. Moreover ω is not a successor ordinal and ω is not finite. Thus, ω is the first limit ordinal.

DEFINITION 2.21. Assume the Axiom of Infinity and for each $n \in \mathbb{N}$ let

$$B^n = {}^nB = \{F \mid F : n \to B\}.$$

Then let

$$B^{<\omega} = {}^{<\omega}B := \bigcup\{B^n : n \in \omega\}.$$

PROOF. (Justification of existence) Use the Power Set Axiom or the Axiom of Replacement. $\square$

REMARK 2.22. Let $\mathcal{L} = (\mathcal{C}, \mathcal{F}, \mathcal{R})$ be a first order language and let B be the set of all logical and non-logical symbols of $\mathcal{L}$. Then the set of formulas of $\mathcal{L}$ is a subset of $B^{<\omega}$. Thus, in particular in a countable first order language (assuming AC) there are only countably many formulas.

Next, we will introduce the notion of an order type.

LEMMA 2.23. Let α, β be ordinals and suppose that $f : (\alpha, \in) \to (\beta, \in)$ is an order preserving bijection (i.e. an isomorphism). Then $\alpha = \beta$ and $f = \mathrm{id}$.

PROOF. Let $\xi \in \alpha$. Then $f(\xi) \in \beta$. Furthermore, since f is order preserving $f(\xi) = \{f(\mu) : \mu < \xi\}$. Suppose $X_0 = \{\xi \in \alpha : f(\xi) \neq \xi\} \neq \varnothing$. Then X_0 has a minimal element μ. Thus for all $\xi < \mu$, $f(\xi) = \xi$ and so

$$f(\mu) = \{f(\xi) : \xi < \mu\} = \{\xi : \xi < \mu\} = \mu,$$

which is a contradiction. Therefore $X_0 = \varnothing$ and so f is the identity. $\square$

THEOREM 2.24. *Let A be a set and let R be a well-order on A. Then there is a unique ordinal α such that $(A, R) \cong (\alpha, \in)$.*

REMARK 2.25. Uniqueness follows from the previous statement.

PROOF. (Existence) For $a \in A$ let $a\downarrow := \{x \in A : xRa\}$ and let

$$G = \{a \in A : \exists \xi_a \in \mathbb{ON}((a\downarrow, R) \cong (\xi_a, \in))\}.$$

Since A is a set, by the Axiom of Comprehension G is also a set. Since $\forall a \in G\ \exists \xi_a$ as above, by Replacement there is a set $X \subseteq \mathbb{ON}$ and a function $f : G \to X$ such that for all $a \in G$, $f(a) = \xi_a$. Then $\in$ is a well-order on $\mathrm{range}(f) \subseteq X$. Moreover $\mathrm{range}(f)$ is a transitive and so it is an ordinal, say α. Then $f : (G, R) \cong (\alpha, \in)$. Note that:

- if $G = A$, then we are done.
- if $G \subseteq A$ and $G \neq A$, let $e = \min_R(A \setminus G)$. Then $e\downarrow = G$ and $f : (e\downarrow, R) \cong (\alpha, \in)$. That is $\xi_e = \alpha$. But, this implies that $e \in G$, which is a contradiction. Thus $G = A$.

$\square$

DEFINITION 2.26. (Order Type) Let R be a well-order on A. Then $\mathrm{type}(A, R)$ is the unique ordinal α such that $(A, R) \cong (\alpha, \in)$. We denote this ordinal by $\mathrm{type}(A, R)$.

2.2. Ordinal Arithmetic.

DEFINITION 2.27. Let α, β be ordinals. Then

(1) The ordinal multiplication of α and β, denoted $\alpha \cdot \beta$, is the ordinal

$$\mathrm{type}(\beta \times \alpha, <_{lex}).$$

(2) The ordinal addition of α and β, denoted $\alpha + \beta$, is the ordinal

$$\mathrm{type}(\{0\} \times \alpha \cup \{1\} \times \beta, <_{lex}).$$

LEMMA 2.28. If R well-orders A and $X \subseteq A$, then R well-orders X and

$$\mathrm{type}(X, R) \leq \mathrm{type}(A, R).$$

PROOF. We can assume that $\mathrm{type}(A, R) = (\alpha, \in)$ and that X, A are sets of ordinals. Let $\delta = \mathrm{type}(X, R)$ and let $f : (X, R) \cong (\delta, \in)$. Suppose $X_0 = \{\xi \in X : f(\xi) > \xi\} \neq \varnothing$ and let $\mu = \min X_0$. Then $f(\mu) > \mu$ and $\forall \xi \in X \cap \mu (f(\xi) \leq \xi)$. Since f is an isomorphism

$$f(\mu) = \{f(\xi) : \xi < \mu\} \leq \mu,$$

which is a contradiction. Therefore for all $\xi \in X$, $f(\xi) \leq \xi$. Then

$$\delta = \{f(\xi) : \xi \in X\} \subseteq \alpha \text{ and so } \delta \subseteq \alpha.$$

□

EXAMPLE 2.29.

(1) $\omega + \omega$

$$0, 1, \cdots, n, n+1, \cdots, \omega = \omega + 0, \omega + 1, \omega + 2, \cdots, \omega + n, \cdots$$

(2) $\omega \cdot 2 = \mathrm{type}(\{0,1\} \times \omega, <_{\mathrm{lex}})$

$$(0,0), (0,1), \cdots, (0,n), \cdots, (1,0), (1,1), \cdots, (1,n), \cdots$$

Thus $\omega + \omega = \omega \cdot 2$ (because the order type is unique!).

(3) However $1 + \omega = \omega$, while $\omega < \omega + 1$. Thus $1 + \omega \neq \omega + 1$.

(4) Also $2 \cdot \omega = \mathrm{type}(\omega \times \{0,1\}, <_{\mathrm{lex}}) = \omega$, while $\omega \cdot 2 = \omega + \omega > \omega$.

(5) More precisely, what is $2 \cdot \omega$?

$$(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), \cdots, (n,0), (n,1), \cdots$$

(6) In particular $2 \cdot \omega \neq \omega \cdot 2$.

(7) Both, ordinal multiplication and ordinal addition are associative, but not commutative.

THEOREM 2.30. *(Transfinite Induction on $\mathbb{ON}$) Let $\psi(\alpha)$ be a formula. If there is an ordinal α such that $\psi(\alpha)$, then there is a least ordinal ξ such that $\psi(\xi)$.*

PROOF. Fix α such that $\psi(\alpha)$. If α is least, then we are done. Otherwise, $X = \{\xi \in \alpha : \psi(\alpha)\} \neq \varnothing$ and so $\xi = \min X$ is as desired. □

THEOREM 2.31. *(Primitive Recursion on $\mathbb{ON}$) Suppose for all s there is a unique y such that $\varphi(s,y)$ and define $G(s)$ to be this unique y. Then there is a formula ψ for which the following two properties are provable:*

(1) $\forall x \exists! y \psi(x,y)$. *Thus, ψ defines a function F, where $F(x)$ is such that $\psi(x,F(x))$.*
(2) $\forall \xi \in \mathbb{ON}(F(\xi) = G(F(\xi)))$.

PROOF.
δ-approximations to F: Let $\delta \in \mathbb{ON}$ and let $\mathrm{App}(\delta,h)$ abbreviate

$$h \text{ is a function}, \mathrm{dom}(h) = \delta, \forall \xi \in \delta h(\xi) = G(h \restriction \xi).$$

Uniqueness: We will show that

$$\delta \le \delta' \wedge \mathrm{App}(\delta,h) \wedge \mathrm{App}(\delta',h') \to h = h' \restriction \delta.$$

In particular, the case $\delta = \delta'$ gives the uniqueness of h. Fix δ, δ', h, h' as above. Suppose $h \neq h' \restriction \delta$. Then $X = \{\xi < \delta : h(\xi) \neq h'(\xi)\} \neq \varnothing$ and so there is $\mu = \min X$. Then for all $\xi < \mu$ $h(\xi) = h'(\xi)$. That is $h \restriction \mu = h' \restriction \mu$. But then $h(\mu) = G(h \restriction \mu) = G(h' \restriction \mu) = h'(\mu)$, which is a contradiction. Therefore $X = \varnothing$ and $h = h' \restriction \delta$.

Existence: By transfinite induction on $\mathbb{ON}$ show that $\forall \delta \exists h \mathrm{App}(\delta,h)$. Suppose not and let $\delta \in \mathbb{ON}$ be least such that $\neg \exists h \mathrm{App}(\delta,h)$. Thus in particular $\forall \xi < \delta \exists h_\xi$ such that $\mathrm{App}(\xi, h_\xi)$.

Case 1: $\delta = \varnothing$ - impossible, since $\mathrm{App}(0,\varnothing)$.

Case 2: If $\delta = \beta+1$ let $f = h_\beta \cup \{\langle \beta, G(h_\beta)\rangle\}$. Then $\mathrm{App}(\delta,f)$ which contradicts our hypothesis.

Case 3: δ is a limit ordinal. Let $f = \bigcup\{h_\xi : \xi < \delta\}$. Then uniqueness implies that f is a function and furthermore $\mathrm{App}(\delta,f)$, which is a contradiction to the choice of δ.

Thus $\forall \delta \in \mathbb{ON} \exists! h \mathrm{App}(\delta,h)$. Let $\psi(x,y)$ be the following formula:

$$(x \notin \mathbb{ON} \wedge y = 0) \vee (x \in \mathbb{ON} \wedge \exists \delta > x \exists h(\mathrm{App}(\delta,h) \wedge h(x) = y)).$$

The uniqueness and existence of h imply that $\forall x \exists! y \psi(x,y)$ and so $\psi(x,y)$ defines a function F. Now, let $\xi \in \mathbb{ON}$. Then pick any $\delta > \xi$ and h such that $\mathrm{App}(\delta,h)$. Then

$$F(\xi) = h(\xi) = G(h \restriction \xi) = G(F \restriction \xi)$$

as desired. □

REMARK 2.32. One can define ordinal addition and exponentiation by transfinite recursion on the ordinals as follows:

Ordinal addition Let $\alpha \in \mathbb{ON}$. Recursively over $\beta \in \mathbb{ON}$ define $\alpha + \beta$ as follows:

(1) $\alpha + 0 = \alpha$,
(2) $\alpha + \beta = S(\alpha + \gamma)$ if $\beta = S(\gamma)$.
(3) $\alpha + \beta = \bigcup_{\gamma \in \beta}(\alpha + \gamma)$ if β is a limit > 0.

Ordinal multiplication Let $\alpha \in \mathbb{ON}$. By recursion over $\beta \in \mathbb{ON}$ define the ordinal $\alpha \cdot \beta$ as follows:

(1) $\alpha \cdot 0 = 0$,

(2) $\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha$, if $\beta = S(\gamma)$,
(3) $\alpha \cdot \beta = \bigcup_{\gamma \in \beta}(\alpha \cdot \gamma)$, if β is a limit > 0.

Exercise 1. The latter two definitions are equivalent to the definitions of ordinal addition and ordinal multiplication respectively, which we gave earlier in the lecture.

Definition 2.33. (Ordinal Exponentiation) Recursively, one can define ordinal exponentiation as follows:

$$\alpha^0 = 1,\ \alpha^{S(\beta)} = \alpha^\beta \cdot \alpha,\ \alpha^\gamma = \sup_{\beta < \gamma} \alpha^\beta \text{ for } \gamma \text{ limit.}$$

3. Cardinal Arithmetic

3.1. Comparing infinities.

Definition 3.1. Let X, Y be sets.

(1) $X \preceq Y$ iff there is an injective function $f : X \to Y$;
(2) $X \approx Y$ iff there is a bijection $f : X \to Y$.

Remark 3.2. Note that

- $\preceq$ is transitive and reflexive, and that
- $\approx$ is an equivalence relations.

So, we can think of different infinite sizes as equivalence classes, consisting of sets any two of which are in bijective correspondence.

Lemma 3.3. If $B \subseteq A$ and there is an injective $f : A \to B$ then $A \approx B$.

Proof. Using the fact that $f(A) \subseteq B \subseteq A$ obtain:

$$A \supseteq B \supseteq f(A) \supseteq f(B) \supseteq f^2(A) \supseteq f^2(B) \supseteq f^3(A) \supseteq \dots$$

Let $f^0 = \mathrm{id}$ and for each $n \in \mathbb{N}$ let

$$H_n = f^n(A) \backslash f^n(B),\ K_n = f^n(B) \backslash f^{n+1}(A).$$

We will show that for each n, the functions

$$f \upharpoonright H_n : H_n \to H_{n+1} \text{ and } f \upharpoonright K_n : K_n \to K_{n+1}$$

are bijections.

Claim 3.4. $f \upharpoonright H_n : H_n \to H_{n+1}$ is a bijection, where $H_n = f^n(A) \backslash f^n(B)$.

Proof. Let $g = f \upharpoonright H_n$. Clearly since f is injective, then also g is injective. We need to show that g is onto. Let $x \in H_{n+1}$. Thus $x \in f^{n+1}(A) \backslash f^{n+1}(B)$. So clearly, there is $y \in f^n(A)$ such that $x = f(y)$. We need to show that $y \notin f^n(B)$. However, if $y \in f^n(B)$ then $f(y) = x \in f^{n+1}(B)$ which is a contradiction. Thus, $x = f(y)$ for some $y \in H_n = f^n(A) \backslash f^n(B)$, i.e. g is a bijection. $\square$

Consider the set $P = \bigcap_{n\in\omega} f^n(A) = \bigcap_{n\in\omega} f^n(B)$. Then

$$A = P \cup H_0 \cup H_1 \cup H_2 \cup \cdots \cup K_0 \cup K_1 \cup \cdots$$

$$B = P \cup H_1 \cup H_2 \cup H_3 \cup \cdots \cup K_0 \cup K_1 \cup \cdots$$

are partitions of A, B. Then the function $k : A \to B$ defined by

- $k \upharpoonright H_n = f \upharpoonright H_n$ for each n,
- $k \upharpoonright P = \mathrm{id}$ and
- $k \upharpoonright K_n = \mathrm{id}$ for each n,

is a bijection from A to B. □

THEOREM 3.5. *(Schröder-Bernstein) $A \approx B$ iff $A \leq B$ and $B \leq A$.*

PROOF. ($\Rightarrow$) If $f : A \to B$ is a bijection, then f witnesses $A \leq B$ and f^{-1} witnesses $B \leq A$.

($\Leftarrow$) Suppose $f : A \to B$ and $h : B \to A$ are injective. Let $\hat{B} = h(B)$. Then $\hat{B} \subseteq A$ and $h : B \to \hat{B}$ is a bijection. Thus, by definition $B \approx \hat{B}$. On the other hand $\hat{B} \subseteq A$ and so $h \circ f : A \to \hat{B}$ witnesses $A \leq \hat{B}$. Thus, by the previous Lemma $A \approx \hat{B}$. Since $B \approx \hat{B}$ we obtain $A \approx B$. □

DEFINITION 3.6. $X \prec Y$ iff $X \leq Y$ and it is not the case that $Y \leq X$.

REMARK 3.7. By the theorem of Schröder-Bernstein, $X \prec Y$ means that X can be mapped injectively into Y, but there is no bijection between X and Y.

LEMMA 3.8. (Cantor's Diagonal Element) If F is a function, $\mathrm{dom}(f) = A$ and $\mathbb{D} = \{x \in A : x \notin f(x)\}$ then $\mathbb{D} \notin \mathrm{ran}(f)$.

PROOF. Suppose $\mathbb{D} \in \mathrm{ran}(f)$. Then there is $x \in A$ such that $\mathbb{D} = f(x)$. There are two possibilities: *If $x \in f(x)$*, then $x \in \mathbb{D}$ (since $f(x) = \mathbb{D}$) and so x satisfies the defining characteristic of $\mathbb{D}$, i.e. x is an element of A such that $x \notin f(x)$. This is a contradiction. *If $x \notin f(x)$*, then since $x \in A$ we have that x satisfies the defining characteristic of $\mathbb{D}$ and so we must have that $x \in \mathbb{D}$, i.e. $x \in f(x)$. Again we reach a contradiction. Therefore $\mathbb{D} \notin \mathrm{ran}(f)$. □

THEOREM 3.9. $A \prec \mathcal{P}(A)$.

PROOF. Clearly $A \leq \mathcal{P}(A)$ witnessed by the mapping $x \mapsto \{x\}$ for each $x \in A$. We claim that $\mathcal{P}(A) \nleq A$. Well, suppose to the contrary that $\mathcal{P}(A) \leq A$. Then by Schröder-Bernstein $\mathcal{P}(A) \approx A$ and so there is a bijection $f : A \to \mathcal{P}(A)$. Then since $\mathbb{D} = \{x \in A : x \notin f(x)\} \in \mathcal{P}(A)$ and f is onto we must have $\mathbb{D} = \{x \in A : x \notin f(x)\} \in \mathrm{ran}(f)$ contradicting Cantor's Diagonal Element Lemma. □

COROLLARY 3.10. $\mathbb{N} \prec \mathcal{P}(\mathbb{N})$.

REMARK 3.11. Characteristic Functions Let A be a set and let $B \subseteq A$. Then we refer to $\chi_B : A \to 2 = \{0, 1\}$ defined by

$$\chi_B(a) = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{otherwise} \end{cases}$$

as the characteristic function of B. The mapping $B \mapsto \chi_B$ where $B \in \mathcal{P}(A)$ is a bijection between ${}^A2$ and $\mathcal{P}(A)$. Thus ${}^A2 \approx \mathcal{P}(A)$. In particular ${}^{\mathbb{N}}2 = 2^{\mathbb{N}} \approx \mathcal{P}(\mathbb{N})$.

REMARK 3.12. $\mathcal{P}(\mathbb{N}) \approx (0,1)$.

DEFINITION 3.13. (Finite, countable and uncountable sizes)

(1) A set A is said to be countable, if $A \preceq \omega$.
(2) A set A is said to be finite if $A \preceq n$ for some $n \in \omega$.
(3) Infinite means not finite. Uncountable means not countable.
(4) A countably infinite set is a countable set which is infinite.

3.2. Cardinal Numbers.

FACT 1.

(1) If $B \subseteq \alpha$ then $\mathrm{type}(B, \in) \leq \alpha$.
(2) If $B \preceq \alpha$, then $B \approx \delta$ for some $\delta \leq \alpha$.
(3) If $\alpha \leq \beta \leq \gamma$ and $\alpha \approx \gamma$ then $\alpha \approx \beta \approx \gamma$.

PROOF. (2) If $B \preceq \alpha$, then $B \approx \delta$ for some $\delta \leq \alpha$ (identify B with a subset of α and apply part (1)). To see item (3) notice that $\alpha \subseteq \beta$ and $\beta \preceq \alpha$ imply that $\alpha \approx \beta$. □

Thus, the ordinals come in blocks of the same size. Informally, the first ordinal in a block is called a cardinal.

DEFINITION 3.14. A cardinal is an ordinal α such that $\xi \prec \alpha$ for all $\xi \in \alpha$.

REMARK 3.15. Thus, an ordinal α fails to be a cardinal iff there is $\xi < \alpha$ such that $\xi \approx \alpha$. We denote by $\mathbb{CD}$ the collection of all cardinals.

THEOREM 3.16.

(1) If $\alpha \geq \omega$ is a cardinal, then α is a limit ordinal.
(2) Every natural number is a cardinal.
(3) If A is a set of cardinals, then $\sup A$ is a cardinal.
(4) ω is a cardinal.

PROOF. (1) Let $\alpha \geq \omega$ be an infinite cardinal. Suppose α is a successor ordinal. Thus $\alpha = \delta + 1 = \delta \cup \{\delta\}$. Then $f : \delta \cup \{\delta\} \to \delta$ defined by $f(\delta) = 0$, $f(n) = n+1$ for all $n \in \omega$ and $f(\xi) = \xi$ for all ξ such that $\omega \leq \xi < \delta$ is a bijection. Thus $\delta \in \alpha$, but $\delta \nprec \alpha$, which is a contradiction to α being a cardinal.

(2) Proceed by induction. Now, 0 is trivially a cardinal. Suppose n is a cardinal and suppose $S(n) = n+1$ is not a cardinal. Then $\exists \xi (\xi < S(n))$ such that $\xi \approx S(n)$. Thus there is a bijection $f : \xi \to S(n) = n \cup \{n\}$. Clearly $\xi \neq 0$ and so $\xi = S(m)$ for some $m < n$. But, then

$$f : m \cup \{m\} \to n \cup \{n\}$$

is a bijection. We have the following options: <u>*If* $f(m) = n$</u>, then $f \upharpoonright m : m \to n$ is a bijection, contradiction to the assumption that n is a cardinal. <u>*Otherwise* $f(m) = j \in n$</u>. Now $n \in \mathrm{ran}(f)$

and so there is $i \in m$ such that $f(i) = n$. Consider the mapping $g : m \to n$ defined by $g(i) = j$ and $g \upharpoonright m \backslash \{i\} = f$. Then g is a bijection, again a contradiction to the assumption that n is a cardinal.

(3) Suppose, by way of contradiction that $\sup A = \bigcup A$ is not a cardinal. Thus there is $\xi < \sup A$ such that $\xi \approx \sup A$. Recall that $\sup A$ is the least ordinal, which is greater or equal each element of A. Thus there is $\alpha \in A$ such that $\xi < \alpha$. However $\xi < \alpha \leq \sup A$ and $\xi \approx \sup A$ implies $\xi \approx \alpha$ which is a contradiction to α being a cardinal.

(4) Note that $\omega = \sup_{n\in\mathbb{N}} n = \bigcup_{n\in\mathbb{N}} n$ and so the claim follows from items (2) and (3) above. □

DEFINITION 3.17.

(1) We say that a set A is well-orderable, if there is a relation R on A such that (A, R) is a well-order.
(2) If A is well-orderable, then the cardinality of A, denoted $|A|$, is the least ordinal α such that $A \approx \alpha$.

REMARK 3.18. The cardinality of a set is always a cardinal number. Under the Axiom of Choice every set can be well-ordered and so under the AC every set is characterised by its cardinality.

LEMMA 3.19.

(1) If A is a set, which can be well-ordered and $f : A \to B$ is an onto mapping, then B can be well-ordered and $|B| \leq |A|$.
(2) Let κ be a cardinal and $B \neq \varnothing$. Then $B \leq \kappa$ if and only if there is an onto mapping $f : \kappa \to B$.

COROLLARY 3.20. (A) set $B \neq \varnothing$ is countable if and only if there is an onto function $f : \omega \to B$.

THEOREM 3.21. *(Hartogs, 1915) Let A be a set. Then there is a cardinal κ such that $\kappa \npreceq A$.*

PROOF. Fix A and let $W = \{(X, R) : X \subseteq A \wedge R \text{ well-orders } X\}$. Then if α is an ordinal, we have that

$$\alpha \preceq A \text{ iff } \exists (X, R) \in W \text{ s.t. } \alpha = \text{type}(X, R).$$

By the Axiom of Replacement $Z = \{\text{type}(X, R) + 1 : (X, R) \in W\}$ is a set. But then $\beta = \sup Z$ is an ordinal. Moreover, for each $\alpha \preceq A$, we have that $\beta > \alpha$. Thus, $\beta \npreceq A$. Take $\kappa = |\beta|$. Then $\kappa \approx \beta$ and $\kappa \npreceq A$. □

DEFINITION 3.22. Let A be a set. Then $\aleph(A)$ denotes the least cardinal κ such that $\kappa \npreceq A$. For ordinals α define $\alpha^+ = \aleph(\alpha)$.

DEFINITION 3.23. By transfinite recursion on $\mathbb{ON}$, define the cardinal numbers $\aleph_\xi$ as follows:

(1) $\aleph_0 = \omega_0 = \omega$
(2) $\aleph_{\xi+1} = \omega_{\xi+1} = (\aleph_\xi)^+$
(3) $\aleph_\eta = \omega_\eta = \sup\{\aleph_\xi : \xi < \eta\}$ whenever η is a limit ordinal.

REMARK 3.24. (The class of all cardinals) The collection of all cardinals is a proper class.

$$\aleph_0 = |\mathbb{N}| < \aleph_1 < \aleph_2 < \ldots < \aleph_n \ldots < \aleph_\omega < \aleph_{\omega+1} < \ldots$$

DISCUSSION 3.25. The cardinality of the real line How large is $\mathbb{R}$? What is $|\mathbb{R}|$? Note that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ and $|\mathcal{P}(\mathbb{N}) = 2^{\aleph_0}$ where $2^{\aleph_0}$ is cardinal exponentiation (to be defined shortly) and is the cardinality of the set of functions from $\mathbb{N}$ to 2.

THEOREM 3.26. *(Hessenberg, 1906) Suppose $\alpha \geq \omega$ is an ordinal. Then $|\alpha \times \alpha| = |\alpha|$. Thus in particular, if $\kappa \geq \omega$ is a cardinal, then $|\kappa \times \kappa| = \kappa$.*

REMARK 3.27. Observe that it is sufficient to prove the claim for cardinal numbers. Indeed. Suppose α is an infinite ordinal and we have proved that $||\alpha| \times |\alpha|| = |\alpha|$. Now $\alpha \approx |\alpha|$, which induces a bijection witnessing $|\alpha| \times |\alpha| \approx \alpha \times \alpha$ and so $||\alpha| \times |\alpha|| = |\alpha|$.

PROOF. Define a relation $\lhd$ on $\mathbb{ON} \times \mathbb{ON}$ as follows: $(\xi_1, \xi_2) \lhd (\eta_1, \eta_2)$ iff

- either $\max\{\xi_1, \xi_2\} < \max\{\eta_1, \eta_2\}$,
- or $\max\{\xi_1, \xi_2\} = \max\{\eta_1, \eta_2\}$ and $(\xi_1, \xi_2) <_{\text{lex}} (\eta_1, \eta_2)$.

Note that $\lhd$ is a well-order. It is sufficient to show that

CLAIM 3.28. For each infinite cardinal κ, $\text{type}(\kappa \times \kappa, \lhd) = \kappa$.

PROOF. Proceed by transfinite induction on κ. Let κ be the least infinite cardinal such that $\text{type}(\kappa \times \kappa, \lhd) \neq \kappa$. Now, let $\delta = \text{type}(\kappa \times \kappa, \lhd)$ and let $F : (\delta, <) \to (\kappa \times \kappa, \lhd)$ be an order preserving bijection. Since $\delta \neq \kappa$, there are two options $\delta > \kappa$ or $\delta < \kappa$.

Suppose $\delta > \kappa$. Then $F(\kappa)$ is defined and so $\exists (\xi_1, \xi_2) \in \kappa \times \kappa$ such that $F(\kappa) = (\xi_1, \xi_2)$. Let $\alpha = \max\{\xi_1, \xi_2\} + 1$. Then since κ is a limit ordinal, $\alpha < \kappa$. Moreover since F is order preserving, $F''\kappa \subseteq \alpha \times \alpha$. Therefore $\kappa \leq \alpha \times \alpha < \kappa$, which is clearly a contradiction. Now, suppose $\delta < \kappa$. Then $\kappa \leq \kappa \times \kappa \approx \delta$, which is a contradiction, since κ is a cardinal.

Therefore there is no such κ, i.e. for each infinite cardinal κ, $|\kappa \times \kappa| = \kappa$. This proves the claim and the theorem. $\square$

$\square$

3.3. Cardinal Arithmetic. Note that

(1) If $A \prec B$ and $C \prec D$, then ${}^AC \preceq {}^BD$.
(2) If $2 \leq C$, then $A \prec \mathcal{P}(A) \preceq {}^AC$, simply because $\mathcal{P}(A) \approx {}^A2 \prec {}^AC$.

LEMMA 3.29.

(1) ${}^C({}^BA) \approx {}^{C \times B}A$
(2) ${}^{(B \cup C)}A \approx {}^BA \times {}^CA$, where B and C are disjoint.

PROOF. (1) Consider the mapping $\Phi : {}^C({}^BA) \to {}^{C \times B}A$ defined by

$$\Phi(f)(c, b) = (f(c))(b).$$

(2) Consider the mapping $\Psi : {}^{B \cup C}A \to {}^BA \times {}^CA$ given by

$$\Psi(f) = (f \upharpoonright B, f \upharpoonright C).$$

□

DEFINITION 3.30. (Cardinal addition, multiplication and exponentiation) Let κ and λ be cardinals. Then:

(1) $\kappa + \lambda$ is defined to be the cardinality of the set $\{0\} \times \kappa \cup \{1\} \times \lambda$.
(2) $\kappa \times \lambda$ is defiend to be the cardinality of the set $\kappa \times \lambda$.
(3) κ^λ is the cardinality of the set ${}^\kappa\lambda := \{f \mid f : \kappa \to \lambda\}$.

LEMMA 3.31. (Monotonicity) Let $\kappa, \kappa', \lambda, \lambda'$ be cardinals such that $\kappa \le \kappa'$, $\lambda \le \lambda'$. Then:

(1) $\kappa + \lambda \le \kappa' + \lambda'$,
(2) $\kappa \cdot \lambda \le \kappa' \cdot \lambda'$,
(3) $\kappa^\lambda \le (\kappa')^{\lambda'}$.

PROOF. (1) Note that $\{0\} \times \kappa \cup \{1\} \times \lambda \subseteq \{0\} \times \kappa' \cup \{1\} \times \lambda'$. Thus $\mathrm{id} : \kappa + \lambda \le \kappa' + \lambda'$ and so $\kappa + \lambda \le \kappa' + \lambda'$.

(2) Similarly $\kappa \times \lambda \subseteq \kappa' \times \lambda'$ and so $\mathrm{id} : \kappa \cdot \lambda \le \kappa' \cdot \lambda'$. Therefore $\kappa \cdot \lambda \le \kappa' \cdot \lambda'$.

(3) Consider the mapping $\varphi : {}^\lambda\kappa \to {}^{(\lambda')}(\kappa')$ defined by

- $\varphi(f) \restriction \lambda = f$ and
- $\varphi(f)(\xi) = 0$ for all $\lambda \le \xi < \lambda'$.

When $\kappa = \kappa' = 0$, note that $0^0 = |{}^0 0| = |\{\varnothing\}| = 1$ and for $\lambda > 0$, $0^\lambda = |{}^\lambda 0| = |\varnothing| = 0$. □

LEMMA 3.32. Let κ, λ, θ be cardinals. The following properties refer to cardinal arithmetic:

(1) $\kappa + \lambda = \lambda + \kappa$,
(2) $\kappa \cdot \lambda = \lambda \cdot \kappa$,
(3) $(\kappa + \lambda) \cdot \theta = \kappa \cdot \theta + \lambda \cdot \theta$,
(4) $\kappa^{(\lambda \cdot \theta)} = (\kappa^\lambda)^\theta$,
(5) $\kappa^{(\lambda + \theta)} = \kappa^\lambda \cdot \kappa^\theta$.

PROOF. To see (1) note that $A \cup B = B \cup A$. To see (2) note that $A \times B = B \times A$. To see (3) observe that $(A \cup B) \times C = A \times C \cup B \times C$. To see (4) note that ${}^C({}^B A) \approx {}^{C \times B} A$. To see (5) observe that ${}^{(B \cup C)}A \approx {}^B A \times {}^C A$ provided that B, C are disjoint. □

EXAMPLE 3.33.

(1) ω, $\omega \cdot \omega$, $\omega + \omega$ are three different ordinals, all of the same cardinality.
(2) ω^ω as ordinal exponentiation is equal to $\sup_{n \in \omega} \omega^n$, which is a countable set.
(3) However, ω^ω as cardinal exponentiation is uncountable: $|{}^\omega\omega| = |\mathcal{P}(\omega)| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ (to be proven shortly).

LEMMA 3.34. Let κ, λ be cardinals and suppose at least one of them is infinite.

(1) Then the cardinal sum of κ and λ is equal to $\max\{\kappa, \lambda\}$.
(2) If none of them is 0, then the cardinal product of κ and λ is equal to $\max(\kappa, \lambda)$.

PROOF. Let $\kappa \le \lambda$. Thus λ is infinite. But then $\lambda \le \kappa + \lambda \le \lambda \times \lambda$. However we proved that $\lambda \times \lambda \approx \lambda$. Therefore $\lambda \le \kappa + \lambda$ and $\kappa + \lambda \le \lambda$. Therefore $\kappa + \lambda = \max\{\kappa, \lambda\} = \lambda$. To see the second claim assume that $\kappa \le \lambda$. Thus λ is infinite. Then $\lambda \le \kappa \times \lambda \le \lambda \times \lambda \approx \lambda$ and so $\kappa \times \lambda \approx \lambda$. □

LEMMA 3.35. If $2 \le \kappa \le 2^\lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$. All exponentiation here is cardinal exponentiation.

PROOF. $2^\lambda \le \kappa^\lambda \le (2^\lambda)^\lambda \le 2^{\lambda \times \lambda} = 2^{\lambda \cdot \lambda} = 2^\lambda$. □ □

COROLLARY 3.36. $2^\omega = \omega^\omega$.

REMARK 3.37. (CH and GCH)

(1) For every ordinal α, $2^{\aleph_\alpha} \ge \aleph_{\alpha+1}$.
(2) The Continuum Hypothesis(abbreviated CH) is the statement that $2^{\aleph_0} = \aleph_1$.
(3) The Generalized Continuum Hypothesis (abbreviated GCH) is the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha \in \mathbb{ON}$.

REMARK 3.38. Thus CH is the statement that the cardinality of the real line is the first uncountable cardinal. If CH holds, then there are no infinite sizes between $|\mathbb{N}|$ and $|\mathbb{R}|$.

4. Cofinality and Lemma of König

4.1. Cofinality.

DEFINITION 4.1. (Cofinality) If γ is a limit ordinal, then the cofinality of γ is defined as follows:

$$\mathrm{cf}(\gamma) = \min\{\mathrm{type}(X) : X \subseteq \gamma \wedge \sup(X) = \gamma\}.$$

We say that γ is a regular cardinal, if $\mathrm{cf}(\gamma) = \gamma$.

REMARK 4.2. Note that $\mathrm{cf}(\gamma) \le \gamma$.

EXAMPLE 4.3. $\aleph_0 < \aleph_1 < \ldots < \aleph_n < \ldots < \aleph_\omega < \ldots$ Then $\mathrm{cf}(\aleph_\omega) = \omega$.

LEMMA 4.4. Let γ be a limit ordinal. Then:

(1) If $A \subseteq \gamma$ and $\sup(A) = \gamma$, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A))$.
(2) $\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$. Thus $\mathrm{cf}(\gamma)$ is a regular ordinal.
(3) $\omega \le \mathrm{cf}(\gamma) \le |\gamma| \le \gamma$.
(4) If γ is a regular ordinal, then γ is a cardinal.

PROOF. (1) Let $\alpha = \mathrm{type}(A)$. Since γ is limit and A is unbounded in γ, α must be limit as well. Let $f : (\alpha, \in) \to (A, \in)$ be an isomorphism.

$\underline{\mathrm{cf}(\gamma) \le \mathrm{cf}(\alpha)}$: If $Y \subseteq \alpha$ is unbounded in α, then $f''(Y)$ is unbounded in γ and $\mathrm{type}(f''(Y)) = \mathrm{type}(Y)$. Now, take $Y \subseteq \alpha$ such that $\mathrm{type}(Y) = \mathrm{cf}(\alpha)$. Then $Y \subseteq \gamma$ is unbounded in γ, $\mathrm{type}(Y) = \mathrm{cf}(\alpha)$. Thus $\mathrm{cf}(\gamma) \le \mathrm{cf}(\alpha)$.

cf$(\alpha) \le$ cf(γ): Let $X \subseteq \gamma$ be unbounded and let type$(X) = \mathrm{cf}(\gamma)$ and consider the mapping $h : X \to A(\subseteq \gamma)$ given by $h(\zeta) = \min\{\eta : \eta \in A \wedge \eta \ge \zeta\}$. Then h is non-decreasing. Consider the set

$$X' = \{\eta \in X : \forall \xi \in X \cap \eta (h(\xi) < h(\eta))\}.$$

Therefore $h \restriction X' : X' \to A$ is order preserving and so injective. Thus $h(X')$ is unbounded in A. However the set A was chosen to be of order type α. Therefore

$$\mathrm{cf}(\alpha) \le \mathrm{type}(X') \le \mathrm{type}(X) = \mathrm{cf}(\gamma).$$

(2) Let $A \subseteq \gamma$ be an unbounded subset of γ of order type cf(γ). Then by part (1) of this Lemma, $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\mathrm{cf}(\gamma))$.

(3) By definition $\omega \le \mathrm{cf}(\gamma)$ and $|\gamma| \le \gamma$. So, we need to show that $\mathrm{cf}(\gamma) \le |\gamma|$. For this purpose, let $\kappa := |\gamma|$ and fix an onto function $f : \kappa \to \gamma$. Recursively, define a function $g : \kappa \to \mathbb{ON}$ as follows:

$$g(\eta) := \max\{f(\eta), \sup\{g(\xi) + 1 : \xi < \eta\}\}.$$

What can we say about g?

(1) $\mathrm{dom}(g) = \mathrm{dom}(f) = \kappa$,
(2) $g(\eta) \ge f(\eta)$ for all $\eta \in \kappa$,
(3) if $\xi < \eta$ then $g(\xi) < g(\eta)$, because $g(\eta) \ge g(\xi) + 1 > g(\xi)$,
(4) If $\eta = \zeta + 1$, then

$$g(\zeta + 1) = \max\{f(\zeta + 1), \sup\{g(\xi) + 1 : \xi \le \zeta\}\} = \max\{f(\zeta + 1), g(\zeta) + 1\}.$$

In particular we have that $g : \kappa \cong \mathrm{ran}(g)$ and so $\mathrm{type}(\mathrm{ran}(g)) = \kappa$.

If ran$(g) \subseteq \gamma$, then since $g(\eta) \ge f(\eta)$ and $\mathrm{ran}(f) = \gamma$, we have ran(g) is unbounded in γ. Therefore $\mathrm{cf}(\gamma) \le \kappa = |\gamma|$ and we are done.

If ran$(g) \nsubseteq \gamma$, we can find $\eta \in \kappa$ least such that $g(\eta) \ge \gamma$. Suppose $\eta = \xi + 1$. Then

$$g(\eta) = g(\xi + 1) = \max\{g(\xi) + 1, f(\eta)\}.$$

However $g(\eta) \ge \gamma$ and $f(\eta) < \gamma$. Thus $g(\eta) = g(\xi) + 1$. By minimality of η, $g(\xi) < \gamma$ and so $g(\xi) + 1 \le \gamma$. Therefore $g(\eta) = g(\xi) + 1 \le \gamma \le g(\eta)$. But then $\gamma = g(\xi) + 1$ is a successor, which is a contradiction! Therefore η is a limit ordinal and $g''\eta$ is unbounded in γ. Moreover $g \restriction \eta : \eta \approx g''\eta$. In particular $\mathrm{type}(g''\eta) \le \eta$. Then $\mathrm{cf}(\gamma) \le \mathrm{type}(g''\eta) \le \eta < \kappa = |\gamma|$.

(4) This is a direct corollary to (3). Indeed, suppose γ is regular. Then $\gamma = \mathrm{cf}(\gamma)$. However by item (3) we have that $\mathrm{cf}(\gamma) \le |\gamma| \le \gamma$. Therefore $\gamma \le |\gamma| \le \gamma$ and so $\gamma = |\gamma|$ is a cardinal. □

DEFINITION 4.5. (Regular and Singular Cardinals) Let γ be an infinite cardinal.

(1) If $\gamma = \mathrm{cf}(\gamma)$, we say that γ is regular.
(2) If $\mathrm{cf}(\gamma) < \gamma$, we say that γ is singular.

REMARK 4.6. By the previous Lemma, part (1), we have that $\mathrm{cf}(\alpha + \beta) = \mathrm{cf}(\beta)$. Indeed, the set $A = \{\alpha + \xi : \xi < \beta\}$ is unbounded in $\alpha + \beta$. Thus, for every limit ordinal $\gamma < \omega_1$, $\mathrm{cf}(\gamma) = \omega$. For every limit ordinal γ such that $\gamma < \omega_2$, either $\mathrm{cf}(\gamma) = \omega$ or $\mathrm{cf}(\gamma) = \omega_1$.

LEMMA 4.7. Let γ be a limit ordinal.

(1) Suppose $\gamma = \aleph_\alpha$, where $\alpha = 0$ or $\alpha = \beta + 1$ is a successor ordinal. Then γ is regular.
(2) If $\gamma = \aleph_\alpha$ for a limit ordinal α, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\alpha)$.

PROOF. (1) If $\alpha = 0$, then $\aleph_\alpha = \aleph_0 = \omega$ and $\omega \le \mathrm{cf}(\omega) \le |\omega| \le \omega$ is regular. Thus, suppose $\gamma = \aleph_{\beta+1}$. Consider any $A \subseteq \aleph_{\beta+1}$ such that $\mathrm{type}(A) < \aleph_{\beta+1}$. It is sufficient to show that A is not unbounded in $\aleph_{\beta+1}$, since then $\aleph_{\beta+1} \le \mathrm{cf}(\gamma)$. But $\mathrm{cf}(\gamma) \le |\gamma| = \aleph_{\beta+1}$ and so $\mathrm{cf}(\aleph_{\beta+1}) = \aleph_{\beta+1}$.

To show that A is not unbounded in γ, consider $\sup A = \bigcup A$. Note that $|A| \le \aleph_\beta$, because $|A| \le \mathrm{type}(A) < \aleph_{\beta+1}$. Moreover, every element of A is of cardinality at most $\aleph_\beta$. Therefore we can view A as a collection of $\le \aleph_\beta$-many sets, each of cardinality at most $\aleph_\beta$. Then, by the Axiom of Choice we obtain that $|\sup A| = |\bigcup A| \le \aleph_\beta$ (see Lemma 4.10). Thus $\sup A < \aleph_{\beta+1}$ (otherwise contradiction to the notion of a cardinal!) Thus A can not be unbounded in $\aleph_{\beta+1}$.

(2) Let $A = \{\aleph_\xi : \xi < \alpha\}$. Then $A \subseteq \aleph_\alpha$ and $\sup A = \aleph_\alpha$. By a previous Lemma $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\mathrm{type}(A))$. However $\mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\alpha)$. Thus $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$. □

EXAMPLE 4.8.

- $\mathrm{cf}(\aleph_n) = \aleph_n$ for each $n \in \omega$, and
- $\mathrm{cf}(\aleph_\omega) = \omega$.

4.2. König's Lemma.

REMARK 4.9. Let A, B be sets such that $A \ne \varnothing$. Then there is an injective function $g : B \to A$ if and only if there is an onto function $f : A \to B$.

LEMMA 4.10. (AC) Let κ be an infinite cardinal. If $\mathcal{F}$ is a family of sets with $|\mathcal{F}| \le \kappa$ and $|X| \le \kappa$ for each $X \in \mathcal{F}$, then $|\bigcup \mathcal{F}| \le \kappa$.

PROOF. Assume $\mathcal{F} \ne \varnothing$ and $\varnothing \notin \mathcal{F}$. Then there is an onto function $f : \kappa \to \mathcal{F}$. Similarly, for each $B \in \mathcal{F}$ fix an onto function

$$g_B : \kappa \to B.$$

This defines an onto mapping $h : \kappa \times \kappa \to \bigcup \mathcal{F}$ given by

$$h(\alpha, \beta) = g_{f(\alpha)}(\beta).$$

Since $|\kappa \times \kappa| = \kappa$, we obtain an onto mapping from κ onto $\bigcup \mathcal{F}$. □

THEOREM 4.11. *(AC) Let θ be a cardinal.*

(1) Suppose θ is regular and $\mathcal{F}$ is a family of sets, such that $|\mathcal{F}| < \theta$ and moreover $|S| < \theta$ for all $S \in \mathcal{F}$. Then $|\bigcup \mathcal{F}| < \theta$.
(2) Suppose $\mathrm{cf}(\theta) = \lambda < \theta$. Then there is a family $\mathcal{F}$ of subsets of θ with $|\mathcal{F}| = \lambda$ and $|\bigcup \mathcal{F}| = \theta$ such that $|S| < \theta$ for all $S \in \mathcal{F}$.

PROOF. (1) Let $X = \{|S| : S \in \mathcal{F}\}$. Then $X \subseteq \theta$, $|X| < \theta$ and so $\mathrm{type}(X) < \theta$. Since θ is regular, $\mathrm{type}(X) < \mathrm{cf}(\theta)$ and so X is not unbounded in θ. Thus $\sup(X) < \theta$. Consider $\kappa := \max\{\sup(X), |\mathcal{F}|\}$. Then $\kappa < \theta$. If κ is infinite, then by Lemma 4.10 $|\bigcup \mathcal{F}| \le \kappa$. If κ is finite, then $\bigcup \mathcal{F}$ is finite. In either of those two cases $|\bigcup \mathcal{F}| < \theta$.

(2) Just take $\mathcal{F}$ to be a subset of θ such that $\operatorname{type}(\mathcal{F}) = \lambda$ and $\sup(\mathcal{F}) = \bigcup \mathcal{F} = \theta$. □

THEOREM 4.12. *(König) Let* $\kappa \geq 2$ *and* λ *be infinite. Then* $\operatorname{cf}(\kappa^\lambda) > \lambda$.

PROOF. Let $\theta = \kappa^\lambda$. Note that θ is infinite and $\theta^\lambda = \kappa^{\lambda \cdot \lambda} = \kappa^\lambda = \theta$. Thus, we can enumerate ${}^\lambda\theta$ is order type θ, i.e. ${}^\lambda\theta = \{f_\alpha : \alpha \in \theta\}$. There are two options. Either $\operatorname{cf}(\kappa^\lambda) \leq \lambda$ or $\operatorname{cf}(\kappa^\lambda) > \lambda$.

If $\operatorname{cf}(\kappa^\lambda) \leq \lambda < 2^\lambda \leq \kappa^\lambda$, then by Theorem 4.11 we have $\theta = \bigcup_{\xi<\lambda} S_\xi$, where each $|S_\xi| < \theta$. Let $g : \lambda \to \theta$ be the function $g(\xi) = \min(\theta \setminus \{f_\alpha(\xi) : \alpha \in S_\xi\})$. Then $g \in {}^\lambda\theta$ and so there is $\alpha \in \theta$ such that $g = f_\alpha$. Take $\xi < \lambda$ such that $\alpha \in S_\xi$. Then $g(\xi) \neq f_\alpha(\xi)$, contradiction.

Therefore $\operatorname{cf}(\kappa^\lambda) > \lambda$. □

EXAMPLE 4.13.

(1) $\operatorname{cf}(2^{\aleph_0}) > \aleph_0 = \omega$ and so $2^{\aleph_0}$ can not be $\aleph_\omega$.
(2) Consistently (using the method of forcing) $2^{\aleph_0}$ is any cardinal of uncountable cofinality, e.g. $\aleph_{2020}$, $\aleph_{\omega+1}$, $\aleph_{\omega_1}$, etc.

THEOREM 4.14. *Assume GCH. Let* κ, λ *be cardinals such that* $\max\{\kappa, \lambda\} \geq \omega$.

(1) Suppose $2 \leq \kappa \leq \lambda^+$. *Then* $\kappa^\lambda = \lambda^+$.
(2) Suppose $1 \leq \lambda \leq \kappa$. *Then* $\kappa^\lambda = \kappa$ *provided that* $\lambda < \operatorname{cf}(\kappa)$ *and* $\kappa^\lambda = \kappa^+$ *provided that* $\lambda \geq \operatorname{cf}(\kappa)$.

PROOF. (1) Since we have GCH, $2^\lambda = \lambda^+$. Then $2 \leq \kappa \leq 2^\lambda$. But then $2^\lambda \leq \kappa^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$ and so $\kappa^\lambda = 2^\lambda$. Thus by GCH we obtain $\kappa^\lambda = \lambda^+$.

(2) Since $1 \leq \lambda \leq \kappa$ we have that $\kappa \leq \kappa^\lambda \leq \kappa^\kappa = 2^\kappa = \kappa^+$ (the latter equality by GCH). Therefore either $\kappa^\lambda = \kappa$ or $\kappa^\lambda = \kappa^+$. By König's Lemma $\operatorname{cf}(\kappa^\lambda) > \lambda$. Thus:

If $\operatorname{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda \neq \kappa$. Therefore $\kappa^\lambda = \kappa^+$. Done!

If $\lambda < \operatorname{cf}(\kappa)$, then every $f : \lambda \to \kappa$ is bounded. Thus for all $f \in {}^\lambda\kappa$ there is $\alpha_f < \kappa$ such that $f \in {}^\lambda\alpha_f$ and so ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$. Now ${}^\lambda\alpha \subseteq \mathcal{P}(\lambda \times \alpha)$ and for $\alpha < \kappa$, $|\lambda \times \alpha| < \kappa$. Therefore $|{}^\lambda\alpha| \leq \kappa$ by GCH. Then by Lemma 4.10 we have also $|{}^\lambda\kappa| \leq \kappa$ and so $\kappa^\lambda = \kappa$. Done! □

DEFINITION 4.15. (The beth function) By recursion on the ordinals define $\beth_\zeta$ as follows:

(1) $\beth_0 = \aleph_0 = \omega$,
(2) $\beth_{\zeta+1} = 2^{\beth_\zeta}$,
(3) $\beth_\eta = \sup\{\beth_\zeta : \zeta < \eta\}$ for η limit ordinal.

REMARK 4.16. CH is equivalent to the statement that $\beth_1 = \aleph_1$ and GCH is equivalent to the statement that $\beth_\xi = \aleph_\xi$ for all $\xi \in \mathbb{ON}$.

DEFINITION 4.17. A cardinal κ is said to be weakly inaccessible if $\kappa > \omega$, κ is regular and $\kappa > \lambda^+$ for all $\lambda < \kappa$. A cardinal κ is strongly inaccessible if $\kappa > \omega$ is regular and $\kappa > 2^\lambda$ for all $\lambda < \kappa$.

REMARK 4.18. If κ is strong inaccessible, then κ is weakly inaccessible. The existence of a strong inaccessible cardinal is not provable in ZFC.

CHAPTER 2

Foundations and Consturctibility

1. Well-founded relations

1.1. Well-foundedness.

Definition 1.1. Let R be a relation on a class A. If $y \in A$, let

$$y \downarrow= \mathrm{pred}_R(y) = \mathrm{pred}_{A,R}(y) = \{x \in A : xRy\}.$$

The relation R is said to be set-like on A iff for all $y \in A$, $y \downarrow$ is a set.

Example 1.2.

(1) If $A = V$, where V denotes the collection of all sets and R is the membership relation, then $y \downarrow= y$ and so $\in$ is set-like.
(2) If $A = V$, where V denotes the collection of all sets and R is the subset relation, then $y \downarrow= \mathcal{P}(y)$. Thus R is set-like if and only if the power set axiom holds.
(3) The membership relation is set-like on the class of all ordinals.
(4) Let A be the class of all pairs of ordinals and R be the lexicographic order. Fix any pair (α, β). Then for any ordinal γ, $(\varnothing, \gamma) \leq_{lex} (\alpha, \beta)$ and so $(\alpha, \beta) \downarrow$ is a proper class.

Definition 1.3. Let A be a class and R a relation on A.

(1) An R-path of n steps in A, where $n \in \mathbb{N}$ and $n \geq 1$ is a function s with domain $n+1$ such that for all $i < n(s(i)Rs(j+1))$. Moreover, s is said to be a path from $s(0)$ to $s(n)$.
(2) The transitive closure of R on A, denoted as $R^* = R^*_A$ is the set of all pairs (a, b) of elements of A such that there is a path from a to b.

Lemma 1.4. Let R be a relation on a class A. Then

(1) The transitive closure R^* of R is a transitive relation on A.
(2) If R is set-like on A, then R^* is set-like on A.

Proof. The relation R^* is transitive on A, since the composition of two paths is a path. Suppose R is set-like on A. For each $n \geq 1$, let

$$D_n(a) = \{x \in A : \exists \text{ path in } A \text{ from } x \text{ to } a \text{ of } n \text{ steps}\}.$$

By induction on n we will show that for each $a \in A$, $D_n(a)$ is a set. Fix $a \in A$. Then $D_0(a) = \varnothing$, $D_1(a) = \mathrm{pred}_R(a)$ which is a set, since R is set-like. Suppose $n \geq 1$ and $D_n(a)$ is a set. Then by the axiom of replacement

$$E = \{\mathrm{pred}_R(y) : y \in D_n(a)\}$$

is a set and so by the union axiom, $\bigcup E = D_{n+1}(a)$ is also a set. Now, by the axiom of replacement

$$F = \{D_n(a) : n \in \omega\}$$

is also a set and so by the union axiom, $\text{pred}_{R^*}(a) = \bigcup F$ is also a set. Therefore R^* is indeed set-like. □

THEOREM 1.5. *(Transfinite Induction on Well-Founded Relations) Assume R is well-founded and set-like on A. Let X be a non-empty subclass of A. Then X has an R-minimal element.*

PROOF. Fix $a \in A$. Then, since R^* is set-like, we have that $Y = \{a\} \cup (\text{pred}_{R^*}(a) \cap X)$ is a set. By definition, R is well-founded and so there is $b = \min_R Y$. If there is y such that yRb then $y \in \text{pred}_{R^*}(a)$. Now, if $y \in X$ then $y \in Y$ and yRb is a contradiction to the minimality of b. Therefore either there is no such y, or $y \notin X$. □

THEOREM 1.6. *(Transfinite Recursion on Well-founded Relations) Let A be a defined class and let R be a defined relation on A, which is set-like and well-founded on A. Suppose for all x, s there is a unique y such that $\varphi(x,s,y)$ and so φ defines a function G with the property that for all x, s, $G(x,s) = y$ where $\varphi(x,s,y)$. Then, there is a formula ψ such that the following are provable:*

(1) $\forall x \exists! y \psi(x,y)$ and so ψ defined a function, which we denote F
(2) for all $a \in A$ we have

$$F(a) = G(a, F \upharpoonright (a\downarrow)) = G(a, F \upharpoonright \mathit{pred}_{A,R}(a)).$$

PROOF. Consider the formula $\text{App}(d,h)$ which states:

(1) h is a function
(2) $d = \text{dom}(h) \subseteq A$
(3) for all $y \in d$, $\text{pred}_{A,R}(y) \subseteq d$
(4) for all $y \in d$, $h(y) = G(y, h \upharpoonright (y\downarrow))$.

Note that item (3) implies that $\text{pred}_{A,R^*}(y) \subseteq d$ for all $y \in d$. By item (4), h is an approximation to F. Since R is set-like, R^* is also set-like and for all $x \in A$, $d_x = \{x\} \cup \text{pred}_{A,R^*}(x)$ is a set. Now, let $\psi(x,y)$ be the following formula

$$x \notin A \wedge y = \varnothing) \vee (x \in A \wedge \exists d, h(\text{App}(d,h) \wedge x \in d \wedge h(x) = y)).$$

Uniqueness of Approximations: Suppose $\text{App}(d,h) \wedge \text{App}(d',h')$. We will show that $\text{App}(d \cap d', h \cap h')$. Note that for all $y \in d \cap d'$ we have $\text{pred}_{A,R}(y) \subseteq d \cap d'$. Furthermore, by induction using item (4) one can show that $h \upharpoonright d \cap d' = h' \upharpoonright d \cap d'$. Indeed, if this is not the case, then there is $y_0 \in d \cap d'$ such that $h(y_0) \neq h'(y_0)$ and without loss of generality, we can assume that y_0 is R-least with this property. But, then by item (4) we have

$$h(y_0) = G(y_0, h \upharpoonright (y_0\downarrow)) = G(y_0, h' \upharpoonright (y_0\downarrow)) = h'(y_0),$$

which is a contradiction.

Existence of Approximations: We want to show that $\forall x \in A \exists d, g(\text{App}(d,h) \wedge x \in d)$. Note that if $\text{App}(d,h) \wedge x \in d$ then $\text{App}(d_x, h_x)$ where $h_x = h \upharpoonright d_x$. We proceed, by induction. Suppose

$$X = \{x \in A : \neg\exists d, h(\text{App}(d,h) \wedge x \in d)\} \neq \varnothing.$$

Let $a = \min_R(X)$. Since a is R-least, for each $x \in \mathrm{pred}_{R,A}(a)$ there are d_x, h_x such that $\mathrm{App}(d_x, h_x)$. Take

$$\tilde{d} = \bigcup\{d_x : x \in \mathrm{pred}_{R,A}(a)\}, \tilde{h} = \bigcup\{h_x : x \in \mathrm{pred}_{R,A}(a)\}.$$

Then $\mathrm{App}(\tilde{d}, \tilde{h})$. Now, take $d := \tilde{d} \cup \{a\}$ and $h := \tilde{h} \cup \{(a, G(a, \tilde{h} \upharpoonright (a\downarrow)))\}$. Then $\mathrm{App}(d, h)$ and since $a \in A$, we reach a contradiction to $a \in X$.

By the uniqueness and existence of the approximating functions we obtain that $\forall x \in A \exists! y \psi(x, y)$. Therefore, ψ defines a function F as desired. $\square$

REMARK 1.7. Note that if F and F' satisfy item (2) of the above theorem, then $F(a) = F(a')$ for all $a \in A$. Indeed, if this is not the case, then $X = \{a \in A : F(a) \neq F'(a)\}$ is non-empty and so we can take $a = \min_R X$. But then, by minimality of a we have that $F \upharpoonright (a\downarrow) = F' \upharpoonright (a\downarrow)$ and so

$$F(a) = G(a, F \upharpoonright (a\downarrow)) = G(a, F' \upharpoonright (a\downarrow)) = F'(a),$$

which is a contradiction.

1.2. Rank.

DEFINITION 1.8. Let R be a relation, which is well-founded and set-like on a class A. For $y \in A$ define

$$\mathrm{rank}(y) := \mathrm{rank}_{A,R}(y) = \bigcup\{S(\mathrm{rank}(x)) : x \in \mathrm{pred}_{A,R}(y)\}.$$

Let $\mathrm{rank}(y) = \varnothing$ for $y \notin A$.

<u>Justification</u> Let $G(x, s) = \cup\{S(t) : t \in \mathrm{range}(s)\}$. Then $G(x, s)$ does not depend on x and is defined for all s, x. Then $F(a) = G(a, F \upharpoonright (a\downarrow)) = \cup\{S(F(c)) : c \in A, cRa\}$.

LEMMA 1.9. Let R be well-founded and set-like on A. Then

(1) For all $y \in A$, $\mathrm{rank}(y)$ is an ordinal and so

$$\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 : x \in \mathrm{pred}_{A,R}(y)\}.$$

(2) If $x \in \mathrm{pred}_{A,R}(y)$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.

PROOF. To see item (1) proceed by induction. Suppose y is R-minimal such that $\mathrm{rank}(y)$ is not an ordinal. However $\mathrm{rank}(y) = \cup\{S(\mathrm{rank}(x)) : x \in \mathrm{pred}_{A,R}(y)\} = \sup\{\mathrm{rank}(x) + 1 : x \in \mathrm{pred}_{A,R}(y)\}$ which is an ordinal and so we reached a contradiction. To see item (2) consider any $x \in \mathrm{pred}_{A,R}(y)$. Then by definition $\mathrm{rank}(y) \geq \mathrm{rank}(x) + 1 > \mathrm{rank}(x)$. $\square$

LEMMA 1.10. Let A be a defined class, R a defined relation on A. If there is a defined function $\Phi : A \to \mathbb{ON}$ such that

$$\text{if } xRy \text{ then } \Phi(x) < \Phi(y)$$

then R is well-founded in A.

PROOF. Let X be a subset of A, $X \neq \varnothing$. Then $\{\Phi(x) : x \in X\}$ is a set (by replacement) of ordinals and so it has an $\in$-minimal elements $\alpha = \Phi(y)$ for some y. Clearly, y is R-minimal in X. Indeed, if zRy and $z \in X$, then $\Phi(z) < \Phi(y)$, which is a contradiction to the minimality of $\Phi(y)$. $\square$

LEMMA 1.11. Let A be a defined class and R a defined relation on A. If R is set-like and well-founded on A, then R^* is well-founded in A.

PROOF. Define $\Phi : A \to \mathbb{ON}$ by $\Phi(x) := \text{rank}_{A,R}(x)$. If xR^*y, then there is a path from x to y of n steps, where $n \geq 1$ and so by Lemma 1.9 $\text{rank}(x) < \text{rank}(y)$, i.e. $\Phi(x) < \Phi(y)$. By Lemma 1.10. R^* is well-founded on A. □

LEMMA 1.12. Let A be a defined class and R a defined relation which is set-like and well-founded on A. Fix $b \in A$ and $\alpha < \text{rank}_{R,A}(b)$. Then, there is $a \in A$ such that $aR^*_A b$ and $\text{rank}_{R,A}(a) = \alpha$.

PROOF. Consider the class

$$X = \{c \in A : \text{rank}(c) > \alpha \text{ and } \neg(\exists u \in \text{pred}_{A,R^*}(c) \wedge \text{rank}_{A,R}(u) = \alpha)\}.$$

Suppose $X \neq \emptyset$. Since R is set-like and well-founded on A, there is $c \in X$ which is R-minimal. Note that $\text{rank}_{A,R}(c) = \sup\{\text{rank}(t) + 1 : t \in \text{pred}_{A,R}(c)\}$. Since $\text{rank}(c) > \alpha \geq 0$, $\text{pred}_{A,R}(c) \neq \emptyset$. If $\text{rank}(t) + 1 \leq \alpha$ for all $t \in \text{pred}_{A,R}(c)$ then $\text{rank}(c) \leq \alpha$ which is a contradiction to the choice of c. Therefore there is $t \in \text{pred}_{A,R}(c)$ such that $\text{rank}(t) + 1 > \alpha$, i.e. $\text{rank}(t) \geq \alpha$. Fix such t.

If $\text{rank}(t) = \alpha$, then tR^*b is a contradiction to $c \in X$.

If $\text{rank}(t) \geq \alpha + 1 > \alpha$, then since $t \in \text{pred}_{A,R}(c)$ and c is R-minimal in X, $t \notin X$. Thus, there is $d \in \text{pred}_{A,R^*}(t)$ such that $\text{rank}(d) = \alpha$. But then $d \in \text{pred}_{A,R^*}(c)$ which is a contradiction to $c \in X$.

Therefore, $X = \emptyset$ and so there is $a \in \text{pred}_{A,R^*}(b)$ such that $\text{rank}_{A,R}(a) = \alpha$. □

LEMMA 1.13. Let α be an ordinal.

(1) Then $\text{rank}_{\mathbb{ON},\in}(\alpha) = \alpha$.
(2) If the Axiom of Foundation holds, then $\text{rank}_{V,\in}(\alpha) = \alpha$.

PROOF. To prove item (1) observe that $\in$ is set-like and well-founded on $\mathbb{ON}$ and so we can define $\text{rank}_{\mathbb{ON},\in}$. We proceed by induction. If the claim is not true, then

$$X = \{\alpha \in \mathbb{ON} : \text{rank}_{\mathbb{ON},\in}(\alpha) \neq \alpha\}$$

is non-empty and so it has an $\in$-minimal element α. Then

$$\text{rank}_{\mathbb{ON},\in}(\alpha) = \sup\{\xi + 1 : \xi < \alpha\} = \alpha,$$

which is a contradiction.

To see item (2) consider $X = \{\alpha \in \mathbb{ON} : \text{rank}_{V,\in}(\alpha) \neq \alpha\}$. If $X \neq \emptyset$, then it has a least element and the proof continues as in part (1). □

LEMMA 1.14. Suppose $A \subseteq B$ and R is well-founded and set-like on B.

(1) If $b \in A$ then $\text{rank}_{A,R}(b) \leq \text{rank}_{B,R}(b)$
(2) If $b \in A$ and $\text{pred}_{B,R^*_B}(b) \subseteq A$, then $\text{rank}_{A,R}(b) = \text{rank}_{B,R}(b)$.

PROOF. (1) Suppose not. Then $X = \{x \in A : \operatorname{rank}_{A,R}(x) > \operatorname{rank}_{B,R}(x)\} \neq \varnothing$. Since R is well-founded on B (and set-like), it is also well-founded on A. Then $X \subseteq A$ has an R-minimal element a. Then

$$\begin{aligned}\operatorname{rank}_{A,R}(a) &= \sup\{\operatorname{rank}_{A,R}(t)+1 : t \in \operatorname{pred}_{A,R}(a)\}\\ &\leq \sup\{\operatorname{rank}_{B,R}(t)+1 : t \in \operatorname{pred}_{A,R}(a)\}\\ &\leq \sup\{\operatorname{rank}_{B,R}(t)+1 : t \in \operatorname{pred}_{B,R}(a)\} = \operatorname{rank}_{B,R}(a),\end{aligned}$$

which is a contradiction.

(2) The second claim is proven similarly. Suppose by way of contradiction that

$$X = \{b \in A : \operatorname{pred}_{B,R^*_B}(b) \subseteq A \wedge \operatorname{rank}_{A,R}(b) < \operatorname{rank}_{B,R}(b)\} \neq \varnothing.$$

Let b be R-minimal in X. Then

$$\begin{aligned}\operatorname{rank}_{A,R}(b) &= \sup\{\operatorname{rank}_{A,R}(t)+1 : t \in \operatorname{pred}_{A,R}(b)\}\\ &= \sup\{\operatorname{rank}_{B,R}(t)+1 : t \in \operatorname{pred}_{B,R}(b)\}\\ &= \operatorname{rank}_{B,R}(b),\end{aligned}$$

which is a contradiction. □

DEFINITION 1.15. Let x be a set. Let

(1) $\bigcup^0 x = x$
(2) For $n \geq 1$ let $\bigcup^{n+1} x = \bigcup\bigcup^n x$.

Finally, let $\operatorname{trcl}(x) = \bigcup\{\bigcup^n x : n \in \omega\}$.

LEMMA 1.16. Let b be a set. Then the membership relation is well-founded on $\operatorname{trcl}(b)$ iff it is well-founded on $\{b\} \cup \operatorname{trcl}(b)$.

PROOF. Note that if $b \in \operatorname{trcl}(b)$ then the two sets coincide and so the statement is trivially true. Suppose $b \notin \operatorname{trcl}(b)$.

($\Rightarrow$). Suppose $\in$ is well-founded on $\operatorname{trcl}(b)$. Let $X \subseteq \{b\} \cup \operatorname{trcl}(b)$. If $b \notin X$, then $X \subseteq \operatorname{trcl}(b)$ and so by hypothesis X has an $\in$-minimal element. If $X = \{b\}$ then $b = \min_\in X$. Thus, suppose $b \in X$ and $X \setminus \{b\} \neq \varnothing$. Since $X \setminus \{b\} \subseteq \operatorname{trcl}(b)$, we can take $c = \min_\in(X \setminus \{b\})$. In particular $c \in \operatorname{trcl}(b)$. If $b \in c$, then since $\operatorname{trcl}(b)$ is a transitive set we obtain that $b \in \operatorname{trcl}(b)$, contrary to our hypothesis. Therefore $b \notin c$ and so $c = \min_\in X$.

($\Leftarrow$) Straightforward, since $\operatorname{trcl}(b) \subseteq \operatorname{trcl}(b) \cup \{b\}$. □

REMARK 1.17. If $b \in^* b$, i.e. $b \in \operatorname{trcl}(b)$ then $\in$ is not well-founded on $\operatorname{trcl}(b)$, since $\in$ is not irreflexive.

DEFINITION 1.18. A set b is said to be well-founded if $\in$ is well-founded on $\operatorname{trcl}(b)$. For a well-founded set b, let $\operatorname{rank}(b) = \operatorname{rank}_{\{b\}\cup\operatorname{trcl}(b),\in}(b)$. WF denotes the class of well-founded sets.

COROLLARY 1.19. Let T be a transitive class and let $\in$ be well-founded on T. Then $T \subseteq \mathrm{WF}$ and $\operatorname{rank}(b) = \operatorname{rank}_{T,\in}(b)$ for all $b \in T$.

PROOF. If $b \in T$ then $\operatorname{pred}_{T,\in^*}(b) = \operatorname{trcl}(b) \subseteq T$. Thus, the statement follows by Lemma 1.14. □

COROLLARY 1.20. The class of all ordinals is a subclass of the class of well-founded sets and so WF is a proper class. Moreover, $\mathrm{rank}(\alpha) = \alpha$ for all $\alpha \in \mathbb{ON}$.

COROLLARY 1.21. The Axiom of Foundation is equivalent to the statement that $\mathrm{V} = \mathrm{WF}$.

1.3. Basic Properties of Well-founded Sets.

LEMMA 1.22.

(1) Suppose b is a well-founded set and $x \in b$. Then x is well-founded and $\mathrm{rank}(x) < \mathrm{rank}(b)$. Thus, in particular, WF is a transitive class.
(2) $\in$ is well-founded on WF.
(3) If b is a set of well-founded sets, then b is well-founded.
(4) Let $b \in \mathrm{WF}$. Then $\mathrm{rank}(b) = \mathrm{rank}_{\mathrm{WF},\in}(b)$.
(5) Let $b \in \mathrm{WF}$. Then $\mathrm{rank}(b) = \sup\{\mathrm{rank}(x) + 1 : x \in b\}$.
(6) Let $b \in \mathrm{WF}$ and $c \subseteq b$. Then $c \in \mathrm{WF}$ and $\mathrm{rank}(c) \leq \mathrm{rank}(b)$.

PROOF. (1) Since $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(b)$, $\in$ is well-founded on the transitive closure of x and so $x \in \mathrm{WF}$. Then $\mathrm{rank}(x) = \mathrm{rank}_{\{x\}\cup\mathrm{trcl}(x),\in}(x)$ (by definition) and by Lemma 1.14

$$\mathrm{rank}_{\{x\}\cup\mathrm{trcl}(x),\in}(x) = \mathrm{rank}_{\{b\}\cup\mathrm{trcl}(b),\in}(x) < \mathrm{rank}_{\{b\}\cup\mathrm{trcl}(b),\in}(b).$$

(2) Exercise!

(3) Suppose x consists of well-founded sets. Then $\mathrm{trcl}(x)$ is a set of well-founded sets and since $\in$ is well-founded on WF, it is well-founded on $\mathrm{trcl}(x)$. Thus x is well-founded by definition.

(4) The claim is immediate from Lemma 1.14.(2) since $\mathrm{trcl}(b) \subseteq \mathrm{WF}$.

(5) Immediate from item (4) and Lemma 1.9.

(6) By (3) $c \in \mathrm{WF}$. By (4)

$$\mathrm{rank}(c) = \sup\{\mathrm{rank}(x) + 1 : x \in c\} \leq \sup\{\mathrm{rank}(x) + 1 : x \in b\} = \mathrm{rank}(b),$$

just because $c \subseteq b$. □

COROLLARY 1.23. Let $x, y \in \mathrm{WF}$. Then

(1) $\{x, y\} \in \mathrm{WF}$ and $\mathrm{rank}(\{x, y\}) = \max(\mathrm{rank}(x), \mathrm{rank}(y)) + 1$.
(2) $\langle x, y\rangle \in \mathrm{WF}$ and $\mathrm{rank}(\langle x, y\rangle) = \max(\mathrm{rank}(x), \mathrm{rank}(y)) + 2$.
(3) If $\mathcal{P}(x)$ exists, then $\mathcal{P}(x) \in \mathrm{WF}$ and $\mathrm{rank}(\mathcal{P}(x)) = \mathrm{rank}(x) + 1$.
(4) $\bigcup x \in \mathrm{WF}$ and $\mathrm{rank}(\bigcup x) \leq \mathrm{rank}(x)$
(5) $x \cup y \in \mathrm{WF}$, $\mathrm{rank}(x \cup y) = \max(\mathrm{rank}(x), \mathrm{rank}(y))$.
(6) $\mathrm{trcl}(x) \in \mathrm{WF}$ and $\mathrm{rank}(trcl(x)) = \mathrm{rank}(x)$.

PROOF. (1) By assumption, $x \in \mathrm{WF}$ and $y \in \mathrm{WF}$, so $\{x, y\} \subseteq \mathrm{WF}$. However, every set consisting of well-founded sets is well-founded (by Lemma 1.22(3)) and so $\{x, y\} \in \mathrm{WF}$. To calculate the rank, proceed as follows:

$$\begin{aligned}
\mathrm{rank}\left(\{x, y\}\right) &= \sup\left\{\mathrm{rank}(z) + 1 : z \in \{x, y\}\right\} \quad \text{by Lemma 1.22(5)} \\
&= \max\left\{\mathrm{rank}(x) + 1, \mathrm{rank}(y) + 1\right\} \\
&= \max\left\{\mathrm{rank}(x), \mathrm{rank}(y)\right\} + 1.
\end{aligned}$$

(2) By (1), we have that both $\{x\}$ and $\{x,y\}$ are in WF. Then again by (1) we have $\langle x,y\rangle = \big\{\{x\},\{x,y\}\big\} \in$ WF. To calculate the rank, note that

$$\begin{aligned}
\operatorname{rank}\big(\langle x,y\rangle\big) &= \sup\big\{\operatorname{rank}(z)+1 : z \in \langle x,y\rangle\big\} \quad \text{by Lemma 1.22(5)}\\
&= \max\big\{\operatorname{rank}(\{x\})+1, \operatorname{rank}(\{x,y\})+1\big\}\\
&= \max\big\{\operatorname{rank}(x)+2, \max\{\operatorname{rank}(x),\operatorname{rank}(y)\}+2\big\} \quad \text{by (1)}\\
&= \max\big\{\operatorname{rank}(x),\operatorname{rank}(y)\big\}+2.
\end{aligned}$$

(3) Since $x \in$ WF, it follows from Lemma 1.22(1) that $x \subseteq$ WF. Note that for every $z \subseteq x$, we have $z \subseteq x \subseteq$ WF, and so $z \in$ WF. Thus $\mathcal{P}(x)$ is a set, consisting of well-founded sets and so $\mathcal{P}(x) \in$ WF. By Lemma 1.22.(6) for each $z \subseteq x$ we have $\operatorname{rank}(z) \le \operatorname{rank}(x)$. Then

$$\operatorname{rank}(x)+1 \le \operatorname{rank}(\mathcal{P}(x)) = \sup\{\operatorname{rank}(z)+1 : z \in \mathcal{P}(x)\} \le \operatorname{rank}(x)+1,$$

where for the first inequality we used $x \in \mathcal{P}(x)$. Thus $\operatorname{rank}(\mathcal{P}(x)) = \operatorname{rank}(x)+1$.

(4) Suppose $z \in \bigcup x$. Then there is $w \in x$ such that $z \in w \in x$. Then $w \in$ WF by Lemma 1.22(1), and so also $z \in$ WF, since it consists of well-founded sets and so $\bigcup x \in$ WF. Furthermore, for every such $z \in \bigcup x$, we have $\operatorname{rank}(z)+1 \le \operatorname{rank}(x)$. Thus

$$\operatorname{rank}\left(\bigcup x\right) = \sup\left\{\operatorname{rank}(z)+1 : z \in \bigcup x\right\} \le \operatorname{rank}(x).$$

(5) We have $x,y \in$ WF, so $x,y \subseteq$ WF, which implies $x \cup y \subseteq$ WF and thus $x \cup y \in$ WF by Lemma 1.22.(3). To compute the rank, using Lemma 1.22(5) we have

$$\begin{aligned}
\operatorname{rank}(x\cup y) &= \sup\big\{\operatorname{rank}(z)+1 : z \in x\cup y\big\}\\
&= \max\big\{\sup\{\operatorname{rank}(z)+1 : z\in x\}, \sup\{\operatorname{rank}(z)+1 : z \in y\}\big\}\\
&= \max\big\{\operatorname{rank}(x),\operatorname{rank}(y)\big\}.
\end{aligned}$$

(6) By assumption, $x \in$ WF, and by induction it follows that every $\bigcup^n x \in$ WF for every $n \ge 1$, by (4) above. Thus $\operatorname{trcl}(x) = \bigcup\{\bigcup^n x : n \ge 0\} \subseteq$ WF, and so $\operatorname{trcl}(x) \in$ WF. By induction, one can show that $\operatorname{rank}(\cup^n x) \le \operatorname{rank}(x)$ for each n and so

$$\begin{aligned}
\operatorname{rank}\big(\operatorname{trcl}(x)\big) &= \sup\big\{\operatorname{rank}(z)+1 : z \in \operatorname{trcl}(x)\big\}\\
&= \sup\big\{\operatorname{rank}(z)+1 : z \in \textstyle\bigcup^n x, \text{ for some } n\big\}\\
&= \sup\big\{\sup\{\operatorname{rank}(z)+1 : z \in \textstyle\bigcup^n x\} : n \ge 0\big\}\\
&= \sup\big\{\operatorname{rank}\big(\textstyle\bigcup^n x\big) : n \ge 0\big\}\\
&= \operatorname{rank}(x).
\end{aligned}$$

□

With this, we can define initial segments of the well-founded universe:

Definition 1.24. Let α be an ordinal and let $R(\alpha) = \{x \in \text{WF} : \operatorname{rank}(x) < \alpha\}$.

LEMMA 1.25. Let b be a set, $\alpha \in \mathbb{ON}$. Then

$$b \in R(\alpha + 1) \text{ iff } b \subseteq R(\alpha).$$

PROOF. ($\Rightarrow$) Let $b \in \mathrm{WF}$ and $\mathrm{rank}(b) < \alpha + 1$. Take $x \in b$. Then x is well-founded, $\mathrm{rank}(x) < \mathrm{rank}(b) \leq \alpha$. Thus, $b \subseteq R(\alpha)$.

($\Leftarrow$) Let $b \subseteq R(\alpha)$. Then in particular, b is a set of well-founded sets and so b is well-founded. For each $x \in b$, we have $\mathrm{rank}(x) < \alpha$. Thus, $\mathrm{rank}(b) = \sup\{\mathrm{rank}(x) + 1 : x \in b\} \leq \alpha < \alpha + 1$. □

LEMMA 1.26. Assume the Power Set Axiom. Then for each $\alpha \in \mathbb{ON}$, $R(\alpha)$ is a set. Moreover:

(1) $R(0) = \varnothing$,
(2) $R(\alpha + 1) = \mathcal{P}(R(\alpha))$, and
(3) $R(\gamma) = \bigcup_{\alpha<\gamma} R(\alpha)$ for γ limit ordinal.

PROOF. By induction on α. If $\alpha = 0$, then $R(0) = \varnothing$. Now, suppose $R(\alpha)$ is a set. Then by the Power Set Axiom $\mathcal{P}(R(\alpha))$ is a set and by the previous Lemma $R(\alpha + 1) = \mathcal{P}(R(\alpha))$. If α is a limit and for each $\gamma < \alpha$, $R(\gamma)$ is a set then by the Replacement and Union Axioms $\bigcup_{\gamma<\alpha} R(\gamma)$ is a set, which by definition of $R(\alpha)$ is exactly $R(\alpha)$. □

REMARK 1.27. The Power set axiom is not necessary to define the notion of a rank. As we will see, the rank of a set is absolute for transitive models of ZF-P.

2. Mostowski Collpase

2.1. Mostowski Collapsing Function.

DEFINITION 2.1. Let R be a relation, which is well-founded and set-like on A. For $y \in A$, define the Mostowski collapsing function $\mathrm{mos}(y)$ as follows:

$$\mathrm{mos}(y) = \mathrm{mos}_{A,R}(y) = \{\mathrm{mos}(x) : x \in \mathrm{pred}_{A,R}(y)\}.$$

Justification: For each pair of sets x, s define $G(x, s) := \mathrm{range}(s)$. Note that G does not depend on x. Now, define

$$F(y) = G(y, F \restriction (y \downarrow)) = \{F(x) : x \in \mathrm{pred}_{A,R}(y)\}.$$

LEMMA 2.2. Let R be a defined relation which is well-founded and set-like on A. Then $\mathrm{mos}'' A$ is transitive.

PROOF. Let $\mathrm{mos}(y) \in \mathrm{mos}'' A$. Then $\mathrm{mos}(y) = \{\mathrm{mos}(x) : x \in \mathrm{pred}_{A,R}(y)\} \subseteq \mathrm{mos}'' A$. Thus, $\mathrm{mos}'' A$ is transitive. □

DEFINITION 2.3. A relation R is said to be extensional if

$$\forall x, y \in A(\mathrm{pred}_{A,R}(x) = \mathrm{pred}_{A,R}(y) \to x = y).$$

Note that if A under the membership relation is a transitive class, then $\in$ is extensional on A.

LEMMA 2.4. Let R be a well-founded and set-like relation on A.

(1) $\mathrm{mos}_{A,R}$ is injective iff R is extensional on A.

(2) If R is extensional on A, then $\mathrm{mos} : (A, R) \cong (\mathrm{mos}'' A, \in)$.

PROOF. (1) Assume $\mathrm{mos}_{A,R}$ is injective, but R is not extensional on A. Thus, there are $a \neq b$ such that $\mathrm{pred}_{A,R}(a) = \mathrm{pred}_{A,R}(b)$. But, then $\mathrm{mos}_{A,R}(a) = \mathrm{mos}_{A,R}(b)$, which is a contradiction.

Suppose, R is extensional on A. By way of contradiction, suppose $X = \{a \in R : \exists y \in A(y \neq a \wedge \mathrm{mos}(a) = \mathrm{mos}(y)\} \neq \varnothing$. Let $a \in X$ be R-minimal. Then, there is $b \in X$ such that $b \neq a$ and $\mathrm{mos}(a) = \mathrm{mos}(b)$. Since R is extensional on A, we must have $\mathrm{pred}_{A,R}(b) \neq \mathrm{pred}_{A,R}(a)$. There are two cases to consider:

Case 1 Suppose there is $c \in \mathrm{pred}_{A,R}(a) \backslash \mathrm{pred}_{A,R}(b)$. However $\mathrm{mos}(c) \in \mathrm{mos}(a) = \mathrm{mos}(b)$ and so there is $d \in \mathrm{pred}_{A,R}(b)$ such that $\mathrm{mos}(c) = \mathrm{mos}(d)$. Since $c \notin \mathrm{pred}_{A,R}(b)$, $c \neq d$. Thus $c \in X$. However $c \in \mathrm{pred}_{A,R}(a)$ is a contradiction to the minimality of a.

Case 2 Otherwise, there is $d \in \mathrm{pred}_{A,R}(b) \backslash \mathrm{pred}_{A,R}(a)$. Just as in Case 1, find $c \in \mathrm{pred}_{A,R}(a)$ such that $\mathrm{mos}(c) = \mathrm{mos}(d)$. But, then $c \in X$ and cRa is a contradiction to the minimality of a. Therefore if R is extensional on A, then $\mathrm{mos}_{A,R}$ is injective.

(2) Straightforward. □

LEMMA 2.5. Assume $\in$ is well-founded and extensional on A. Let $T \subseteq A$ be transitive. Then $\mathrm{mos}_{A,\in}(y) = y$ for all $y \in T$.

PROOF. Suppose not. Then $\{y \in T : \mathrm{mos}(y) \neq y\}$ has an $\in$-minimal element. Now $\mathrm{mos}(a) = \{\mathrm{mos}(y) : y \in \mathrm{pred}_{A,\in}(a)\} = \{y : y \in y\} = a$, which is a contradiction. □

LEMMA 2.6. (Transitive $\in$-models are unique) Let A, B be transitive sets with $A \in \mathrm{WF}$. Let $f : (A, \in) \cong (B, \in)$ be an isomorphism. Then $f = \mathrm{id}_A$ and hence $A = B$.

PROOF. Let $a \in A$ and $b = f(a)$. Then since A, B are transitive, we have

$$\forall y(y \in b \leftrightarrow \exists x \in a(f(x) = y)).$$

Thus, $f(a) = \{f(x) : x \in a\}$. But, A is well-founded and so $f = \mathrm{mos}_{A,\in}$. By the previous Lemma $f = \mathrm{id}$ and so $A = B$. □

REMARK 2.7. If two countable transitive models are isomorphic, then they coincide.

COROLLARY 2.8. Let A be a well-founded set and let B be a set such that

$$(\mathrm{trcl}(A) \cup \{A\}, \in) \cong (\mathrm{trcl}(B) \cup \{B\}, \in).$$

Then $A = B$.

DEFINITION 2.9. Let κ be a cardinal. Then $H(\kappa) = \{x \in \mathrm{WF} : |\mathrm{trcl}(x)| < \kappa\}$. In particular, $\mathrm{HC} = H(\aleph_1)$ denotes the set of hereditarily countable sets.

REMARK 2.10. In particular, $\mathrm{HC} = H(\aleph_1)$ denotes the set of hereditarily countable sets and $\mathrm{HF} = H(\aleph_0)$ the set of hereditarily finite sets. Note that $H(\omega) = R(\omega)$.

LEMMA 2.11. Let κ be an infinite cardinal. Then $|H(\kappa)| = 2^{<\kappa}$ and $H(\kappa) \subseteq R(\kappa)$.

PROOF. Let $x \in H(\kappa)$ and let $\alpha = \text{rank}(x)$. Since $\text{trcl}(x)$ is a transitive set, for each $\xi < \alpha$ there is $z \in \text{trcl}(x)$ such that $\text{rank}(z) = \xi$. However, this implies that $\alpha = \{\text{rank}(z) : z \in \text{trcl}(x)\}$ and since $|\,\text{trcl}(x)| < \kappa$, we obtain $\alpha < \kappa$. Thus, in particular $x \in R(\kappa)$.

We will show that $|H(\kappa)| = 2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}$ in two steps. First we show that $|H(\kappa)| \geq 2^{<\kappa}$. If $\lambda < \kappa$, then $\mathcal{P}(\lambda) \subseteq H(\kappa)$, we get that $|H(\kappa)| \geq 2^\lambda$. But, this is true for each $\lambda < \kappa$ and so $|H(\kappa)| \geq 2^{<\kappa}$.

To see that $|H(\kappa)| \leq 2^{<\kappa}$ consider the mapping $F : H(\kappa) \to \cup\{\mathcal{P}(\lambda \times \lambda) : \lambda < \kappa\}$ defined as follows. Let $x \in H(\kappa)$ and let $\lambda = |\,\text{trcl}(x) \cup \{x\}|$. Thus $\lambda < \kappa$. Assuming the Axiom of Choice, we can find $F(x) \subseteq \lambda \times \lambda$ such that $(\lambda, F(x)) \cong (\text{trcl}(x) \cup \{x\}, \in)$. By Corollary 2.8, the function F is injective and so

$$|H(\kappa)| \leq |\bigcup\{\mathcal{P}(\lambda \times \lambda) : \lambda < \kappa\}| = \sup_{\lambda<\kappa} 2^\lambda = 2^{<\kappa}.$$

□

REMARK 2.12. If κ is an uncountable cardinal, then $|R(\kappa)| = \beth_\kappa$. By the above Lemma $|H(\kappa)| = 2^{<\kappa}$ and so $H(\kappa)$ is much smaller than $R(\kappa)$. Note also that $|\text{HC}| = 2^{\aleph_0} = \beth_1$ and $|R(\omega_1)| = \beth_{\omega_1}$.

3. The Consistency of Foundation

We will make use of the following notation and theories.

REMARK 3.1.

(1) ZFC^- denotes the axiomatic system ZFC without the axiom of foundation;
(2) Z denotes the axiomatic system *ZFC* without the axiom of choice and without the axiom of replacement;
(3) ZF-P denotes the axiomatic system *ZFC* without the axiom of choice and the power set axiom;
(4) BST denotes the set $\{$Axiom 1-5$\} \cup \{$Power set axiom $\vee$ Replacememt$\}$.
(5) If Γ is a sub-theory of ZFC then Γ^- denotes the same theory without the axiom of foundation;
(6) In the discussion below all theories are extensions of BST^-.

Our next goal is to provide a proof of the following statement.

THEOREM 3.2. *Let Γ be one of the theories ZF-P, ZFC-P, ZF, ZFC. Let Γ^- be*

$$\Gamma \backslash \{\textit{Axiom of Foundation}\}.$$

Then there is a finitistic proof of $\text{Con}(\Gamma^-) \to \text{Con}(\Gamma)$. *That is if we can find a contradiction from Γ, then we can find a contradiction from Γ^-.*

3.1. Relative interpretation.

DEFINITION 3.3. Let Λ be a set of axioms in $\mathcal{L}_\in$ and let $\mathcal{L}$ be a finite, conservative (only defined notions are allowed) extension of $\mathcal{L}_\in$. A relative interpretation of $\mathcal{L}$ is a class A definable by a formula $\alpha(x)$ such that $\Lambda \vdash \exists x \alpha(x)$ such that

(1) for every n-ary function symbol f in $\mathcal{L}$, where $n > 0$, there is a formula $\varphi(x_1, \cdots, x_n, y)$ such that $\Lambda \vdash \forall x_1 \cdots x_n \in A \exists! y \in A \varphi(\bar{x}, y)$ (thus, φ is the intended interpretation of f);
(2) for every n-ary predicate symbol$p$, where $n > 0$, there is a formula $\varphi(x_1, \cdots, x_n)$ with $\mathrm{Fr}(\varphi) = \{x_1, \cdots, x_n\}$ such that if $\bar{a} = (a_1, \cdots, a_n)$ then $\bar{a}$ is in the intended interpretation of p if and only if $\Lambda \vdash \wedge_{j=1}^{n} \alpha(a_j) \wedge \varphi(a_1, \cdots, a_n)$;
(3) for every constant symbol c, there is a formula φ such that $\Lambda \vdash \exists! y(\alpha(y) \wedge \varphi(y))$;
(4) for every 0-ary predicate symbol p, there is a closed sentence φ such that the intended interpretation of p is true iff $\Lambda \vdash \neg\varphi$ and the intended interpretation is false iff $\Lambda \vdash \varphi$.

REMARK 3.4. The above relative interpretation extends in a natural way to all terms and formulas in the language, by substituting all non-logic symbols with their relative interpretations; $\forall x$ with $\forall x \in A$ and $\exists x$ with $\exists x \in A$. Relative interpretations are usually clear from context.

DISCUSSION 3.5. Suppose A has a relative interpretation of $\mathcal{L}$ in Λ.

(1) Whenever $\psi, \varphi_1, \cdots, \varphi_n$ are closed and $\{\varphi_1, \cdots, \varphi_k\} \vdash \psi$, then $\Lambda \vdash (\varphi_1^A \wedge \cdots \wedge \varphi_k^A) \to \psi^A$.
(2) Let Γ be a set of sentences and suppose for each $\varphi \in \Gamma$, we have $\Lambda \vdash \varphi^A$. Then the consistency of Λ implies the consistency of Γ. Indeed, if we can derive a contradiction from Γ, then we can derive a contradiction from Λ.

3.2. Δ_0 formulas.

DEFINITION 3.6.

(1) An $\in$-model for $\mathcal{L}_\in$ is any structure $\mathfrak{A} = (A, E)$ where $E = \{(a, b) \in A \times A : a \in b\} (= \in^{\mathfrak{A}})$.
(2) A transitive model is any $\in$-model the universe of which is a transitive set.

DEFINITION 3.7. Let $\mathfrak{A} \subseteq \mathfrak{B}$ and φ a $\mathcal{L}_\in$-formula. Then a formula φ is said to be absolute between $\mathfrak{A}$ and $\mathfrak{B}$ if for every assignment σ in A we have

$$\mathfrak{A} \vDash \varphi[\sigma] \text{ iff } \mathfrak{B} \vDash \varphi[\sigma].$$

DEFINITION 3.8. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_\in$. The set of Δ_0-formulas of $\mathcal{L}$ is defined as follows:

(1) All atomic formulas are Δ_0-formulas.
(2) if φ is a Δ_0 formula, y is a variable, τ is a term such that y does not occur in τ, then $\forall y \in \tau\varphi$ and $\exists y \in \tau\varphi$ are Δ_0-formulas.
(3) If φ is a Δ_0-formula, then so is $\neg\varphi$.
(4) If φ and ψ are Δ_0-formulas, then so are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$.

LEMMA 3.9. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_\in$ and assume $\mathfrak{A} \subseteq \mathfrak{B}$ are models of $\mathcal{L}$, the universe A of $\mathfrak{A}$ is a transitive set and $\in_{\mathfrak{A}} = \{(a, b) \in A \times A : a \in b\}$, $\in_{\mathfrak{B}} = \{(a, b) \in B \times B : a \in b\}$. Then all Δ_0 formulas of $\mathcal{L}$ are absolute between $\mathfrak{A}$ and $\mathfrak{B}$.

PROOF. Induction on φ. The case in which φ is atomic is straightforward and so are the inductive steps, regarding logical connectives. Assume $\varphi(\bar{x}, z)$ is

$$\exists y(y \in \tau(\bar{x}, z)) \wedge \psi(\bar{x}, y, z)$$

where ψ is Δ_0 and $\mathfrak{A} \preceq_\psi \mathfrak{B}$. Since $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, we have that whenever $\bar{a}$ and c are from A then $\tau^{\mathfrak{A}}[\bar{a}, c] = \tau^{\mathfrak{B}}[\bar{a}, c]$. Then, by definition of the satisfaction relation we have:

$$\begin{array}{lll} \mathfrak{A} \vDash \varphi[\bar{a}, c] & \text{iff } \exists b \in A\big(b \in \tau^{\mathfrak{A}}[\bar{a}, c] \wedge \mathfrak{A} \vDash \psi[\bar{a}, b, c]\big) & \text{by definition of } \varphi \\ & \text{iff } \exists b \in B\big(b \in \tau^{\mathfrak{B}}[\bar{a}, c] \wedge \mathfrak{B} \vDash \psi[\bar{a}, b, c]\big) & \text{since } \mathfrak{A} \preceq_\psi \mathfrak{B}, \tau^{\mathfrak{A}}[\bar{a}, c] \subseteq A \subseteq B \\ & \text{iff } \mathfrak{B} \vDash \varphi[\bar{a}, c] & \text{by definition.} \end{array}$$

□

EXAMPLE 3.10. Examples of formulas in $\mathcal{L}_\in$ which are logically equivalent to Δ_0-formulas:

(1) $(x \subseteq y)$; $\forall z(z \in x \to z \in y)$ is logically equivalent to $\forall z \in x(z \in y)$;
(2) $x = \varnothing$; $\forall z(z \notin x)$ is logically equivalent to $\forall z \in x(z \neq z)$;
(3) $y = S(x)$; $x \in y \wedge x \subseteq y \wedge \forall z \in y(z = x \vee z \in x)$
(4) $y = v \cap w$: $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$ which is equivalent to $(y \subseteq v \wedge y \subseteq w \wedge \forall x \in v(\forall x \in w(x \in y)))$;
(5) $\mathrm{Sing}(x)$: $\exists y \in x \forall z \in x(z = y)$.

DEFINITION 3.11. A formula φ is said to be absolute for A if $A \preceq_\varphi V$.

REMARK 3.12.

(1) If $\mathfrak{A}$ has a relative interpretation of $\mathcal{L}$, where $\mathcal{L}$ is a finite extension of $\mathcal{L}_\in$ in Λ, then Δ_0-formulas are absolute between $\mathfrak{A}$ and $\mathfrak{B}$, whenever $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is transitive.
(2) Let $\bar{x} = (x_1, \cdots, x_n)$ be an n-tuple of variables. Suppose $\varphi(\bar{x})$ and $\psi(\bar{x})$ are $\mathcal{L}_\in$-formulas and $\forall x(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. Then $M \preceq_\varphi V$ iff $M \preceq_\psi V$. In particular, if M is transitive, to show that a given φ is absolute for M, it suffices to show that φ is equivalent to a Δ_0-formula.

LEMMA 3.13. Let M be a model of the Axioms of Extensionality, Comprehension, Pairing and Union. Then $\varnothing^M$, S^M, $\cap^M$ are defined and if M is transitive then these are also absolute for M.

3.3. Axioms 1-6 in WF.

LEMMA 3.14. (ZF$^-$-P) If M is a transitive class, then the Axiom of Extensionality holds in M.

PROOF. We have to show that $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$. We relativize this to M: $\forall x \in M \forall y \in M(\forall z \in M(z \in x \leftrightarrow z \in y) \to x = y)$. Now, fix x, y in M. Since M is transitive $x, y \subseteq M$. Now, if $\forall z \in M(z \in x \leftrightarrow z \in y)$, then in fact we have $\forall z(z \in x \leftrightarrow z \in y)$ which by the Axiom of Extensionality implies $x = y$. □

LEMMA 3.15. (ZF$^-$-P) If M is a class consisting of well-founded sets, then the Foundation Axiom holds in M.

PROOF. The Foundation Axiom states:

$$\forall x \exists y(y \in x) \to \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y)).$$

Relativizing the above to M we get:

$$\forall x \in M \exists y \in M(y \in x) \to \exists y \in M(y \in x \wedge \neg \exists z \in M(z \in x \wedge z \in y)).$$

Fix $x \in M$ and suppose $\exists y_0 \in M$ such that $y_0 \in x$. Since M consists of well-founded sets, x is well-founded. Let $\mu(x)$ be a first order formula defining M. Then $\Delta = \{z \in x : \mu(z)\}$ is a set (by the Axiom of Comprehension). Since x is well-founded, we can take $y = \min_{\in} \Delta$. Then since $y \in \Delta$, we have $\mu(y)$ and so $y \in M$. Moreover, if $\exists z \in M(z \in x \wedge z \in y)$ then z would contradict the minimality of y and so we are done. □

LEMMA 3.16. (ZF$^-$-P) If $\forall z \in M \forall y \subseteq z(y \in M)$, then the Comprehension Axiom holds in M.

PROOF. Fix a formula φ. The comprehension axiom for φ is:

$$\forall z \exists y \forall x(x \in y \leftrightarrow x \in z \wedge \varphi(x))$$

Now $\varphi = \varphi(x, z, x_0, \cdots, x_{n-1})$ and we must show that

$$\forall z, x_0, \cdots, x_{n-1} \in M \exists y \in M \forall x \in M(x \in y \leftrightarrow x \in z \wedge \varphi^M(x, z, \bar{x})).$$

By Comprehension in V, $y = \{x \in z : \varphi^M(x, z, \bar{x})\}$ is a set and $y \subseteq z$. By hypothesis $y \in M$ and so the relativized instance of comprehension holds in M. □

LEMMA 3.17. (ZF$^-$-P) If $x, y \in M(\{x, y\} \in M)$ then the pairing axiom holds in M.

PROOF. Recall the Pairing axiom $\forall x, y \exists z(x \in z \wedge y \in z)$. Relativized to M this is

$$\forall x, y \in M \exists z \in M(x \in z \wedge y \in z).$$

Since by assumption for all $x, y \in M$ the pair $\{x, y\} \in M$, we can just take $z = \{x, y\}$ above. □

LEMMA 3.18. If $\forall \mathcal{F} \in M(\bigcup \mathcal{F} \in M)$ then the union axiom holds in M.

PROOF. Straightforward. □

LEMMA 3.19. (ZF$^-$-P) Suppose M is a transitive class and for all functions f the following holds: if $\operatorname{dom}(f) \in M$ and $\operatorname{ran}(f) \subseteq M$, then $\operatorname{ran}(f) \in M$. Then, the Replacement Axiom holds in M.

PROOF. Recall the Replacement Axiom: For each formula φ without B free:

$$\forall A \forall x \in A \exists! y \varphi(x, y) \to \exists B \forall x \in A \exists y \in B \varphi(x, y).$$

Let $A \in M$. Now, suppose $\forall x \in M(x \in A \to \exists! y \in M \varphi^M(x, y))$. By Comprehension in V, $\Delta = \{x \in A : \mu(x)\}$ is a set. We are given that $\forall x \in \Delta \exists! y(\varphi^M(x, y) \wedge \mu(y))$, where again $\mu(y)$ is the defining formula for M. By Replacement in V, there is a function f such that $\operatorname{dom}(f) = \Delta$ and for all $x \in \Delta$, $f(y)$ is the unique y such that $\varphi^M(x, y) \wedge \mu(y)$. We extend f to a function f' such that $\operatorname{dom}(f') = A$ by defining $f' \restriction \Delta = f$ and $f'(y) = a_0$ for each $y \in A \backslash \Delta$, where $a_0 \in \operatorname{ran}(f)$ is fixed. Then $\operatorname{dom}(f') = A \in M$, $\operatorname{ran}(f') = \operatorname{ran}(f) \subseteq M$ and so by hypothesis, $\operatorname{ran}(f') \in M$. Then, take $B = \operatorname{ran}(f')$. □

COROLLARY 3.20. (ZF$^-$-P) Axioms $1-6$ holds in WF.

PROOF. The sufficient conditions given in the previous six lemmas hold in WF. □

3.4. The Power Set Axiom, Axiom of Infinity and Axiom of Choice in WF. Recall the Power Set Axiom: $\forall x \exists y \forall z (z \subseteq x \to z \in y)$. Since $\subseteq$ is defined by a Δ_0 formula in $\mathcal{L}_\in$, the formula $x \subseteq y$ is absolute for transitive classes.

LEMMA 3.21. (ZF$^-$) Let M be a transitive class.

(1) If for all $x \in M$, $\mathcal{P}(x) \cap M \in M$, then PSA holds in M.
(2) If $(PSA)^M$ and M satisfies Comprehension, then $\forall x \in M (\mathcal{P}(x) \cap M \in M)$.

PROOF. Note that $(PSA)^M$ is the formula

$$\forall x \in M \exists y \in M \forall z \in M (z \subseteq x \to z \in y),$$

where we used absoluteness of $\subseteq$. To obtain (1) take $y = \{z \subseteq x : \mu(z)\} = \mathcal{P}(x) \cap M \in M$. To obtain (2) consider any $x \in M$. By $(PSA)^M$, there is $y \in M$ such that $\mathcal{P}(x) \cap M \subseteq y$. However being a subset is absolute and so $\Delta = \{z \in y : z \subseteq x \wedge \mu(z)\} = \mathcal{P}(x) \cap M$. □

COROLLARY 3.22. (ZF$^-$) The Power Set Axiom holds in WF.

PROOF. Let $x \in \mathrm{WF}$. If $z \subseteq x$, then $z \in \mathrm{WF}$ (since a set of well-founded sets is well-founded). Therefore $\mathcal{P}(x) \cap \mathrm{WF} = \mathcal{P}(x) \in \mathrm{WF}$, where we also used the Power Set Axiom in V. Then by the above Lemma, $(PSA)^{\mathrm{WF}}$. □

LEMMA 3.23. (ZF$^-$-P) Let M be a transitive class, such that Extensionality, Comprehension, Pairing and Union hold in M.

(1) If $\omega \in M$, then the Axiom of Infinity holds in M.
(2) The Axiom of Choice holds in M iff every disjoint family of non-empty sets in M has a choice set in M.

PROOF. (1) The Axiom of Infinity holds iff $\exists x (\varnothing \in x \wedge \forall y \in y (S(y) \in x))$. Let $\varphi(x)$ be the following formula: $\varnothing \in x \wedge \forall y \in x (S(y) \in x))$. Thus, (Axiom of Infinity)^M iff $\exists x \in M (\varphi(x)^M)$. However $\varphi(x)$ is Δ_0 in the notions $\varnothing$, S, both of which are absolute for M. Thus $\varphi(x)^M = \varphi(x)$. Since $\omega \in M$and $\varphi(\omega)$ holds, we get (AXiom of Infinity)^M.

(2) Let $\mathrm{df}(F)$ be the following formula saying that F is a non-empty set of pairwise disjoint non-empty sets $\varnothing \notin F \wedge \forall x \in F (x \neq \varnothing) \wedge \forall x \in F \forall y \in F (x \neq y \to x \cap y = \varnothing)$ and let $\mathrm{cs}(C, F)$ be the following formula saying that C is a choice function for F, $\forall x \in F (\mathrm{Sing}(C \cap x))$. Note that both $\mathrm{df}(F)$ and $\mathrm{cs}(C, F)$ are Δ_0 (as so are $\varnothing$, $\cap$, Sing) and so they are absolute for M. Therefore $(\mathrm{AC})^M$ is equivalent to $\forall \mathcal{F} \in M \exists C \in M (\mathrm{df}(F) \to \mathrm{cs}(C, F))$. □

COROLLARY 3.24. (ZF$^-$-P)

(1) The Axiom of Infinity holds in WF.
(2) $AC \Rightarrow (AC)^{WF}$.

PROOF. (1) Since $w \in \mathrm{WF}$, the statement holds by the previous Lemma. To see (2) assume AC and let $\mathcal{F} \in WF$ such that $\mathrm{df}(\mathcal{F})$. Then by the Axiom of Choice there is a set C such that $\mathrm{cs}(\mathcal{F}, C)$. Note that $C \cap \bigcup \mathcal{F} \in WF$ is also a choice set for $\mathcal{F}$ and so again the statement holds by the previous Lemma. □

Now, we can prove Theorem 3.2.

THEOREM. Let Γ be one of the theories ZF-P, ZFC-P, ZF, ZFC. Let Γ^- be

$$\Gamma\setminus\{\text{Axiom of Foundation}\}.$$

Then there is a finitistic proof of $\mathrm{Con}(\Gamma^-) \to \mathrm{Con}(\Gamma)$. That is if we can find a contradiction from Γ, then we can find a contradiction from Γ^-.

PROOF. We can work in Γ^- and using the above established results prove each axiom of Γ relativized to WF. □

3.5. Set models of large ZFC framents.

THEOREM 3.25. *(ZF^-) Let $\gamma > \omega$ be a limit ordinal. Then*
(1) $R(\gamma) \vDash ZF\setminus\{\text{Axiom of Replacement}\}$.
(2) $AC \Rightarrow R(\gamma) \vDash ZFC\setminus\{\text{Axiom of Replacement}\}$.

PROOF. (1) We proceed by discussing each axiom.

Extensionality By Lemma 3.14, it suffices to show that $R(\gamma)$ is transitive. Suppose $x \in R(\gamma)$ and let $y \in x$. Then $\mathrm{rank}(y) < \mathrm{rank}(x) < \gamma$ and so $y \in R(\gamma)$. Therefore $x \subseteq R(\gamma)$.

Foundation Since $R(\gamma) \subseteq \mathrm{WF}$, by Lemma 3.15 the Axiom of foundation holds in $R(\gamma)$.

Comprehension Let $z \in R(\gamma)$ be arbitrary and let $y \subseteq z$. Then $\mathrm{rank}(y) \le \mathrm{rank}(z) < \gamma$ and thus $y \in R(\gamma)$. By Lemma 2.3.16 it follows that the comprehension axiom schema holds in $R(\gamma)$.

Pairing Let $x, y \in R(\gamma)$ be arbitrary. Then $\mathrm{rank}(x), \mathrm{rank}(y) < \gamma$ and thus $\mathrm{rank}\{x,y\} = \max(\mathrm{rank}(x), \mathrm{rank}(y)) + 1 < \gamma$, by Corollary 2.1.23 and since γ is a limit ordinal. Thus by Lemma 2.3.17 the pairing axiom holds in $R(\gamma)$.

Union Let $\mathcal{F} \in R(\gamma)$ be arbitrary. Then $\mathrm{rank}(\mathcal{F}) < \gamma$, so $\mathrm{rank}(\bigcup\mathcal{F}) \le \mathrm{rank}(\mathcal{F})$. Thus $\bigcup\mathcal{F} \in R(\gamma)$. By Lemma 2.3.18 it follows that the Union Axiom holds in $R(\gamma)$.

Infinity We have already shown that $R(\gamma)$ is a transitive class which satisfies Extensionality, Comprehension, Pairing and Union. Furthermore, $\mathrm{rank}(\omega) = \omega < \gamma$. Thus $\omega \in R(\gamma)$. By Lemma 3.23, it follows that the Axiom of Infinity holds in $R(\gamma)$.

Power Set Let $x \in R(\gamma)$ be arbitrary. Then $\mathrm{rank}(x) < \gamma$, so $\mathrm{rank}\left(\mathcal{P}(x)\right) = \mathrm{rank}(x) + 1 < \gamma$ by Corollary 1.23.(3). Furthermore, $\mathrm{rank}\left(\mathcal{P}(x) \cap R(\gamma)\right) \le \mathrm{rank}\left(\mathcal{P}(x)\right) < \gamma$ by Lemma 1.22. Thus $\mathcal{P}(x) \cap R(\gamma) \in R(\gamma)$. It now follows from Lemma 3.22 that the Power Set Axiom holds in $R(\gamma)$.

(2) Let C be a choice set for $\mathcal{F} \in R(\gamma)$. Consider $C' = C \cap \bigcup\mathcal{F}$. Then $\mathrm{rank}(C') \le \mathrm{rank}(\bigcup\mathcal{F}) \le \mathrm{rank}(\mathcal{F}) < \gamma$ and is also a choice set for $\mathcal{F}$. Thus the Axiom of Choice holds in $R(\gamma)$. □

REMARK 3.26.
(1) If $\gamma > \omega$ and $R(\gamma) \vDash$ Axiom of Replacement then $\gamma = \beth_\gamma$ and $|\{\delta < \gamma : \delta = \beth_\delta\}| = \gamma$.
(2) If γ is a successor ordinal then $R(\gamma)$ does not satisfy Pairing, as $R(\gamma) \vDash \exists x \forall y (x \notin y)$.

We will make use of the following Lemma:

LEMMA 3.27. Let x, y be sets. Then

(1) if $x \in y$, then $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(y)$,
(2) if $x \subseteq y$, then $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(y)$,
(3) $\mathrm{trcl}(\{x, y\}) = \mathrm{trcl}(x) \cup \mathrm{trcl}(y) \cup \{x, y\}$,
(4) $\mathrm{trcl}(\bigcup x) \subseteq \mathrm{trcl}(x)$.

PROOF. Straightforward. □

THEOREM 3.28. *(ZF^-) Let κ be a regular uncountable cardinal. Then*

(1) $H(\kappa) \models ZF \smallsetminus \{Power\ Set\ Axiom\}$.
(2) $AC \Rightarrow H(\kappa) \models ZFC \smallsetminus \{Power\ Set\ Axiom\}$.

PROOF. (1) By the above Lemma and the various closure criteria, $H(\kappa)$ satisfies Extensionality, Foundation, Comprehension, Pairing and Union. Since $\omega \in H(\kappa)$, Lemma 3.24 implies that $H(\kappa) \models$ Axiom of Infinity.

(2) Let C be a choice set for $\mathcal{F} \in H(\kappa)$. Then $\mathrm{trcl}(C \cap \bigcup \mathcal{F}) \subseteq \mathrm{trcl}(\mathcal{F})$ and so $\mathrm{trcl}(C \cap \bigcup \mathcal{F})| < \kappa$. Therefore $C \cap \bigcup \mathcal{F} \in H(\kappa)$. □

THEOREM 3.29.

(1) If κ is a regular, uncountable cardinal and κ is not strongly inaccessible, then the Power Set Axiom is false in $H(\kappa)$.
(2) If κ is strongly inaccessible, then $R(\kappa) = H(\kappa) \models$ Power Set Axiom.

PROOF. (1) By Lemma 3.21, it suffices to find $x \in H(\kappa)$ such that $\mathcal{P}(x) \cap H(\kappa) \notin H(\kappa)$. Since κ is not strongly inaccessible, there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Let $x := \lambda$. Then $\lambda = \mathrm{trcl}(\lambda)$ and thus $\lambda \in H(\kappa)$. Furthermore, for every $y \in \mathcal{P}(x)$, $y \subseteq \lambda$ and so $y \in H(\kappa)$. Thus $\mathcal{P}(x) \cap H(\kappa) = \mathcal{P}(x)$. Finally, we have $\kappa \leq 2^\lambda = |\mathcal{P}(\lambda)| \leq |\mathrm{trcl}(\mathcal{P}(\lambda))|$, and so $\mathcal{P}(x) \notin H(\kappa)$. Therefore $H(\kappa)$ does not satisfy the Power Set Axiom. □

THEOREM 3.30. *(ZF^-) $HF = R(\omega) = H(\omega) \models ZFC \backslash \{Axiom\ of\ Infinity\}$. In fact, the Axiom of Infinity is false in HF.*

PROOF. Let $\varphi(x_0)$ be the formula $\varnothing \in x \wedge \forall y \in x(S(y) \in x)$. However, there is no $x_0 \in HF$ such that $\varphi(x_0)$. Therefore the Axiom of Infinity does not hold in HF. To see that the Axiom of Choice holds in HF, note that HF can be well-ordered and so every non-empty set of pairwise disjoint non-empty sets has a choice function. □

4. Elementary Submodels and Definability

4.1. Tarksi-Vaught and Löwenheim-Skolem.

Recall the following:

LEMMA 4.1. (Tarski-Vaught) Let $\mathfrak{A}$, $\mathfrak{B}$ be structures. The following are equivalent:

(1) $\mathfrak{A} \preceq \mathfrak{B}$
(2) For all existential formulas $\varphi(\bar{x})$ of $\mathcal{L}$, i.e. formulas of the form $\exists y \psi(\bar{x}, y)$ and all $\bar{a}$ from A: if $\mathfrak{B} \models \varphi[\bar{a}]$, then there is $b \in A$ such that $\mathfrak{B} \models \psi[\bar{a}, b]$.

LEMMA 4.2. (Downward Löwenheim-Skolem Theorem) ZFC$^-$ Let $\mathfrak{B}$ be a $\mathcal{L}$-structure and let κ be such that $\max(|\mathcal{L}|, \aleph_0) \le \kappa \le |\mathfrak{B}|$. Let $S \subseteq B$, $|S| \le \kappa$. Then, there is $\mathfrak{A} \preceq \mathfrak{B}$ such that $S \subseteq A$ and $|A| = \kappa$.

PROOF. Let φ be an existential formula with n free variables $(x_1, \cdots, x_n)$. Let $\bar{x} = (x_1, \cdots, x_n)$. Thus $\varphi(\bar{x})$ is of the form $\exists y \psi(y, \bar{x})$. Define a function $f_\varphi : B^n \to B$ as follows: if $\mathfrak{B} \models \varphi(\bar{a})$ for some $\bar{a} \in B^n$, then $\exists b \in B$ such that $\mathfrak{B} \models \psi[b, \bar{a}]$. For each $\bar{a}$ choose such $b \in B$ and define $f_\varphi(\bar{a}) = b$. If for a given $\bar{a}$ there is no such b, then pick an arbitrary element of B.

Let $\mathcal{F} = \{f_\varphi : \varphi \text{ is existential in } \mathcal{L}\}$. Then $|\mathcal{F}| \le \kappa$ since $|\mathcal{L}| \le \kappa$. Take any S' such that $S \subseteq S' \subseteq B$ such that $|S'| = \kappa$. Now take A to be the closure of S' under $\mathcal{F}$. That is $A = \bigcup_{n \in \omega} S'_n$ where $S'_0 = S'$, $S'_1 = S' \cup \{f_\varphi(\bar{a}) : \bar{a} \in [S'_0]^{<\omega}\}$, $S'_{n+1} = S'_n \cup \{f_\varphi(\bar{a}) : \bar{a} \in [S'_n]^{<\omega}\}$. Then $|A| = \kappa$.

It remains to show that A is the universe of an elementary substructure of $\mathfrak{B}$, which is straightforward with the use of the Tarski-Vaught Criterion. □

EXERCISE 2. (ZFC$^-$) Let $\gamma > \omega_1$ be a limit ordinal. Show that there is a countable, transitive model M and ordinals $\alpha, \beta \in M$ such that $M \equiv R(\gamma)$ and $(\alpha \approx \beta)^M$ is false, while $(\alpha \approx \beta)^{R(\gamma)}$ is true.

Hint Let A be a countable set such that ω, ω_1 are in A and $A \preceq R(\gamma)$ (take for example the Skolem-hull of any countable set $A_0 \subseteq R(\gamma)$ which contains ω, ω_1. Consider the Mostowski Collapse M of A.

4.2. Definable Subsets.

DEFINITION 4.3. Let $\mathfrak{A}$ be a structure for $\mathcal{L}$ with $P \subseteq A$. Fix $k > 0$.

(1) $S \subseteq A^k$ is definable over $\mathfrak{A}$ with parameters in P iff $\exists n \ge 0$ and there is a formula $\varphi(x_1, \cdots, x_k, y_1, \cdots, y_n)$ of $\mathcal{L}$ with $k+n$ free variables such that for some $\bar{b} = (b_1, \cdots, b_n) \in P$, $S = \{\bar{a} \in A^n : \mathfrak{A} \models \varphi[a_1, \cdots, a_k, b_1, \cdots, b_n]\}$.
(2) $S \subseteq A^k$ is definable over $\mathfrak{A}$ with parameters iff S is definable over $\mathfrak{A}$ with parameters in A and S is definable over $\mathfrak{A}$ without parameters iff S is definable over $\mathfrak{A}$ with parameters in $\varnothing$.
(3) For $a \in A$, we say that a is definable with or without parameters in P if $\{a\}$ is definable with or without parameters in P.

EXAMPLE 4.4. Note that $\mathcal{P}(\mathbb{R}) = 2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$. Since $\mathcal{L}_\in$ is countable, there are only $\aleph_0$-many subsets of $\mathbb{R}$ which are definable without parameters and $|\mathbb{R}^{<\omega}| = |\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0}$ many subsets of $\mathbb{R}$ which are definable with parameters. Recall that $\mathbb{R}^{<\omega} = \bigcup_{n \in \omega} \mathbb{R}^n$.

REMARK 4.5.

(1) Let P be a set of parameters in $\mathfrak{A}$. If every element of P is definable over $\mathfrak{A}$ without parameters, then every set definable with parameters in P is also definable without parameters.
(2) Every heraditarily finite set a is a definable element of HF. Every subset of HF which is definable with parameters in HF is definable also without parameters. The definable subsets of HF are called arithmetical.

DEFINITION 4.6. Let A be a set and $P \subseteq A$. Then

(1) $D(A,P) = \{X : X \subseteq A, X \text{ is definable over } (A,\in) \text{ with parameters from } P\}$.
(2) $D^{+}(A) = D(A,A)$, $D^{-}(A) = D(A,\varnothing)$.
(3) If $D^{+}(A) = D^{-}(A)$, then we denote them by $D(A)$.
(4) $D(\varnothing) = D^{+}(\varnothing) = D^{-}(\varnothing) = \{\varnothing\}$.

REMARK 4.7. Note that every finite subset of A is in $D^{+}(A)$. Indeed, if $a = \{b_1, \cdots, b_n\}$ then a is definable via the formula $x = y_1 \vee \cdots \vee x = y_n$.

5. Absoluteness and Reflection

From now on, except explicitly stated otherwise, we assume the Axiom of Foundation and thus, unless explicitly stated otherwise, we work in ZFC. Recall that the following are transitive models of BST: $R(\gamma)$ for $\gamma > \omega$ and $H(\kappa)$ for κ regular uncountable.

LEMMA 5.1. Each of the following notions is given by a formula, which is equivalent to a Δ_0 formula in BST. Thus, each of those notions are absolute to transitive models of BST:

(1) x is a transitive set;
(2) x is an ordinal, x is a successor ordinal; x is a limit ordinal
(3) $x = \varnothing$;
(4) x is a natural number;
(5) $x = \omega$.

LEMMA 5.2. If M is a transitive model of BST then the following are absolute for M:

(1) $\varnothing$, S, $\cap$ (2-ary intersection function), $\cup$ (2-ary union function),
(2) 1-ary union and intersection given by $\cap\varnothing = \varnothing$ and $\cup\varnothing = \varnothing$
(3) The ternary relation $\{x,y\} = z$
(4) The 2-ary unordered pairing function $\{x,y\}$, the 1-ary singleton function $\{x\}$, the 2-ary ordered pairing function $\langle x,y\rangle$.
(5) The properties; z is an ordered pair; x is a relation;
(6) $\mathrm{dom}(x)$, $\mathrm{ran}(x)$
(7) The properties f is a function, f is an injection, f is a surjection, f is a bijection.
(8) The binary relation $x \times y$.
(9) All relational properties of a relation R on a set A: R is transitive, reflexive, irreflexive, trichotomy, symmetry, partial order, total order, equivalence relation.

LEMMA 5.3. HC is a model of ZFC-P together with the statement that all sets are countable and the statement that $\mathcal{P}(\omega)$ does not exist.

PROOF. Recall that $\mathrm{HC} = H(\aleph_1) = \{x : |\mathrm{trcl}(x)| < \aleph_1\}$. Observe that

$$\forall x \in \mathrm{HC} \exists f \in \mathrm{HC}(f : x \to \omega)$$

is injective. However being an injective function is absolute and so

$$\forall x \in \mathrm{HC} \exists f \in \mathrm{HC}(f : x \preceq \omega)^{\mathrm{HC}}.$$

Thus (All sets are countable)$^{\mathrm{HC}}$.

If $\mathrm{HC} \vDash \mathcal{P}(\omega)$ exists, then by the above observation $\mathrm{HC} \vDash \exists f : \mathcal{P}(\omega) \preceq \omega$. By absoluteness, this gives that $\mathcal{P}(\omega)$ is countable, which is a contradiction. $\square$

LEMMA 5.4. The function $\alpha + \beta$ and $\alpha \cdot \beta$ are absolute for transitive models of ZF-P.

PROOF. If M is a transitive model of ZF-P with $\alpha, \beta \in M$ then $\alpha +^M \beta$ and $\alpha \cdot^M \beta$ are defined. Let $\gamma = \alpha \cdot^M \beta$. We want to show that $\gamma = \alpha \cdot \beta$. Let $f \in M$ be such that

$$M \vDash f : (\beta \times \alpha, <_{\mathrm{lex}}) \cong (\gamma, \in).$$

Being a lexicographic order, and being an isomorphism are absolute and so

$$f : (\beta \times \alpha, <_{\mathrm{lex}}) \cong (\gamma, \in).$$

But then $\gamma = \alpha \cdot \beta = \mathrm{type}(\beta \times \alpha, <_{\mathrm{lex}})$. The proof for $\alpha +^M \beta$ is similar. $\square$

LEMMA 5.5. The notions "R well-orders A" and "R is well-founded on A" are absolute for transitive models of ZF-P.

PROOF. Let M be a transitive model of ZF-P. Being a total order is absolute, so we will verify the absoluteness of being a well-founded.

Let A, R be such that R is a well-order on A suppose A, R are elements of M. We have to verify if $M \vDash$ (R is a well-order on A). Suppose this is not the case. Let $\psi(A, R, X)$ be the formula

$$X \subseteq A \wedge X \neq \varnothing \wedge X \text{ has no } R \text{ minimal element},$$

which is the same as

$$X \subseteq A \wedge X \neq \varnothing \wedge \forall z \in X \exists y \in X (yRx).$$

Since by hypothesis (R is not well-founded on A)^M, then $\exists X \in M$ such that $(\psi(A, R, X))^M$. But $\psi(A, R, X)$ is absolute and so $\psi(A, R, X)$ is true, contradiction to R being well-founded on A.

Suppose (R is well-founded on A)^M. Now, since $M \vDash$ ZF-P, then $M \vDash$ ($\exists$ a rank function). That is there is $\Phi \in M$ such that

$$M \vDash \big(\Phi \text{ is a function}, \mathrm{dom}(\Phi) = A, \forall x \in A \Phi(x) \in \mathbb{ON}, xRy \to \Phi(x) < \Phi(y)\big).$$

The above statement is absolute and so there is such a function in V. Therefore R is well-founded on A. Indeed, if $X \subseteq A$, then any $a \in X$ with $\Phi(a) = \min\{\Phi(x) : x \in X\}$ is R-minimal in X. $\square$

COROLLARY 5.6. The properties "R well-orders A" and "R is well-founded on A" are absolute for $R(\gamma)$, for any limit γ.

LEMMA 5.7. Let M be a transitive model of BST. Then:

(1) $[M]^{<\omega} \subseteq M$
(2) $\mathrm{HF} \subseteq M$
(3) ${}^{<\omega}M \subseteq M$.

PROOF. (1) Consider the function $f : \langle x, y\rangle \mapsto x \cup \{y\}$. Then f is absolute and moreover if $x, y \in M$ then $f(x, y) \in M$. Note that $M \subseteq M$. For each $x, y \in M$, the pair $\{x, y\} \in M$ by absoluteness of the pairing function. Note that $z \in [M]^{n+1}$ iff $z = x \cup \{y\} = f(x, y)$ for some $x \in [M]^n$ and $y \in M \backslash x$. Now, if we assume that $[M]^n \subseteq M$ then by absoluteness of f, whenever $x, y \in M$ we have also $f(x, y) = x \cup \{y\} \in M$ and so $[M]^{n+1} \subseteq M$.

(2) By induction on n, we can show that $R(n) \subseteq M$ for each natural number. Thus HF $\subseteq M$.

(3) Recall that ${}^{<\omega}M = \bigcup\{f : n \to M : f$ is a function, $n \in \omega\}$. For each such f note that f is a finite subset of M and so by (1), $f \in M$. □

REMARK 5.8. Let $\mathrm{Fin}(x)$ be the formula $\exists n, f(\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x))$ and let $\mathrm{HrdFin}(x)$ be the formula $\exists n, t, f(x \subseteq t \wedge \mathrm{tran}(t) \wedge \mathrm{nat}(n) \wedge \mathrm{bij}(f, n, t))$, where $\mathrm{tran}(x)$ says that x is transitive, $\mathrm{nat}(x)$ says that x is a natural number and $\mathrm{bij}(f, x, y)$ says that f is a bijection from x onto y. Thus, $\mathrm{Fin}(x)$ says that x is finite and $\mathrm{HrdFin}(x)$ says that x is hereditarily finite. Note that $\mathrm{Fin}(x)$ and $\mathrm{HrdFin}(x)$ are absolute for transitive models of BST. Indeed, fix M transitive model of BST. Then:

- Suppose $x \in M$ and $M \models \mathrm{Fin}(x)$. That is $M \models \exists n \exists f \in M(\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x))$. However nat and bij are absolute and so x is finite.
- Suppose $\mathrm{Fin}(x)$. Thus there are n and f such that $\mathrm{nat}(n) \wedge \mathrm{bij}(f, n, x)$. Suppose $x \in M$. Now $n \in M$ and since M is transitive also $x \subseteq M$. Thus $f \in {}^n x \subseteq {}^n M \subseteq M$ (by item (1) of the previous Lemma). Thus $f \in M$ and so $(\mathrm{Fin}(x))^M$.

The proof that $\mathrm{HrdFin}(x)$ is absolute is similar.

COROLLARY 5.9. The following are absolute for transitive models of ZF-P:the 0-ary function HF; the 0-ary function ω; the 1-ary function $[x]^{<\omega}$ and ${}^{<\omega}x$. So if M is transitive and $M \models$ ZF-P and $x \in M$ then all finite subsets of x are in M and all finite tuples of x are in M.

5.1. Absoluteness of recursively defined notions.

THEOREM 5.10. *Let A be a defined class, R a defined 2-ary relation on A which is well-founded and set-like, and let G be a defined 2-ary function. Let F be a defined 1-ary function such that*

$$\forall a \in A(F(a) = G(a, F \restriction (a \downarrow))$$

and $F(a) = \varnothing$ for $a \notin A$. Let M be a transitive model of ZF-P such that R, A, G are absolute for M, $(R$ is set like on $A)^M$ and for all $a \in M$, $a \downarrow = pred_R(a) \subseteq M$. Then $F^M(a)$ is defined for all $a \in M$ and F is absolute for M.

PROOF. Note that $(R$ is well-founded on $A)^M$ and since $\mathrm{pred}_R(a) = (\mathrm{pred}_R(a))^M$ for each $a \in M$, also $(R$ is set-like on $A)^M$. The existence and uniqueness of F were proved in ZF-P and so F^M is defined. Suppose there is $a \in M$ such that $F^M(a) \neq F(a)$ and pick a which is R-minimal in $\{x \in M : F^M(x) \neq F(x)\}$. Then since $(\mathrm{pred}_R(a))^M = \mathrm{pred}_R(a)$ and $\forall x \in \mathrm{pred}_R(a)$, $F(x) = F^M(x)$, we obtain that $F^M \restriction (\mathrm{pred}_R(a))^M = F \restriction (pred_R(a))$ and so

$$F^M(a) = G(a, F^M \restriction (\mathrm{pred}_R(a))^M) = G(a, F \restriction \mathrm{pred}_R(a)) = F(a),$$

which is a contradiction. □

COROLLARY 5.11. The functions $\alpha + \beta$, $\alpha \cdot \beta$, α^β, rank(x), $D(A, P)$, $D^+(A)$ and $D^-(A)$ are absolute for transitive models of ZF-P.

5.2. Upwards and downwards absoluteness.

DEFINITION 5.12.

(1) A formula φ is Σ_1 iff φ is of the form $\exists y_1 \cdots \exists y_n \psi$ for some $n \geq 0$ and ψ which is Δ_0.
(2) A formula φ is Π_1 iff φ is of the form $\forall y_1 \cdots y_n \psi$ for some $n \geq 0$ and ψ which is Δ_0.

LEMMA 5.13. Let M be a transitive model of BST. Consider an extension $\mathcal{L}_\in$ in which all new non-logical symbols are absolute for M. Let $\varphi(\bar{x})$ and $\psi(\bar{x})$ be a Σ_1 and a Π_1 formulas in $\mathcal{L}$ where $\bar{x} = (x_1, \cdots, x_n)$. Then for all $\bar{a} \in M^n$:

(1) if $\varphi^M(\bar{a})$ then $\varphi(\bar{a})$ (upwards absoluteness);
(2) if $\psi(\bar{a})$ then $\psi^M(\bar{a})$ (downwards absoluteness).

PROOF.
(1) Let $\varphi(\bar{a})$ be of the form $\exists y_1 \cdots \exists y_k \psi(\bar{x}, \bar{y})$ where ψ is Δ_0. If $\varphi^M(\bar{a})$ holds, then there are $b_1, \cdots, b_k$ in M such that $\psi^M(\bar{a}, \bar{b})$ where $\bar{b} = (b_1, \cdots, b_k)$. However ψ is Δ_0 and so $\psi(\bar{a}, \bar{b})$ holds as well. Therefore $\varphi(\bar{a})$ holds.
(2) Let $\psi(\bar{a})$ be of the form $\forall y_1 \cdots \forall y_k \psi(\bar{x}, \bar{y})$ and suppose $\psi(\bar{a})$ holds for some $\bar{a}$ in M. Thus, whenever $\bar{b} = (b_1, \cdots, b_k) \in M^k$ we have that $\psi(\bar{a}, \bar{b})$. However by absoluteness of ψ we have that $\psi^M(\bar{a}, \bar{b})$ and so $M \vDash \psi(\bar{a})$. □

EXAMPLE 5.14.

(1) Note that "R is well-founded on A" can be expressed by a Π_1 formula in absolute notions and so it is downwards absolute. On the other hand "R is well-founded on A" can be expressed by a Σ_1-formula in absolute notions, $\exists \Phi(\Phi$ is a rank function$)$ and so it is upwards absolute.
(2) Being countable is upwards absolute for transitive models of ZF-P. Indeed, given a set x the formula $\exists f : x \preceq \omega$ says that x is a countable set. Thus, being countable can be expressed via a Σ_1 formula in absolute notions (for transitive models of ZF-P). Note that being countable is not necessarily absolute.

5.3. Reflection Theorems.

THEOREM 5.15. *(Tarski-Vaught Criteria for Classes) Let $\varphi_0, \varphi_1, \cdots, \varphi_{n-1}$ be a sub-formula closed list of formulas, i.e. for each $i \in n$ every subformula of φ appears in this list and no formula uses universal quantifier. Let $A \subseteq B$ be classes, A non-empty. The following are equivalent:*

(1) for all $i \in n$, $A \preceq_{\varphi_i} B$
(2) if $\varphi_i = \varphi_i(x_1, \cdots, x_n)$ is an existential formula of the form $\exists y \varphi_j(\bar{x}, y)$, then for all $\bar{a} = (a_1, \cdots, a_n) \in A^n$ we have $(\varphi_i^B(\bar{a}) \to \exists b \in A \varphi_j^B(\bar{a}, b))$.

PROOF. (1) $\Rightarrow$ (2) Fix φ_i, $\bar{a} \in A^n$. Then since $A \preceq_{\varphi_i} B$, we have that $\varphi_i^B(\bar{a}) \to \varphi_i^A(\bar{a})$. By definition of φ_i we get $\exists b \in A \varphi_j^A(\bar{a})$. But φ_j is also absolute and so we have $\exists b \in A \varphi_j^B(\bar{a})$.

(2) $\Rightarrow$ (1) We proceed by induction on the length of the formulas appearing in the given list. Consider φ_i and assume for each φ_j such that φ_j is shorter than φ_i the claim holds, i.e. φ_j is absolute between A and B. Atomic formulas, as well as formulas obtained via logical connectives from formulas which are absolute, are absolute. Thus suppose $\varphi_i = \exists y \varphi_j(\bar{a}, y)$ and let $\bar{a} = (a_1, \cdots, a_n) \in A^n$. Then

$$\varphi_i^B(a) \to \exists b \in B \varphi_j^B(a) \to \exists b \in A \varphi_j^B(\bar{a}, b) \to \exists b \in A \varphi_j^A(\bar{a}, b) \to \varphi_i^A,$$

where in the second implication we used (2) and in the third implication we used the inductive hypothesis on φ_j. On the other hand:

$$\varphi_i^A \to \exists b \in A \varphi_j^A(\bar{a}, b) \to \exists b \in A \varphi_j^B(\bar{a}, b) \to \exists b \in B \varphi_j^B(a) \to \varphi_i^B(a),$$

where the second implication used the absoluteness of φ_j and the third implication used the fact that A is a subclass of B. □

THEOREM 5.16. *(Reflection Theorem) Let $\varphi_0, \varphi_1, \cdots, \varphi_{n-1}$ be any list of formulas of $\mathcal{L}_\in$, B a non-empty class and $\forall \xi \in \mathbb{ON}$ let $A(\xi)$ be a set. Further, assume that:*

(1) if $\xi < \eta$ then $A(\xi) \subseteq A(\eta)$,
(2) $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ for limit η,
(3) $B = \bigcup_{\xi \in \mathbb{ON}} A(\xi)$. Then $\forall \xi \exists \eta > \xi$ such that η is a limit, $A(\eta) \neq \varnothing$ and for each $i \in n$, φ_i is absolute between $A(\eta)$ and B.

PROOF. Without loss of generality $\varphi_0, \cdots, \varphi_{n-1}$ is subformula closed and none of the formulas contains universal quantifiers. Indeed, we can always extend the list by adding all subformulas and substitute each universal quantifier "$\forall$" with "$\neg\exists$". *What we want to do is: climb up the hierarchy to gather all the witnesses!* For each existential formula $\varphi_i(x)$ of the form $\exists y \varphi_j(x_1, \cdots, x_{n_i}, y)$ define $F_i : B^{n_i} \to \mathbb{ON}$ as follows:

$$F_i(\bar{a}) = \begin{cases} \min\{\zeta : \exists b \in A(\zeta) \varphi^B(\bar{a}, b)\} & \text{if } \varphi_i^B(\bar{a}) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

Now, for each $\xi \in \mathbb{ON}$ define

$$G_i(\xi) = \sup\{F_i(a_1, \cdots, a_{n_i}) : \bar{a} = (a_1, \cdots, a_{n_i}) \in (A(\xi))^{n_i}\}$$

and let

$$K(\xi) = \max\{\xi + 1, \max_{i \in n} G_i(\xi)\}$$

where $G_i(\xi) = 0$ if φ_i is not existential. *Thus $K(\xi)$ is the least ordinal greater than ξ such that $A(K(\xi))$ contains all witnesses to existential formulas with parameters in $A(\xi)$.*

Fix ξ and recursively define an increasing sequence $\langle \zeta_n \rangle_{n \in \omega}$ as follows. Let

$$\zeta_0 = \min\{\zeta : \zeta > \xi \wedge A(\zeta) \neq \varnothing\}$$

and for each n, $\zeta_{n+1} = K(\zeta_n)$. Then take $\eta = \sup_{i \in \omega} \zeta_i$. Then $A(\eta)$ contains all witnesses to existential formulas (from the given list) with parameters in $A(\eta)$, i.e.

$$F_i : (A(\eta))^{n_i} \to A(\eta).$$

But, then by the Tarski-Vaught Criteria we have have that for all i, $A(\eta) \preceq_{\varphi_i} B$. $\square$

COROLLARY 5.17. Let $\Lambda = \{\varphi_0, \cdots, \varphi_{n-1}\}$ be a finite set of axioms of ZF. Recall that Z is the set of all Axioms 1-8, except the Axiom of Replacement and ZC is the set of all Axioms 1-8, again except Replacement. Then:

(1) $\mathrm{ZFC} \vdash \exists\eta(R(\eta) \vDash \mathrm{Z} \cup \Lambda)$
(2) $\mathrm{ZFC} \vdash \exists\eta(R(\eta) \vDash \mathrm{ZC} \cup \Lambda)$
(3) $\mathrm{ZFC} \vdash \exists M(M \vDash \mathrm{ZC} \cup \Lambda \wedge |M| = \aleph_0 \wedge M$ is transitive).

REMARK 5.18. In particular, Λ might be finitely many instances of Replacement.

PROOF. (1) – (2). By the Reflection Theorem there is $\eta > \omega$ limit such that for each $i \in n$, $R(\eta) \preceq_{\varphi_i} V$. Since for each i, φ_i is an axiom, $(\varphi_i)^V$ and so $R(\eta) \vDash \varphi_i$ for each i. Recall that ZF^- proves that $R(\eta) \vDash \mathrm{Z}$ (i.e. $\mathrm{ZF}^- \vdash R(\eta) \vDash \mathrm{Z}$) and respectively ZFC^- proves that $R(\eta) \vDash \mathrm{ZC}$. But then $\mathrm{ZF} \vdash R(\eta) \vDash \mathrm{Z} \cup \Lambda$ and $\mathrm{ZFC} \vdash R(\eta) \vDash \mathrm{ZC} \cup \Lambda$.

(3) To obtain a countable, transitive model for $\mathrm{ZC} \cup \Lambda$ find a countable elementary submodel $\mathcal{N}$ of $R(\eta)$ (using a Skolem hull, i.e. gathering existential witnesses) and take $M = \mathrm{mos}''_{R(\eta),\in}\mathcal{N}$. Then $\mathcal{M} \cong \mathcal{N}$ and so $\mathcal{M} \vDash \mathrm{ZC} \cup \Lambda$. $\square$

COROLLARY 5.19. Let $\Lambda = \{\varphi_0, \cdots, \varphi_{n-1}\}$ be a set of $\mathcal{L}_\in$-formulas. Then

$$\mathrm{ZFC} \vdash \exists C(C \vDash \mathrm{ZC} \wedge |C| = \aleph_0 \wedge \wedge_{j<n})\varphi_j^C \leftrightarrow \varphi_j.$$

PROOF. Use Reflection to find a limit $\eta > \omega$ such that $\wedge_{j<n} R(\eta) \preceq_{\varphi_j} V$ and the Downwards-Löwenheim-Skolem Theorem to get a countable elementary submodel C of $R(\eta)$. $\square$

THEOREM 5.20. *(ZFC) Let $\kappa > \omega$ be a regular cardinal and for each $\xi \leq \kappa$, let $A(\xi)$ be a set such that:*

(1) if $\xi < \eta$ then $A(\xi) \subseteq A(\eta)$
(2) $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ for limit $\eta \leq \kappa$
(3) $|A(\xi)| < \kappa$ for all $\xi < \kappa$ and $|A(\kappa)| = \kappa$.

Then $\forall\xi < \kappa\exists\eta$ such that $\xi < \eta < \kappa$, η is a limit, $A(\eta) \neq \varnothing$ and $A(\eta) \preceq A(\kappa)$.[1]

PROOF. Let $\{\varphi_i\}_{i\in\omega}$ enumerate all existential and all quantifier free $\mathcal{L}_\in$-formulas. For each i such that φ_i is existential, define

$$F_i : (A(\kappa))^{n_i} \to \kappa,$$

where $\varphi_i = \exists y\varphi_j(\bar{x}, y)$ and $\bar{x} = (x_0, \cdots, x_{n_{i-1}})$ and $\bar{x} = (x_0, \cdots, x_{n_{i-1}})$ just as before, i.e.

$$F_i(\bar{a}) = \begin{cases} \min\{\zeta < \kappa : \exists b \in A(\zeta)\varphi^{A(\kappa)}(\bar{a}, b)\} & \text{if } A(\kappa) \vDash \varphi_i(\bar{a}) \\ 0 & \text{otherwise} \end{cases}$$

and let

$$G_i(\xi) = \begin{cases} \sup\{F_i(a_1, \cdots, a_{n_i}) : (a_1, \cdots, a_{n_i}) \in (A(\xi))^{n_i}\} & \text{if } \varphi_i \text{ is existential} \\ 0 & \text{otherwise.} \end{cases}$$

[1]Here we consider the sets $A(\xi)$ as $\in$-models for $\mathcal{L}_\in$.

Since $|A(\xi)| < \kappa$ for all $\xi < \kappa$ and κ is regular, we obtain $G_i(\xi) < \kappa$ for all $\xi < \kappa$. Define

$$K(\xi) = \max\{\xi + 1, \sup\{G_i(\xi) : i < \omega\}\}.$$

Then since κ is regular, uncountable, $K(\xi) < \kappa$ for all ξ. Just as in the Reflection Theorem take $\zeta_0 = \min\{\zeta : \zeta > \xi, A(\zeta) \neq \varnothing\}$ and for all $n \geq 0$, define $\zeta_{n+1} = K(\zeta_n)$. Then $\eta = \lim_n \zeta_n$ is as desired (indeed, since κ is regular, $\eta > \kappa$). □

COROLLARY 5.21. (ZFC) If κ is strongly inaccessible, then

$$\{\eta < \kappa : R(\eta) \preceq R(\kappa)\}$$

is unbounded in κ.

6. The Constructible Sets

Consider $\mathcal{L}_\in$. Let A be a set and let $P \subseteq A$. Recall the definitions of $D(A,P)$, $D^+(A) = D(A,A)$, $D^-(A) = D(A,\varnothing)$ and $D(A)$.

DEFINITION 6.1. (The Constructible Hierarchy) Define $L(\delta)$ recursively on $\delta \in \mathbb{ON}$ as follows:

(1) $L(0) = \varnothing$,
(2) $L(\beta + 1) = D^+(L(\beta))$,
(3) $L(\gamma) = \bigcup_{\alpha<\gamma} L(\alpha)$ for limit γ.

Then $L = \bigcup\{L(\alpha) : \alpha \in \mathrm{ON}\}$ is called the Constructible Universe.

LEMMA 6.2.

(1) For each ordinal α, $L(\alpha) \subseteq R(\alpha)$.
(2) For each $\alpha \in \mathbb{ON}$, $L(\alpha)$ is a transitive set.
(3) For each $\alpha, \beta \in \mathbb{ON}$ such that $\alpha \subseteq \beta$, $L(\alpha) \subseteq L(\beta)$.
(4) For each $\alpha \in \mathbb{ON}$, $L(\beta) \cap \mathbb{ON} = \beta$.

PROOF. (1) We proceed by induction on α. Note that $L(0) = R(0) = \varnothing$. Suppose $L(\alpha) \subseteq R(\alpha)$. Then $L(\alpha + 1) \subseteq \mathcal{P}(L(\alpha)) \subseteq \mathcal{P}(R(\alpha)) = R(\alpha + 1)$. If γ is a limit and for all $\beta < \gamma$, $L(\beta) \subseteq R(\beta)$, then $\bigcup_{\beta<\gamma} L(\beta) \subseteq \bigcup_{\beta<\gamma} R(\beta)$.

(2) Again we proceed by induction on α. If $\alpha = 0$, or α is a limit and $\forall \beta < \alpha$, $L(\beta)$ is transitive, then clearly $L(\alpha)$ is transitive. Thus, suppose $L(\beta)$ is transitive and let $b \in L(\beta + 1)$. Then $b \subseteq L(\beta)$, as b is a definable subset of $L(\beta)$. However:

Claim: $L(\beta) \subseteq L(\beta + 1)$.

Proof: Indeed. Let $c \in L(\beta)$. Then by hypothesis, $c \subseteq L(\beta)$ and furthermore $c = \{z \in L(\beta) : z \in c\}$. Thus, c is definable over $L(\beta)$ with parameter the set c, i.e. $c \in L(\beta + 1)$. □

But, then since $b \subseteq L(\beta)$ and $L(\beta) \subseteq L(\beta + 1)$, we obtain $b \subseteq L(\beta + 1)$, i.e. $L(\beta + 1)$ is transitive.

(3) Fix $\alpha \in \mathbb{ON}$. By induction on $\beta \geq \alpha$, we will show that $L(\alpha) \subseteq L(\beta)$. Well, if $\beta = \alpha$, then we are done. Suppose $\beta > \alpha$ and $L(\alpha) \subseteq L(\beta)$. Since $L(\beta) \subseteq L(\beta + 1)$, we obtain $L(\alpha) \subseteq L(\beta + 1)$. If $\beta > \alpha$ is a limit, then since $L(\beta) = \bigcup_{\gamma<\beta} L(\gamma)$, we obtain directly that $L(\alpha) \subseteq L(\beta)$.

(4) Note that $\mathbb{ON} \cap L(\omega) = \mathbb{ON} \cap R(\omega) = \mathrm{HF} \cap \mathbb{ON} = \omega$. If γ is a limit and for all $\alpha < \gamma$, $\mathbb{ON} \cap L(\alpha) = \alpha$, then $\mathbb{ON} \cap \bigcup_{\alpha<\gamma} L(\alpha) = \bigcup_{\alpha<\gamma} \alpha = \gamma$. Thus, consider the successor case. Note that

$$L(\beta+1) \cap \mathbb{ON} \subseteq R(\beta+1) \cap \mathbb{ON} = \beta+1 = \beta \cup \{\beta\}.$$

Thus, it is sufficient to show that $\beta \in L(\beta+1)$. However

$$\beta = \{a \in L(\beta) : L(\beta) \models \varphi[a]\},$$

where φ is $\mathcal{L}_\in$-formula saying that a is an ordinal. Thus β is definable over $L(\beta)$ and so $\beta \in L(\beta+1)$. □

Remark 6.3. Note that for each set x, $\mathrm{rank}(x) = \alpha$ iff $x \in R(\alpha+1) \setminus R(\alpha)$. We will define an analogous notion of an L-rank, denoted by ρ.

Definition 6.4. For $x \in L$, the L-rank of x, denoted $\rho(x)$ is the least α such that $x \in L(\alpha+1)$.

Remark 6.5. Note that for each $\alpha \in \mathbb{ON}$, we have $L(\alpha) = \{x \in L : \rho(x) < \alpha\}$ and

$$L(\alpha+1) \setminus L(\alpha) = \{x \in L : \rho(x) = \alpha\}.$$

Lemma 6.6. For each $\alpha \in \mathbb{ON}$, $L(\alpha) \in L$ and $\rho(L(\alpha)) = \rho(\alpha) = \alpha$.

Proof. Note that $L(\alpha) = \{x \in L(\alpha) : (x = x)^{L(\alpha)}\}$. Thus $L(\alpha) \in D^-(L(\alpha)) \subseteq L(\alpha+1)(= D^+(L(\alpha)))$. On the other hand $L(\alpha) \notin L(\alpha)$, just because $L(\alpha)$ is a set and so $\rho(L(\alpha)) = \alpha$.

Since $L(\alpha) \cap \mathbb{ON} = \alpha$, we have $\alpha \notin L(\alpha)$ (otherwise we would obtain $\alpha \in \alpha$, which is a contradiction). Also, $\alpha + 1 = \alpha \cup \{\alpha\} \subseteq L(\alpha+1)$ and so $\alpha \in L(\alpha+1)$. That is $\alpha \in L(\alpha+1) \setminus L(\alpha)$ and so $\rho(\alpha) = \alpha$. □

Lemma 6.7. Every finite subset of $L(\alpha)$ is in $L(\alpha+1)$.

Proof. Let $A \in [L(\alpha)]^{<\omega}$. Thus $A = \{a_1, \cdots, a_n\}$ for some $n \in \omega$ and $a_j \in L(\alpha)$. Then

$$A = \{x \in L(\alpha) : L(\alpha) \models \varphi(x)\}$$

where $\varphi(x)$ is the formula $x = a_1 \vee \cdots \vee x = a_n$. □

Lemma 6.8. $L(\alpha) = R(\alpha)$ for all $\alpha \leq \omega$ and $L(\omega+1)$ is a proper subset of $R(\omega+1)$.

Proof. Since every finite subset of $L(n)$ is in $L(n+1)$, we obtain that $L(n) = R(n)$ for all $n \in \omega$. But, then

$$L(\omega) = \bigcup_{n\in\omega} L(n) = R(\omega) = \bigcup_{n\in\omega} R(n).$$

Now consider $L(\omega+1)$ and $R(\omega+1)$. While $R(\omega+1) = \mathcal{P}(R(\omega))$ is uncountable, the set $L(\omega+1)$ is countable (because there are only countably many formulas). □

Lemma 6.9. Assume AC. Then $|D^+(A)| = |A|$ for all infinite A.

Proof. For all $a \in A$, $\{a\} \in D^+(A)$. Indeed, $\{a\} = \{x \in A : (A, \in) \models x = a\}$. Thus $|A| \leq |D^+(A)|$. On the other hand

$$|D^+(A)| \leq |[A]^{<\omega}| \cdot \aleph_0 = |A|,$$

since there are $|[A]^{<\omega}|$-many sets of parameters and only $\aleph_0$-many formulas. □

LEMMA 6.10. Assume AC. Then $|L(\alpha)| = |\alpha|$ for all $\alpha \geq \omega$.

PROOF. By induction on α. If $\alpha = \omega$, then $L(\alpha) = R(\alpha) = \mathrm{HF}$ and so $|L(\omega)| = |\omega| = \omega$.

Suppose $|L(\alpha)| = |\alpha|$. Now $|L(\alpha+1)| = |L(\alpha)| = |\alpha| = |\alpha+1|$ because $\alpha \geq \omega$ and there are only $\aleph_0$-many formulas.

Suppose γ is a limit and $|L(\alpha)| = |\alpha|$ for all $\alpha < \gamma$. Then, $|L(\gamma)| = |\bigcup_{\alpha<\gamma} L(\alpha)| \leq \gamma$. However $\gamma \subseteq L(\gamma)$ and so $|\gamma| \leq |L(\gamma)|$. Thus $|\gamma| = |L(\gamma)|$. ☐

REMARK 6.11. Thus, $|L(\omega_1)| = \omega_1$, while $|R(\omega_1)| = \beth_{\omega_1}$. That is $L(\omega_1)$ is much smaller than $R(\omega_1)$.

6.1. ZF holds in L.

LEMMA 6.12. Suppose $x, y \in L$. Then:
(1) $\{x,y\} \in L$, $\rho(\{x,y\}) = \max(\rho(x), \rho(y)) + 1$
(2) $\langle x,y\rangle \in L$ and $\rho(\langle x,y\rangle) = \max(\rho(x), \rho(y)) + 2$
(3) $\bigcup x \in L$ and $\rho(\bigcup \rho) \leq \rho(x)$,
(4) $x \cup y \in L$ and $\rho(x \cup y) \leq \max(\rho(x), \rho(y))$.

PROOF. (1) Let $\alpha = \max\{\rho(x), \rho(y)\}$. Thus $x, y \in L(\alpha+1)$ and $\{x,y\} \nsubseteq L(\alpha)$. Therefore $\{x,y\} \notin L(\alpha+1)$, since $L(\alpha+1)$ is transitive. However $\{x,y\} \in D^+(L(\alpha+1))$ and so $\{x,y\} \in L(\alpha+2)$. Therefore $\rho(\{x,y\}) = \alpha+1$.

(2) Since $\langle x,y\rangle = \{\{x\},\{x,y\}\}$.

(3) Let $x \in L$ and let $\alpha \in \mathbb{ON}$ such that $x \in L(\alpha+1) = D^+(L(\alpha))$. Thus, there are $b_1, \cdots, b_n$ in $L(\alpha)$ and a formula φ such that

$$x = \{a \in L(\alpha) : L(\alpha) \models \varphi[a, b_1, \cdots, b_n]\}.$$

But, then $z \in \bigcup x$ iff $z \in L(\alpha)$ and $L(\alpha) \models \exists v(z \in v \wedge \varphi[v, b_1, \cdots, b_n])$, i.e.

$$\bigcup x = \{z \in L(\alpha) : L(\alpha) \models \exists v(z \in v \wedge \varphi[v, b_1, \cdots, b_n])\}.$$

Thus $\bigcup x \in D^+(L(\alpha)) = L(\alpha+1)$.

(4) Straightforward from (3). ☐

LEMMA 6.13. If M be a transitive class such that the Comprehension Axiom holds in M and moreover for every subset $x \subseteq M$ there is a set $y \in M$ such that $x \subseteq y$, then then all axioms of ZF hold in M.

PROOF. Recall that we are working in ZFC.

Extensionality and Pairing Since M is a transitive class, the Axiom of Extensionality holds in M by Lemma 2.3.14. Since $M \subseteq \mathrm{WF}$, the Axiom of Foundation holds in M by Lemma 2.3.15.

Pairing Suppose $x, y \in M$. Then $\{x,y\} \subseteq M$, so by assumption there is $z \in M$ such that $\{x,y\} \subseteq z$. Since M satisfies every instance of Comprehension, the following set is in also in M:

$$z' := \{w \in z : w = x \vee w = y\} = \{x,y\}$$

Thus the pairing axiom holds in M by Lemma 2.3.17.

Union Suppose $\mathcal{F} \in M$. Since M is transitive, we have $\bigcup \mathcal{F} \subseteq M$, and so by assumption there is $y \in M$ such that $\bigcup \mathcal{F} \subseteq y$. Since M satisfies every instance of comprehension, the following set is also in M:

$$\bigcup \mathcal{F} \;=\; \{z \in y : \exists A \in \mathcal{F}(z \in A)\}.$$

Thus the union axiom holds in M by Lemma 2.3.18.

Infinity Note that by assumption, M is necessarily nonempty since $\varnothing \subseteq M$ so there is $y \in M$ such that $\varnothing \subseteq y$. By Comprehension, we have $\varnothing \in M$. Furthermore, by Comprehension, Union and Pairing, we can define the successor function on M. Since $\varnothing \in M$ and M is closed under the successor function, we have $\omega \subseteq M$. By assumption, there is some $y \in M$ such that $\omega \subseteq y$. By applying comprehension to y, we get that $\omega \in M$. Finally, by Lemma 2.3.23 it follows that the axiom of infinity holds in M.

Power Set Let $x \in M$ be arbitrary. Then $\mathcal{P}(x) \cap M \subseteq M$, so by assumption there is $y \in M$ such that $\mathcal{P}(x) \cap M \subseteq y$. By comprehension the following set is also in M:

$$\mathcal{P}(x) \cap M \;=\; \{z \in y : z \subseteq x\}$$

By Lemma 2.3.21 it follows that the Power Set Axiom holds in M.

Replacement We will use the criterion in Lemma 2.3.19. Suppose f is a function, $\mathrm{dom}(f) \in M$ and $\mathrm{ran}(f) \subseteq M$ (note: we are *not* assuming that $f \in M$ or $f \subseteq M$, although these things will follow from the other assumptions). By assumption we can take $y \in M$ such that $\mathrm{ran}(f) \subseteq y$. By the other axioms we have already checked for M (including Power Set), it follows that M is closed under taking Cartesian products of sets. Thus $\mathrm{dom}(f) \times y \in M$ and $\mathcal{P}(\mathrm{dom}(f) \times y) \in M$. However, $f \in \mathcal{P}(\mathrm{dom}(f) \times y)$, and so $f \in M$ since M is transitive. Now since f is in M, we can recover $\mathrm{ran}(f)$ by applying comprehension in M to y:

$$\mathrm{ran}(f) \;=\; \{x \in y : \exists z \text{ such that } \langle z, x\rangle \in f\}$$

Thus $\mathrm{ran}(f) \in M$. □

THEOREM 6.14. *All axioms of ZF hold in L.*

PROOF. By the above Lemma, since L is transitive, it is sufficient to show that

(1) the Comprehension Axiom holds in L.
(2) for every $x \subseteq L$, there is $y \in L$ such that $x \subseteq y$.

To see (1) consider an arbitrary formula φ such that $y \notin \mathrm{Fr}(\varphi)$. We have to show that:

$$\forall z, v_0, \cdots, v_{n-1} \in L \exists y \in L \forall x \in L(x \in y \leftrightarrow x \in z \wedge \varphi^L(x, z, \bar{v}))$$

Now, fix $z, v_0, \cdots, v_{n-1}$ in L and let $y := \{x \in z : \varphi^L(x, z, \bar{v})\}$. We have to show that $y \in L$. Find α such that $z, v_0, \cdots, v_{n-1}$ are in $L(\alpha)$ and $\beta \geq \alpha$ such that $L(\beta) \preceq_\varphi L$ (use Reflection). Then

$$y = \{x \in L(\beta) : \psi^{L(\beta)}(x, z, \bar{v})\} \in D^+(L(\beta)) = L(\beta + 1) \subseteq L,$$

where $\psi(x, z, \bar{v})$ is the formula $\varphi(x, z, \bar{v}) \wedge x \in z$. □

6.2. The Axiom of Constructibility in L. The Axiom of Constructibility is the assertion $V = L$, i.e. the assertion that $\forall x \exists \delta (x \in L(\delta))$.

LEMMA 6.15. If M is a transitive model of ZF-P^-, then the function $L(\delta)$ is absolute for M. That is $\forall \delta \in \mathbb{ON} \cap M (L(\delta)^M = L(\delta))$.

PROOF. By absoluteness of recursively defined functions. $\square$

COROLLARY 6.16. The Axiom of Constructibility holds in L.

PROOF. We have to show that $(\forall x \exists \delta (x \in L(\delta))^L)^L$. That is, we have to show that $\forall x \in L \exists \delta \in \mathbb{ON}^L (x \in L(\delta)^L)$, which is true by the definition of L. $\square$

DEFINITION 6.17. Let M be a transitive set model. Define $o(M) = M \cap \mathbb{ON}$ to be the set of ordinals in M. Thus, since M is transitive, $o(M)$ is the first ordinal not in M.

LEMMA 6.18. If M is a transitive, set model of Pairing, Union and Comprehension, then $o(M)$ is a limit ordinal.

PROOF. Let $\alpha \in o(M)$. Then $\alpha + 1 = \alpha \cup \{\alpha\}$ can be defined using only Pairing, Union and Comprehension. Thus, $\alpha + 1 \in o(M)$. $\square$

LEMMA 6.19. Let M be a transitive set model of ZF-P. Then M is a model of the Axiom of Constructibility if and only if $M = L(o(M))$.

PROOF. ($\Leftarrow$) If $M = L(o(M))$, then $\forall x \in M \exists \delta \in \mathbb{ON} \cap M (= o(M))$, such that $x \in L(\delta)$. But $x \in L(\delta)$ iff $(x \in L(\delta))^M$ and so $M \models \forall x \exists \delta (x \in L(\delta))$, i.e. $M \models (V = L)$.

($\Rightarrow$) Thus, suppose $M \models V = L$ and M is transitive. Let $\gamma = o(M)$. Then by absoluteness of $L(\delta)$ for $\delta < \gamma$, we obtain $L(\delta) \in M$ for each $\delta < \gamma$. Therefore $L(\gamma) \subseteq M$. On the other hand $M \models V = L$, i.e. $M \models \forall x \exists \delta (x \in L(\delta))$, i.e.

$$\forall x \in M \exists \delta \in M (x \in L(\delta))^M$$

and since $L(\delta)$ is absolute, we obtain

$$\forall x \in M \exists \delta \in o(M)(x \in L(\delta))$$

and so $M \subseteq L(\gamma) = \bigcup_{\delta < \gamma} L(\delta)$. Thus $M = L(\gamma)$. $\square$

6.3. Axiom of Choice and GCH in L. We know, that $L \models \text{ZF} + V = L$. Thus, to show $L \models \text{AC} + \text{GCH}$, it is enough to show that $\text{ZF} + V = L \models \text{AC} \wedge \text{GCH}$.

DISCUSSION 6.20. We can assume that for all symbols of $\mathcal{L}_\in$, and so all formulas, are herediatrily finite sets. Let $E \subseteq \omega \times \omega$ be defined via $(m, n) \in E$ iff 2 does not divide $\llcorner m2^{-n} \lrcorner$, where $\llcorner m2^{-n} \lrcorner$ denotes the greatest integer less than or equal to $\frac{m}{2^n}$. Let $\Gamma : R(\omega) \to \omega$ be defined by $\Gamma(y) := \sum\{2^{\Gamma(x)} : x \in y\}$. Here are some examples:

$$\Gamma(\varnothing) = \sum \varnothing = 0, \Gamma(1) = \Gamma(\{\varnothing\}) = 2^0 = 1, \Gamma(\{1\}) = 2^1 = 2, \text{ etc.}$$

Then $\Gamma : (R(\omega), \in) \cong (\omega, E)$ (is an isomorphism) and $\Gamma^- = \text{mos}_{(\omega, E)}$ is the Mostowski collapsing function on (ω, E).

DEFINITION 6.21. Consider the language $\mathcal{L}_\in$ of set theory.

(1) List all variables $\{v_i : i \in \omega\}$ so that $\forall i, j(i < j \to \Gamma(v_i) < \Gamma(v_j))$.
(2) A formula φ is said to be good, if there is $n \in \omega$ such that $\mathrm{Fr}(\varphi) = \{v_0, \cdots, v_n\}$.
(3) List all good formulas $\{\varphi_i : i \in \omega\}$ so that $\forall i, j(i < j \to \Gamma(\varphi_i) < \Gamma(\varphi_j))$.
(4) For φ_i a good formula, let $n_i + 1$ denote the number of its free variables.

DEFINITION 6.22.

(1) Let $A \neq \varnothing$, $i \in \omega$ and $\bar{b} \in A^{n_i}$. Define $D(A, i, \bar{b})$ to be the set definable over $(A, \in)$ from the formula φ_i with parameter $\bar{b}$. That is
$$D(A, i, \bar{b}) = \{a \in A : A \vDash \varphi_i[b_0, \cdots, b_{n_i}, a]\}.$$
(2) Note that $D^+(A) = \{D(A, i, \bar{b}) : i \in \omega, \bar{b} \in A^{n_i}\}$. Then for $S \in D^+(A)$, define $i(S)$ to be the least index i such that S is definable from φ_i with some parameter $\bar{b} \in A^{n_i}$. That is $i(S)$ is the least i such that $S = D(A, i, \bar{b})$ for some $\bar{b} \in A^{n_i}$.
(3) For $A \neq \varnothing$ and R a well-order on A, let $R^{(n)}$ be the induced lexicorgraphic order on A^n. That is for $\bar{b}_1 \neq \bar{b}_2$, where $\bar{b}_1 = (b^1_1, \cdots, b^1_n)$ and $\bar{b}_2 = (b^1_1, \cdots, b^2_n)$, we have that $\bar{b}^1 R^{(n)} \bar{b}^2$ if $b^1_j R b^2_j$, where $j = \min\{i : b^1_i \neq b^2_i\}$. Now, for $S \in D^+(A)$ let $\bar{p}(S, R)$ be the $R^{n_{i(S)}}$-least parameter $\bar{b} \in A^{n_{i(S)}}$ such that $S = D(A, i(S), \bar{b})$.
(4) Define a well-order $W = W(A, R)$ on $D^+(A)$ as follows: $S_1 W S_2$ iff either $i(S_1) < i(S_2)$, or $i(S_1) = i(S_2)$ and $\bar{p}(S_1, R) R^{n_{i(S)}} \bar{p}(S_2, R)$.
(5) Since $D^+(\varnothing) = \{\varnothing\} = \{\varnothing\}$, the empty order is the only well-order of $\varnothing$ and of $\{\varnothing\}$. Thus, define $W(\varnothing, \varnothing) = \varnothing$ and if R is not a well-order of A, then $W(A, R) = \varnothing$.

LEMMA 6.23. $W(A, R)$ is a well-order of $D^+(A)$.

DEFINITION 6.24. By recursion on the ordinals, define a well-order $\lhd_\delta$ on $L(\delta) \times L(\delta)$ as follows: $x \lhd_\delta y$ iff $\rho(x) < \rho(y)$, or $\rho(x) = \rho(y)$ and $(x, y) \in W(L(\rho), \lhd_\rho)$ where $\rho = \rho(x) = \rho(y)$[2]. Extend these relations $\lhd_\delta$ to a relation $<_L$ on all of L as follows:
$$x <_L y \text{ iff } \quad \begin{aligned} &\rho(x) < \rho(y), \text{ or} \\ &\rho(x) = \rho(y) \text{ and } x \lhd_{\rho+1} y \text{ where } \rho = \rho(x) = \rho(y). \end{aligned}$$

THEOREM 6.25.

(1) $<_L$ is a well-order of L.
(2) $\lhd_\delta = <_L \cap (L(\delta) \times L(\delta))$.
(3) If $V = L$, then $<_L$ well-orders V and so AC holds.

PROOF. Straightforward. □

LEMMA 6.26. (AC) If κ is a regular uncountable cardinal, then $L(\kappa) \vDash \text{ZF-P} + V = L$.

PROOF. Replacement Let $M = L(\kappa)$ and let A be a set in M such that
$$\forall x \in M(x \in A \to \exists! y \in M \varphi^M(x, y)).$$

[2] Recall that $\rho(x)$ is the least α such that $x \in L(\alpha + 1)$ and so $L(\alpha + 1) \backslash L(\alpha) = \{x \in L : \rho(x) = \alpha\}$.

We need to find a set $B \in M$ such that $(\forall x \in A \exists y \in B \varphi(x,y))^M$. Since κ is a limit ordinal, we can find $\alpha < \kappa$ such that $A \in L(\alpha)$. Then $|A| \le |L(\alpha)| < \kappa$. Define a function f such that $\mathrm{dom}(f) = A$ and for all $x \in A$, $f(x)$ is the unique $y \in M$ such that $\varphi(x,y)$. Then $\forall x \in A$, $\rho(f(x)) < \kappa$ and thus $\beta = \sup\{\rho(f(x)) + 1 : x \in A\} < \kappa$, because κ is regular and $|A| < \kappa$. Take $B = L(\beta)$. Then $B \in L(\beta + 1)$ and so $B \in L(\kappa)$.
Comprehension Let $\varphi(x, z, v_0, \cdots, v_{n-1})$ be a formula, $y \in \mathrm{Fr}(\varphi)$. We must verify that:

$$\forall z, v_0, \cdots, v_{n-1} \in L(\kappa) \exists y \in L(\kappa) \forall x \in L(\kappa)(x \in y \leftrightarrow x \in z \wedge \varphi^{L(\kappa)}(x, z, \bar{v})).$$

Now, fix $z, v_0, \cdots, v_{n-1} \in L(\kappa)$. Thus, there is $\alpha < \kappa$ such that $z, v_0, \cdots, v_{n-1} \in L(\alpha)$. Then, we can find $\beta > \alpha$ such that $L(\beta) \preceq_\varphi L(\kappa)$. Take $y = \{x \in L(\beta) : \psi^{L(\beta)}(x, z, \bar{v})\}$ where $\psi(x, z, \bar{v}) = \varphi(x, z, \bar{v}) \wedge x \in z$. Then $y \in L(\beta + 1) \subseteq L(\kappa)$.
All other axioms: Use the sufficient conditions, which we obtained earlier.
V=L To verify that $L(\kappa) \models V = L$, observe that $L(\kappa) = L(o(L(\kappa)))$ and so by an earlier results, we obtain $L(\kappa) \models (V = L)$. □

THEOREM 6.27. *If $V = L$ then for every cardinal $\kappa \ge \omega$ the following holds:*

$$(*)_\kappa \ \ L(\kappa) = H(\kappa).$$

Therefore, $V = L$ implies GCH.

PROOF. We work under the assumption that $V = L$. Let λ be an arbitrary infinite cardinal. Then $\mathcal{P}(\lambda) \subseteq H(\lambda^+)$ and so if $H(\kappa) = L(\kappa)$ for each cardinal $\kappa \ge \omega$, we obtain that

$$2^\lambda = |\mathcal{P}(\lambda)| \le |H(\lambda^+)| = |L(\lambda^+)| = \lambda^+.$$

However $\lambda^+ \le 2^\lambda$ (by definition) and so $2^\lambda = \lambda^+$.

Thus, it is sufficient to show that for all cardinals $\kappa \ge \omega$, $L(\kappa) = H(\kappa)$. If $\kappa = \omega$, then $H(\kappa) = L(\kappa) = R(\kappa) = \mathrm{HF}$, so in this case we are done. If κ is an uncountable limit ordinal, then $L(\kappa) = \bigcup_{\lambda<\kappa} L(\lambda^+)$ and $H(\kappa) = \bigcup_{\lambda<\kappa} H(\lambda^+)$. Thus, it is sufficient to show that $H(\kappa) = L(\kappa)$ for κ a successor cardinal of the form λ^+.

Let κ be an uncountable cardinal and let $x \in L(\kappa)$, $\kappa > \omega$. Then, by definition of $L(\kappa)$, we can find α such that $\omega \le \alpha < \kappa$ and such that $x \in L(\alpha)$. But, then $\mathrm{trcl}(x) = \bigcup\{\bigcup^n x : n \in \omega\} \subseteq L(\alpha)$ and so $|\mathrm{trcl}(x)| \le |L(\alpha)| = |\alpha| < \kappa$. Therefore $x \in H(\kappa)$. Thus, for all uncountable cardinals κ, $L(\kappa) \subseteq H(\kappa)$.

Now, let λ be an infinite cardinal. We will show that $H(\lambda^+) \subseteq L(\lambda^+)$. Let $b \in H(\lambda^+)$ and let $T = \mathrm{trcl}(\{b\})$. Thus, $b \in T$ and $|T| \le \lambda$. Since we work under the assumption that $V = L$, we can pick a regular uncountable $\theta > \rho(T)$. Then $T \subseteq L(\theta)$ and by one of the previous theorems $L(\theta) \models \text{ZF-P} + V = L$. By the Downward Löwenheim-Skolem theorem we can find an elementary submodel $A \preceq L(\theta)$ such that $T \subseteq A$, $|A| = |T| \le \lambda$. Thus, by elementarity $A \models \text{ZF-P} + V = L$. Let $(B, \in)$ be the Mostowski Collapse of $(A, \in)$. Since $T \subseteq A$ is transitive, $\mathrm{mos}_{A,\in} \upharpoonright T = \mathrm{id}$ and so $b = \mathrm{mos}_{(A,\in)}(b) \in B$. Since $(B, \in) \cong (A, \in)$ we have that $(B, \in) \models \text{ZF-P} + V = L$. But then $B = L(o(B))$ using the fact that B is transitive. However $|B| = |o(B)| = |A| \le \lambda$ and so $o(B) < \lambda^+$. Therefore $L(o(B)) \subseteq L(\lambda^+)$ and so $b \in L(\lambda^+)$. Thus $H(\lambda^+) \subseteq L(\lambda^+)$. □

Consequently, we have the following theorem.

THEOREM 6.28.

(1) If $\mathrm{Con}(ZF)$ *then* $\mathrm{Con}(ZFC + V = L)$.
(2) If $\mathrm{Con}(ZF)$ *then* $\mathrm{Con}(ZFC + GCH)$.

LEMMA 6.29. (AC) If κ is weakly inaccessible, then in L, κ is strongly inaccessible and $L(\kappa) \models \mathrm{ZFC} + V = L$.

PROOF. Being a cardinal $(\forall\alpha < \kappa\forall f : \alpha \to \kappa(f \text{ is not onto}))$ and being weakly inaccessible $(\forall\lambda < \kappa(\lambda^+ < \kappa))$ are Π_1 properties and so they are downwards absolute. Under GCH, being weakly inaccessible and strongly inaccassible are notions which coincide. Thus, say κ is weakly inaccessible in V. However, $L \subseteq V$ and by Π^1_1-absoluteness, $L \models (\kappa \text{ is weakly inaccessible})$ and since $L \models \mathrm{GCH}$, we have that $(\kappa \text{ is strongly inaccessible})^L$. However, since AC holds by assumption by one of our earlier theorems today for every uncountable λ, $L(\lambda) \models \text{ZF-P} + V = L$. Thus, $L(\kappa) \models \text{ZF-P} + V = L$. On the other hand working in ZFC^-, we proved that if κ is strongly inaccessible then $R(\kappa) = H(\kappa) \models \mathrm{ZFC}$. Now, assuming $V = L$ (or working in L) we obtain $L(\kappa) = H(\kappa) = R(\kappa) \models \mathrm{ZFC} + V = L$. However $\models$ is recursively defined and so absolute, which implies that in V, $L(\kappa) \models \mathrm{ZFC} + V = L$ as desired. □

COROLLARY 6.30. (AC) If there is a weakly inaccessible cardinal, then there is a countable transitive M such that $M \models \mathrm{ZFC} + V = L$.

PROOF. Let κ be weakly inaccessible. Then, by the above theorem $L(\kappa) \models \mathrm{ZFC} + V = L$. Take a countable elementary submodel M' of $L(\kappa)$ and let $M = \mathrm{mos}''_{(M',\in)} M'$. □

LEMMA 6.31. If M is a transitive model for ZF, then $L(o(M)) \models \mathrm{ZFC} + V = L$.

PROOF. Working in ZF, we can prove $L \models \mathrm{ZFC} + V = L$. Since $M \models \mathrm{ZF}$, we obtain $M \models (L^M \models \mathrm{ZFC} + V = L)$. However

$$\begin{aligned} L^M &= \{x \in M : \exists\delta \in \mathbb{ON} \cap M(x \in L(\delta))^M\} \\ &= \{x \in M : \exists\delta \in \mathbb{ON} \cap M(x \in L(\delta))\} \\ &= L(o(M)). \end{aligned}$$

By absoluteness of $\models$ we obtain $L(o(M)) \models \mathrm{ZFC} + V = L$. □

LEMMA 6.32. Let Λ be a finite set of axioms of ZFC. Then

$$\mathrm{ZFC} \models \exists M(M \models \lambda + V = L \wedge |M| = \aleph_0 \wedge M \text{ is transitive}).$$

PROOF. Apply Reflection to $L = \bigcup_{\xi\in\mathbb{ON}} L(\xi)$ to get a limit ordinal η such that $L(\eta) \models \Lambda + V = L$. Take a countable elementary submodel of $L(\eta)$ and then its transitive closure. □

7. Appendix

7.1. More on Relative Consistency Proofs.

DEFINITION 7.1. A theory Λ is said to be *strictly stronger proof-theoretically than* Γ, denoted $\Gamma \lhd \Lambda$ iff $\Lambda \vDash \mathrm{Con}(\Gamma)$.

EXAMPLE 7.2. To show that $\Gamma \lhd \Lambda$ we will work in Λ to produce a model for Γ. For example, working in ZFC we can show that HC is a model for ZFC-P. Note that by Gödel's Second Incompleteness Theorem, $\lhd$ is not reflexive.

DEFINITION 7.3.

(1) A theory Λ is said to be *stronger proof-theoretically than* Γ, denoted $\Gamma \leq \Lambda$ iff there is a finitistic proof of $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$ (such proofs are referred to as relative consistency proofs).
(2) Theories Γ and Λ are said to be *proof- theoretically equivalent*, denoted $\Gamma \sim \Lambda$ iff $\Gamma \leq \Lambda$ and $\Lambda \leq \Gamma$.

REMARK 7.4. Note that $\leq$ is reflexive and transitive, and $\sim$ is an equivalence relation.

LEMMA 7.5.

(1) If $\Gamma \lhd \Lambda$ then $\Gamma \leq \Lambda$
(2) $\Gamma \leq \Lambda$ and $\Lambda \lhd \Theta$ imply $\Gamma \lhd \Theta$
(3) The relation $\lhd$ is transitive.
(4) If $\Gamma \leq \Lambda$ and $\Lambda \lhd \Gamma$ then $\neg\,\mathrm{Con}(\Gamma)$ and $\neg\,\mathrm{Con}(\Lambda)$.

PROOF.
(1) Suppose $\Lambda \vdash \mathrm{Con}(\Gamma)$ and suppose that $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$ is not true. Thus we have $\mathrm{Con}(\Lambda)$ and $\neg\,\mathrm{Con}(\Gamma)$, i.e. we have a finitistic proof of $\neg\,\mathrm{Con}(\Gamma)$. But then, $\Lambda \vdash \neg\,\mathrm{Con}(\Gamma)$, which will produce a contradiction in Λ.
(2) By hypothesis $\Theta \vdash \mathrm{Con}(\Lambda)$. Since there is a finitistic proof of $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$, we get that $\theta \vdash \mathrm{Con}(\Gamma)$.
(3) Suppose $\Gamma \lhd \theta$ and $\theta \lhd \Lambda$. Then by (1), $\Gamma \leq \theta$. Now by (2) we get $\Gamma \lhd \Lambda$.
(4) By part (2), we have $\Gamma \lhd \Gamma$ and hence $\neg\,\mathrm{Con}(\Gamma)$. However $\Gamma \leq \Lambda$, i.e. $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$. Therefore $\neg\,\mathrm{Con}(\Lambda)$. ☐

REMARK 7.6.

(1) $\mathrm{ZFC}^- \leq \mathrm{ZFC}$. By the theorem of von Neumann, $\mathrm{ZFC} \leq \mathrm{ZFC}^-$. Therefore $\mathrm{ZFC}^- \sim \mathrm{ZFC}$. By the same theorem $\mathrm{ZF}^- \sim \mathrm{ZF}$. Obtaining the Constructible Universe later, we will also have $\mathrm{ZFC} + \mathrm{GCH} \leq \mathrm{ZF}$ and so we have $\mathrm{ZF}^- \sim \mathrm{ZF} \sim \mathrm{ZFC}^- \sim \mathrm{ZFC} \sim \mathrm{ZFC} + \mathrm{GCH}$.
(2) Using the method of forcing, we will see that ZFC is proof-theoretically equivalent to ZFC plus various additional axioms about Lebesuge measure, category, and others.

CHAPTER 3

Infinitary Combinatorics

1. Martin's axiom

1.1. Maximal Almost Disjoint Families.

DEFINITION 1.1. Let κ be an infinite cardinal.

(1) Two subsets x, y of κ are said to be almost disjoint if $|x \cap y| < \kappa$.
(2) $\mathcal{A} \subseteq [\kappa]^\kappa$ is κ-almost disjoint if any two distinct elements of $\mathcal{A}$ are κ-almost disjoint.
(3) A family $\mathcal{A}$ is maximal κ-almost disjoint if $\mathcal{A}$ is κ-almost disjoint and maximal under inclusion. We say that $\mathcal{A}$ is κ-m.a.d.
(4) $\mathfrak{a}(\kappa) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is } \kappa\text{-m.a.d.}, |\mathcal{A}| \geq \kappa\}$.

REMARK 1.2. In the special case $\kappa = \omega$, we simply say that $\mathcal{A}$ is almost disjoint and speak about maximal almost disjoint families. The cardinal $\mathfrak{a} = \mathfrak{a}(\omega)$ is known as the almost disjointness number.

REMARK 1.3.

(1) Let $\mathcal{A} \subseteq [\omega]^\omega$ be almost disjoint. Suppose $\mathcal{A}$ is maximal. Then, there is no almost disjoint family $\mathcal{B}$ such that $\mathcal{A}$ is properly contained in $\mathcal{B}$. With other words, if $X \in [\omega]^\omega \backslash \mathcal{A}$, then $\mathcal{A} \cup \{X\}$ is not almost disjoint. That is, there is $A \in \mathcal{A}$ such that $|X \cap A| = \omega$.
(2) Suppose $\mathcal{A} \subseteq [\omega]^\omega$ is a finite partition of ω. That is, the elements of $\mathcal{A}$ have pairwise empty intersection and $\bigcup \mathcal{A} = \omega$. Then, $\mathcal{A}$ is an almost disjoint family. Is $\mathcal{A}$ maximal?

THEOREM 1.4. *Let $\kappa \geq \omega$ be a regular cardinal.*

(1) If $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is almost disjoint and $|\mathcal{A}| = \kappa$, then $\mathcal{A}$ is not maximal.
(2) There is a κ-m.a.d. family $\mathcal{B} \subseteq [\kappa]^\kappa$ of cardinality $\geq \kappa^+$.

PROOF. (1) Let $\mathcal{A} = \{A_\xi : \xi < \kappa\}$ be an almost disjoint family. For each $\xi < \kappa$, define $B_\xi = A_\xi \backslash \bigcup_{\eta<\xi}(A_\xi \cap A_\eta)$. Since $|A_\xi| = \kappa$ and for each $\eta < \kappa$, $|A_\xi \cap A_\eta| < \kappa$ we have that $B_\xi \neq \varnothing$. Now, for each ξ, pick $b_\xi \in B_\xi$ and let $A_\kappa = \{b_\xi : \xi < \kappa\}$. Note that $A_\kappa \cap A_\eta \subseteq \{b_\xi : \xi \leq \eta\}$. Thus A_κ is a set, which is κ-almost disjoint from every element of $\mathcal{A}$ and so $\mathcal{A}$ is not κ-maximal.

(2) Take any partition $\mathcal{A}$ of κ into κ-many unbounded (in κ) subsets. Then $\mathcal{A}$ is κ-almost disjoint. By item (1) $\mathcal{A}$ is not maximal. However, by Zorn's Lemma (and so the Axiom of Choice), there is a maximal κ-almost disjoint family $\mathcal{B}$ extending κ. Then $\mathcal{B}$ is κ-m.a.d. of cardinality $\geq \kappa$. □

EXERCISE 3. Write an explicit proof of the existence of $\mathcal{B}$ in item (2) of the above theorem, using Zorn's Lemma.

THEOREM 1.5. *If $\kappa \geq \omega$ and $2^{<\kappa} = \kappa$, then there is an almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ of cardinality 2^κ.*

PROOF. Let $I = \{x \subseteq \kappa : \sup(x) < \kappa\}$. Since $2^{<\kappa} = \kappa$, $|I| = \kappa$. Now, for $x \subseteq \kappa$ define

$$A_x = \{x \cap \alpha : \alpha < \kappa\}$$

and so if $|X| = \kappa$, then $|A_x| = \kappa$.

CLAIM. If $x, y \subseteq \kappa$ are distinct, then $|A_x \cap A_x| < \kappa$.

PROOF. Let $x, y \subseteq \kappa$, $x \neq y$. Fix $\beta \in x \backslash y$ (without loss of generality). Then $A_x \cap A_y \subseteq \{x \cap \alpha : \alpha \leq \beta\}$. Indeed, if $\gamma \in \kappa \backslash (\beta + 1)$, then $\beta \in x \cap \gamma$ and for each $\gamma' \in \kappa$ we have that $\beta \notin y \cap \gamma'$. Thus $|A_x \cap A_y| \leq |\beta| < \kappa$. $\square$

Then $\mathcal{A} = \{A_x : x \in [\kappa]^\kappa\}$ is a κ-a.d. family of cardinality 2^κ. Since $|I| = \kappa$ there is a bijection $f : I \to \kappa$. Then for each $x \in [\kappa]^\kappa$, let $A'_x = \{f(x \cap \alpha) : \alpha < \kappa\}$. Thus $A'_x \in [\kappa]^\kappa$ and $\mathcal{A}' = \{A'_x : x \in [\kappa]^\kappa\}$ is an a.d. family of subsets of κ of cardinality 2^κ. $\square$

REMARK 1.6. By the above theorem, there is a maximal almost disjoint family of cardinality $2^\omega (= |\mathbb{R}|)$.

1.2. Δ-system lemma.

DEFINITION 1.7. A family $\mathcal{A}$ of sets is a Δ-system, if there is a set r such that the intersection of any two pairwise distinct elements a, b of $\mathcal{A}$ is the set r. The set r is called the root of the Δ-system.

THEOREM 1.8. *If $\mathcal{A}$ is an uncountable family of finite sets, then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ forms a Δ-system.*

EXERCISE 4. Prove the above theorem.

We will prove the following more general statement.

THEOREM 1.9. *Let $\kappa \geq \omega$ be a cardinal, $\theta > \kappa$ regular such that for all $\alpha < \theta(|\alpha^{<\kappa}| < \theta)$. If $\mathcal{A}$ is a set such that $|\mathcal{A}| \geq \theta$ and for all $x \in \mathcal{A}$ we have that $|x| < \kappa$, then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \theta$ and $\mathcal{B}$ forms a Δ-system.*

PROOF. Without loss of generality $|\mathcal{A}| = \theta$. By hypothesis, $\forall x \in \mathcal{A}(|x| < \kappa < \theta)$ and so $|\bigcup \mathcal{A}| = \theta$. Now, for all $x \in \mathcal{A}$, let $\alpha_x = \text{type}(x)$. Note that $\alpha_x < \kappa$. Thus, $\mathcal{A} = \bigcup_{\alpha < \kappa} \mathcal{A}^\alpha$ where

$$\mathcal{A}^\alpha = \{x \in \mathcal{A} : \alpha_x = \alpha\}.$$

Since $\kappa < \theta$ and θ is regular, there is $\alpha_0 < \kappa$ such that $|\mathcal{A}^{\alpha_0}| = \theta$. So, let $\mathcal{A}_0 = \mathcal{A}^{\alpha_0}$. It is sufficient to find the desired family $\mathcal{B}$ as a subset of $\mathcal{A}_0$.

CLAIM 1.10. $\bigcup \mathcal{A}_0$ is unbounded in θ. That is $\forall \alpha \in \theta \exists \beta \in \bigcup \mathcal{A}_0$ such that $\alpha \leq \beta$.

PROOF. Fix $\alpha < \theta$. Since by hypothesis of the theorem $|\alpha^{<\kappa}| < \theta$, there are less than θ-many elements of $\mathcal{A}_0$ contained in α. Thus there is $x \in \mathcal{A}_0$ such that $x \nsubseteq \alpha$, i.e. there is $\beta \in x$ such that $\beta \geq \alpha$. Then, $\beta \in \bigcup \mathcal{A}_0$. $\square$

For each $x \in \mathcal{A}_0$, $\text{type}(x) = \alpha_0$. Now, for each $\xi < \alpha_0$, denote by $x(\xi)$, the ξ-th element of x.

CLAIM 1.11. There is $\xi < \alpha_0$ such that $C_\xi = \{x(\xi) : x \in \mathcal{A}_0\}$ is unbounded in θ.

PROOF. Otherwise, for all $\xi < \alpha_0$, there is $\beta_\xi < \theta$ such that $C_\xi \subseteq \beta_\xi$. But, then $\bigcup \mathcal{A}_0 \subseteq \sup_{\xi<\alpha_0} \beta_\xi$ and so $|\bigcup \mathcal{A}_0| \leq \sup_{\xi<\alpha_0} \beta_\xi < \theta$ (by regularity of θ and $\alpha_0 < \kappa < \theta$). □

Let $\xi_0 = \min\{\xi : C_\xi$ is unbounded in $\theta\}$. By minimality of ξ_0, we get

$$\alpha_1 = \sup\{x(\eta) + 1 : \eta < \xi_0, x \in \mathcal{A}_0\} < \theta.$$

Thus, in particular $x(\eta) < \alpha_1$ for all $x \in \mathcal{A}_0$.

CLAIM 1.12. There is a family $\mathcal{A}_1 \subseteq \mathcal{A}_0$ such that $|\mathcal{A}_1| = \theta$ and for all $x, y \in \mathcal{A}_1$ the intersection $x \cap y \subseteq \alpha_1$.

PROOF. By transfinite induction, we can construct a sequence

$$\tau = \langle x_\mu : \mu < \theta \rangle$$

of elements in $\mathcal{A}_0$ such that for all μ, $x_\mu(\xi_0) > \max\{\mu, \bigcup_{\nu<\mu} x_\nu\}$. Take $\mathcal{A}_1 = \{x_\mu : \mu < \theta\}$. □

For each $y \in [\alpha_1]^{<\kappa}$, let $\mathcal{A}_{1,y} = \{x \in \mathcal{A}_1 : x \cap \alpha_1 = y\}$. Then $\mathcal{A}_1 = \bigcup\{\mathcal{A}_{1,y} : y \in [\alpha_1]^{<\kappa}\}$. However, by hypothesis $|\alpha_1^{<\kappa}| < \theta$ and so $\exists y \in [\alpha_1]^{<\kappa}$ such that $|\mathcal{A}_{1,y}| = \theta$.

CLAIM 1.13. For all distinct $a, b \in \mathcal{A}_{1,y}$ we have $|a \cap b| = y$.

PROOF. Fix a, b distinct in $\mathcal{A}_{1,y}$. Then $a \cap b \subseteq \alpha_1$ since $a, b \in \mathcal{A}_1$. Moreover

$$a \cap b = a \cap b \cap \alpha_1 = a \cap \alpha_1 \cap b \cap \alpha_1 = y \cap y = y.$$

□

Clearly $\mathcal{B} = \mathcal{A}_{1,y}$ is a Δ-system with root the set y. □

1.3. Martin's axiom.

DISCUSSION 1.14. Suppose CH fails. Then we can ask:

(1) If $\omega \leq \kappa < 2^\omega$, does $2^\kappa = 2^\omega$?
(2) If $\omega \leq \kappa < 2^\omega$, does every a.d. family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality κ fail to be maximal?
(3) Is it true that every set $A \subseteq \mathbb{R}$ such that $|A| < 2^\omega = |\mathbb{R}|$ is of Lebesgue measure zero?
(4) Is it true that every set $A \subseteq \mathbb{R}$ such that $|A| < 2^\omega$ is meager?
(5) Let S_∞ denote the group of all permutations of $\mathbb{N}$. A subgroup $\mathcal{G}$ of S_∞ is said to be cofinitary if for every $f \in \mathcal{G}\backslash\{\text{id}\}$, the set $\text{fix}(g) = \{n \in \omega : g(n) = n\}$ is finite. A cofinitary group if said to be maximal, abbreviated mcg, if it is not properly contained in another cofinitary group. Is it true that every cofinitary group $G \leq S_\infty$ of cardinality strictly smaller than $\mathfrak{c}$ is not maximal?

Under the assumption of CH the answer to each of the above questions is "yes". However, if CH does not hold, each of those answers is independent of ZFC.

DEFINITION 1.15.

(1) A **partial order** $\langle \mathbb{P}, \leq \rangle$ is a pair such that $\mathbb{P} \neq \varnothing$ and $\leq$ is a relation on $\mathbb{P}$ which is transitive and reflexive.
(2) $\langle \mathbb{P}, \leq \rangle$ is a partial order in the **strict sense** iff in addition for all p, q if $p \leq q$ and $q \leq p$ then $p = q$.
(3) If $p \leq q$ we say that p **extends** q, or p **is stronger than** q, or q **is weaker than** p. We denote $p < q$ the fact that $p \leq q$ and $p \neq q$.

DEFINITION 1.16. Let $\langle \mathbb{P}, \leq \rangle$ be a p.o.

(1) A **chain** in $\mathbb{P}$ is a set $C \subseteq \mathbb{P}$ such that for all $p, q \in C (p \leq q \vee q \leq p)$.
(2) $p \not\perp q$ iff there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. We say that p and q are **compatible**, also that they have a common extension.
(3) $p \perp q$ iff p and q do not have a common extensions, i.e. there is no r such that $r \leq p$ and $r \leq q$. We say that p, q are **incompatible**.
(4) An **antichain** in $\mathbb{P}$ is a subset A of $\mathbb{P}$ such that for all $p, q \in A$ if $p \neq q$ then $p \perp q$.

DEFINITION 1.17. A partial order $\langle \mathbb{P}, \leq \rangle$ has the **countable chain condition** iff every non-empty antichain in $\mathbb{P}$ is countable.

EXAMPLE 1.18.

(1) Let $\mathbb{P} = \omega_1$ with $\alpha < \beta$ iff $\alpha \in \beta$. Every antichain in $\mathbb{P}$ has cardinality 1.
(2) Let $X \neq \varnothing$. Consider the power set $\mathcal{P}(X)$ of X with extension relation $p \leq q$ iff $p \subseteq q$. Thus $p \perp q$ iff $p \cap q = \varnothing$. Thus $\mathcal{A} \subseteq \mathcal{P}(X)$ is an antichain iff for any two distinct a, b in $\mathcal{A}$ the intersection $a \cap b$ is empty. Then $(\mathcal{P}(X), \subseteq)$ has the c.c.c. iff $|X| \leq \omega$.

DEFINITION 1.19. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order.

(1) A set $D \subseteq \mathbb{P}$ is **dense** iff for all $p \in \mathbb{P}$ there is $q \leq p$ such that $q \in D$.
(2) A non-empty subset G of $\mathbb{P}$ is a **filter** iff
   - for all p, q in G there is $r \in G$ such that $r \leq p$ and $r \leq q$;
   - for all $p \in G$ and all $q \in \mathbb{P}$, if $p \leq q$ then $q \in G$.

DEFINITION 1.20.

(1) $\mathrm{MA}(\kappa)$ is the statement: Whenever $\langle \mathbb{P}, \leq \rangle$ is a non-empty ccc partial order and $\mathcal{D}$ is a family of $\leq \kappa$ many dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that for all $D \in \mathcal{D}(G \cap D \neq \varnothing)$.
(2) MA is the statement: $\forall \kappa < 2^{\omega}(\mathrm{MA}(\kappa))$.

REMARK 1.21. Martin's axiom is consistent with $\mathbb{R}$ being arbitrarily large. Moreover MA implies that the answer to each of Questions 1-5 from Discussion 1.14 is yes.

1.4. Cohen Forcing.

DEFINITION 1.22. Let $\mathbb{P}$ be the partial order consisting of all subsets p of $\omega \times 2$, where $|p| < \omega$ and p is a function. Define $p \leq q$ iff $q \subseteq p$.

DISCUSSION 1.23.

(1) Observe that $p \not\perp q$ iff $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$.
(2) If $p \not\perp q$ then $p \cup q \leq p, q$.
(3) Since $|\mathbb{P}| = \aleph_0$, the partial order has the countable chain condition.
(4) If G is a filter in $\mathbb{P}$, then since for any two elements p, q of G the functions p, q coincide on their common domain and so

$$\bigcup G = \bigcup\{p : p \in G\}$$

is a function, which we denote f_G.
(5) Note that it is possible that $\mathrm{dom}(f_G)$ is finite, or empty. However if G meets significantly many dense sets, then f is indeed a function.
(6) For each n, define $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$. Note that D_n is dense. Take an arbitrary $q \in \mathbb{P}$. If $n \notin \mathrm{dom}(q)$ then $q' = q \cup \{(n, k)\} \leq q$ for any k. Therefore if $G \cap D_n \neq \varnothing$, then $\mathrm{dom}(f_G) = \omega$.
(7) For each $h \in {}^{\omega}2$ let $E_h = \{p \in \mathbb{P} : p \neq h \restriction \mathrm{dom}(p)\}$. Note that E_h is dense. Indeed, take any $p \in \mathbb{P}$ and suppose $p = h \restriction \mathrm{dom}(p)$. Let $n \in \omega \backslash \mathrm{dom}(p)$ and $k \neq h(n)$. Then $p' = p \cup \{(n, k)\} \leq p$ and $p' \in E_h$.
(8) If G is a filter and $G \cap E_h \neq \varnothing$ for each $h \in {}^{\omega}2$, then $f_G \neq h$ for each $h \in {}^{\omega}2$. Indeed, pick such an h. Then there is $p \in G \cap E_h$, so there is $n \in \mathrm{dom}(p)$ such that $p(n) \neq h(n)$. However, since $p \in G$, $f_G(n) = p(n)$. Thus, $f_G(n) \neq h(n)$. However f_G is a function and we just claimed that $h \notin {}^{\omega}2$, which is a contradiction. The problem is that there is no filter G such that $G \cap E_h \neq \varnothing$ for all $f \in {}^{\omega}2$.

LEMMA 1.24.
(1) If $\kappa' < \kappa$ then $\mathrm{MA}(\kappa')$ implies $\mathrm{MA}(\kappa)$.
(2) $\mathrm{MA}(2^{\omega})$ is false.
(3) $\mathrm{MA}(\omega)$ is true.

PROOF. Part (1) is clear by definition. Part (2) was just shown. To see item (3) consider any ccc partial order $\mathbb{P}$ and let $\{D_n\}_{n\in\omega}$ be a dense subset of $\mathbb{P}$. Recursively, define a sequence $\{p_n\}_{n\in\omega} \subseteq \mathbb{P}$ such that $p_0 \in D_0$, $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Then $G = \{q \in \mathbb{P} : \exists n \in \mathbb{N}(p_n \leq q)\}$ is a filter meeting all D_n's. $\square$

REMARK 1.25. The Continuum Hypothesis implies Martin's axiom. Note also, that Martin's Axiom is consistent with arbitrarily large continuum.

EXAMPLE 1.26. Consider the partial order $\mathbb{P}$ consisting of all finite functions p such that

$$p \subseteq \omega \times \omega_1$$

(again, we identify p with its graph). Let $G \subseteq \mathbb{P}$ be a filter meeting every dense set $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$ for each $n \in \omega$. Then $f_G = \bigcup G : \omega \to \omega_1$. Now, for each $\alpha \in \omega_1$ consider the set $D^{\alpha} = \{p \in \mathbb{P} : \alpha \in \mathrm{ran}(p)\}$ and note that D^{α} is dense. If G is a filter and $G \cap D^{\alpha} \neq \varnothing$ for all α and $G \cap D_n \neq \varnothing$ for all $n \in \omega$, then f_G is a function from ω onto ω_1, which is clearly not possible. Note that $\{(0, \alpha) : \alpha < \omega_1\}$ is an antichain of size ω_1 and so the partial order is not c.c.c.

1.5. MA and the continuum. The following forcing notion is well-known and has broad applications in the study of the set theoretic properties of the real line.

DEFINITION 1.27. Mathias forcing with respect to a filter $\mathcal{F} \subseteq [\omega]^\omega$ is denoted $\mathbb{M}(\mathcal{F})$ and consists of all pairs (s, A) where $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, $\max s < \min A$ and has extension relation defined as follows: $(s_1, A_1) \leq (s_0, A_0)$ if $s_0 \subseteq s_1$, $s_1 \backslash s_0 \subseteq A_0$ and $A_1 \subseteq A_0$.

The partial order $\mathbb{M}(\mathcal{F})$ has the countable chain condition. In fact is satisfies the following property:

DEFINITION 1.28. A partial order $\mathbb{P}$ is σ-centered if for each $n \in \omega$, there is $\mathbb{P}_n \subseteq \mathbb{P}$ such that

$$\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$$

and for all $p, q \in \mathbb{P}_n \exists r \in \mathbb{P}_n (r \leq p, q)$.

Indeed, $\mathbb{M}(\mathcal{F}) = \bigcup_{s\in[\omega]^{<\omega}} \mathbb{P}_s$, where $\mathbb{P}_s = \{(s_0, A_0) \in \mathbb{M}(\mathcal{F}) : s_0 = s\}$. Note that:

CLAIM 1.29. If $\mathbb{P}$ is σ-centered, then $\mathbb{P}$ is ccc.

PROOF. Let $\mathbb{P}$ be σ-centered and $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$, where for each $n \in \omega$, the partial order $\mathbb{P}_n$ is centered. Let $\mathcal{A} \subseteq \mathbb{P}$, $|\mathcal{A}| = \omega_1$. Then, there is $n \in \omega$ such that $|\mathcal{A} \cap \mathbb{P}_n| > \aleph_0$, as otherwise

$$|\mathbb{P} \cap \mathcal{A}| = |\bigcup_{n\in\omega} \mathbb{P}_n \cap \mathcal{A}| \leq \bigcup_{n\in\omega} |\mathbb{P}_n \cap \mathcal{A}| \leq \aleph_0,$$

which is a contradiction. But then $|\mathcal{A} \cap \mathbb{P}_n| \geq 2$ and so there are $p, q \in \mathcal{A} \cap \mathbb{P}_n$. By hypothesis $p \not\perp q$ and so $\mathcal{A}$ is not an antichain. ☐

Thus, $\mathbb{M}(\mathcal{F})$ is ccc. In fact, $\mathbb{M}(\mathcal{F})$ is Knaster, which by definition, means that from every family of $\aleph_1$ conditions of the partial order, one can find a subfamily of cardinality $\aleph_1$ in which any two distinct elements are pairwise compatible.

LEMMA 1.30. The following sets are dense in $\mathbb{M}(\mathcal{F})$:

(1) For each $n \in \omega$, $D_n = \{(s, A) : \exists m > n (m \in s)\}$.
(2) For each $X \in \mathcal{F}$, the set $D_X = \{(s, A) : A \subseteq X\}$.

PROOF. To see item (1), fix $n \in \omega$ and let $(s, A) \in \mathbb{M}(\mathcal{F})$ be an arbitrary condition. Since A is infinite, we can find $m \in A$ such that $m > n$ and $m > \max s$. Then $(s \cup \{m\}, A \backslash (m + 1))$ is an extension of (s, A) from D_n. To see item (2) fix $X \in \mathcal{F}$ and consider an arbitrary $(s, A) \in \mathbb{M}(\mathcal{A})$. Since $\mathcal{F}$ is a filter, $Y = X \cap A \in \mathcal{F}$. Then $(s, Y) \in D_X$ and $(s, Y) \leq (s, A)$ as desired. ☐

LEMMA 1.31. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a filer on ω, let G be a filter of the partial order $\mathbb{M}(\mathcal{F})$ and let $\sigma_G = \bigcup\{s : \exists A (s, A) \in G\}$.

(1) If $G \cap D_n \neq \emptyset$ for each $n \in \omega$, where D_n is as in Lemma 1.30, then $|\sigma_G| = \omega$.
(2) If $G \cap D_X \neq \emptyset$ form some $X \in \mathcal{F}$, where D_X is defined as in Lemma 1.30, then $\sigma_G \subseteq^* X$, i.e. $\sigma_G \backslash X$ is finite.

Proof. To see item (1) note that if $(s, A) \in G$ then $s \subseteq \sigma_G$. Therefore, if $(s, A) \in G \cap D_n$ then since there is $m > n$ such that $m \in s$, we obtain that there is $m > n$ with $m \in \sigma_G$. To see item (2) note that if $(s, A) \in G$, then $\sigma_G \subseteq A$. Then, if $(s, A) \in G \cap D_X$, $\sigma_G \subseteq^* X$. □

Now, we are ready to obtain the following theorem:

Theorem 1.32. *Martin's Axiom implies that the almost disjointness number* $\mathfrak{a}$ *is equal to* 2^ω.

Let $\mathcal{A}$ be an infinite almost disjoint family and let $|\mathcal{A}| < \mathfrak{c}$. Note that the set $\mathcal{I}(\mathcal{A})$ which is defined as the downwards closure (i.e. closures with respect to subsets) of $\{\bigcup \mathcal{A}_0 : \mathcal{A}_0 \in [\mathcal{A}]^{<\omega}\}$ is an ideal. Moreover, the set of complements of elements of $\mathcal{I}(\mathcal{A})$ is a filter, referred to as the dual filter and will be denoted here as $\mathcal{F}(\mathcal{A})$. Note that for every $A \in \mathcal{A}$, $\omega \backslash A \in \mathcal{F}(\mathcal{A})$. Now, suppose G is filer for the partial order $\mathbb{M}(\mathcal{F}(\mathcal{A}))$ such that G has a non-empty intersection with every element of the families $\{D_n\}_{n\in\omega}$ and $\{D_X : \omega \backslash X \in \mathcal{A}\}$, where D_n and D_X are defined as in Lemma 1.30. Then, by the above considerations, σ_G is an infinite subset of ω and $\sigma_G \subseteq^* \omega \backslash A$ for every $A \in \mathcal{A}$. Then $\sigma_G \in [\omega]^\omega$ and $|\sigma_G \cap A| < \omega$ for all $A \in \mathcal{A}$. That is $\mathcal{A} \cup \{\sigma_G\}$ is an almost disjoint family and so $\mathcal{A}$ is not maximal.

Definition 1.33. (Almost Disjoint Forcing) Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$. The almost disjoint set partial order $\mathbb{P}_\mathcal{A}$ consisits of all pairs $(s, F) \in [\omega]^{<\omega} \times [\mathcal{A}]^{<\omega}$ with extension relation defined as follows:

$$(s', F') \leq (s, F) \text{ iff } s \subseteq s', F \subseteq F', \forall x \in F(x \cap s' \subseteq s).$$

Remark 1.34. The conditions of the above partial order are intended to describe a set, which is almost disjoint from the elements of $\mathcal{A}$.

Lemma 1.35. Let (s_1, F_1) and (s_2, F_2) be conditions in $\mathbb{P}_\mathcal{A}$. Then the following are equivalent:

(1) (s_1, F_1) and (s_2, F_2) are compatible;
(2) for all $x \in F_1(x \cap s_2 \subseteq s_1)$ and for all $x \in F_2(x \cap s_1 \subseteq s_2)$;
(3) for all $x \in F_1$ and all $n \in x \backslash s_1$, we have that $n \notin s_2$ and for all $x \in F_2$ and all $n \in x \backslash s_2$ we have that $n \notin s_1$.

Definition 1.36. Let G be a $\mathbb{P}_\mathcal{A}$-filter and let $d_G = \bigcup\{s : \exists F(s, F) \in G\}$.

Lemma 1.37. If $G \subseteq \mathbb{P}_\mathcal{A}$ is a filter and $(s, F) \in G$ then for all $x \in F(d_G \cap x \subseteq s)$.

Proof. Let $x \in F$. To show that $d_G \cap x \subseteq s$, it suffices to show that $d_G \backslash s \cap x = \varnothing$. So, let $n \in d_G \backslash s$. Then (by definition of d_G) there is $(s', F') \in G$ such that $n \in s'$. Without loss of generality we can assume that $(s', F') \leq (s, F)$. Then $n \in s' \backslash s$. By definition of the extension relation $\leq$, $s' \backslash s \cap x = \varnothing$ and so $n \notin x$. That is $d_G \backslash s \cap x = \varnothing$. □

Corollary 1.38. Let $x \in \mathcal{A}$. Then $D_x = \{(s, F) \in \mathbb{P}_\mathcal{A} : x \in F\}$ is dense. If $G \cap D_x \neq \varnothing$, then by the previous Lemma $|d_G \cap x| < \omega$.

Proof. We only need to show that D_x is dense. So, let $p \in \mathbb{P}_\mathcal{A}$. Then $p = (s, F) \in [\omega]^{<\omega} \times [\mathcal{A}]^{<\omega}$. If $x \in F$ then $(s, F) \in D_x$. If $x \notin F$, then observe that $(s, F \cup \{x\}) \leq (s, F)$ and clearly $(s, F \cup \{x\}) \in D_x$. □

LEMMA 1.39. $\mathbb{P}_{\mathcal{A}}$ is ccc.

PROOF. In fact $\mathbb{P}_{\mathcal{A}}$ is σ-centered. Indeed, for $a \in [\omega]^{<\omega}$ let

$$\mathbb{P}_a = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : s = a\}.$$

Then $\mathbb{P}_a$ is centered and $\mathbb{P}_{\mathcal{A}} = \bigcup\{\mathbb{P}_a : a \in [\omega]^{<\omega}\}$. □

LEMMA 1.40 (Solovay's Lemma). Assume $\mathsf{MA}(\kappa)$. Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}(\omega)$ where $|\mathcal{A}| \leq \kappa$, $|\mathcal{C}| \leq \kappa$. Suppose for all $y \in \mathcal{C}$ and for all $\mathcal{F} \in [\mathcal{A}]^{<\omega}$ we have that $|y \setminus \bigcup \mathcal{F}| = \omega$. Then there is $d \in [\omega]^{\omega}$ such that

$$\forall x \in \mathcal{A}(|x \cap d| < \omega) \text{ and } \forall x \in \mathcal{C}(|x \cap d| = \omega).$$

PROOF. For $y \in \mathcal{C}$, $n \in \omega$, let $E^y_n = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : s \cap y \notin n\}$.

CLAIM. E^y_n is dense in $\mathbb{P}_{\mathcal{A}}$.

PROOF. Let $(s, F) \in \mathbb{P}_{\mathcal{A}}$. By hypothesis $|y \backslash \bigcup F| = \omega$ and so there is $m \in y \backslash \bigcup F$ such that $m > n$. Then $(s \cup \{m\}, F)$ is an extension of (s, F) from E^y_n. □

Consider the collection of dense sets $\{D_x\}_{x \in \mathcal{A}} \cup \{E^y_n\}_{y \in \mathcal{C}, n \in \omega}$. Since this is a collection of at most κ-many dense sets, by MA(κ) there is a filter G meeting all of them. But then

$$d = d_G = \bigcup\{s : \exists F (s, F) \in G\}$$

is such that $\forall x \in \mathcal{A}(d_G \cap x \text{ is finite})$ and $\forall y \in \mathcal{C}(y \cap d_G \text{ is infinite})$. □

COROLLARY 1.41. Let $\mathcal{A} \subseteq [\omega]^{\omega}$ be an a.d. family such that $|\mathcal{A}| = \kappa$, where $\omega \leq \kappa < 2^{\omega}$. Assume MA$(\kappa)$. Then $\mathcal{A}$ is not maximal.

PROOF. Since $\mathcal{A}$ is infinite, for each finite $\mathcal{F} \subseteq \mathcal{A}$, the set $\omega \backslash \bigcup \mathcal{F}$ is infinite. Indeed, suppose there is a finite subset $\mathcal{F}$ of $\mathcal{A}$ such that $\omega \backslash \bigcup \mathcal{F}$ is finite. Take any $A \in \mathcal{A} \backslash \mathcal{F}$. Then, there is $A_0 \in \mathcal{F}$ such that $A \cap A_0$ is infinite, since otherwise $|A| < \omega$. However, this is a contradiction to $\mathcal{A}$ being an a.d. family. Therefore, we can apply Solovay's Lemma to $\mathcal{A}$ and $\mathcal{C} = \{\omega\}$. Thus, there is a set d such that $|d| = \omega$ and $|d \cap x| < \omega$ for each $x \in \mathcal{A}$. Thus, $\mathcal{A}$ is not maximal. □

THEOREM 1.42. *Let $\omega \leq \kappa < 2^{\omega}$ and assume MA(κ). Then $2^{\kappa} = 2^{\omega}$.*

PROOF. Fix $\kappa < 2^{\omega}$. Since there is an a.d. family of cardinality 2^{ω}, there is also an a.d. family of cardinality κ. Fix such a family $\mathcal{B}$. Define $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\mathcal{B})$ as follows:

$$\Phi(d) = \{x \in \mathcal{B} : |d \cap x| < \omega\}.$$

We will show that Φ is an onto mapping.

Note that by Corollary 1.41, the family $\mathcal{B}$ is not maximal. Then, there is $d \in \mathcal{P}(\omega)$ such that for all $b \in \mathcal{B}(|d \cap b| < \omega)$ and so $\Phi(d) = \mathcal{B}$. Now, consider any $\mathcal{B}_0$ which is a proper subset of $\mathcal{B}$ and let $\mathcal{C} = \mathcal{B} \backslash \mathcal{B}_0$. We can apply Solovay's Lemma to $\mathcal{B}_0$ and $\mathcal{C}$. Then, there is $d \in \mathcal{P}(\omega)$ such that for all $x \in \mathcal{B}_0(|x \cap d| < \omega)$, while for all $d \in \mathcal{B} \backslash \mathcal{B}_0(|x \cap d| = \omega)$. That is $\Phi(d) = \mathcal{B}_0$.

Therefore Φ is indeed onto and so $|\mathcal{P}(\mathcal{B})| = 2^{\kappa} \leq |\mathcal{P}(\omega)| = 2^{\omega}$. However, by monotonicity of exponentiation, we have $2^{\omega} \leq 2^{\kappa}$ and so $2^{\kappa} = 2^{\omega}$. □

COROLLARY 1.43. MA implies that 2^ω is regular.

PROOF. Let $\omega \le \kappa < 2^\omega$. By König's Lemma $\mathrm{cf}(2^\kappa) > \kappa$. Since $2^\kappa = 2^\omega$, we obtain that for each κ such that $\omega \le \kappa < 2^\kappa$,

$$\kappa < \mathrm{cf}(2^\kappa) = \mathrm{cf}(2^\omega) \le 2^\omega.$$

Therefore $2^\omega = \mathrm{cf}(2^\omega)$, i.e. 2^ω is regular. □

2. Applications

2.1. Application to measure. The collection all Lebesgue measure zero sets, forms a σ-ideal, which we denote $\mathcal{N}$. A countable set is of measure zero, while the real line itself is not of measure zero. Thus, of interest becomes the following cardinal value:

$$\mathrm{add}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{N}, \bigcup \mathcal{F} \notin \mathcal{N}\}.$$

We will show that MA implies that $\mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$. More precisely:

THEOREM 2.1. *Assume $\mathsf{MA}(\kappa)$. Then $\mathrm{add}(\mathcal{N}) > \kappa$.*

PROOF. In the following μ denotes the Lebesuge measure on the real line $\mathbb{R}$. We have to show that if $\{M_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{N}$, then $\bigcup_{\alpha<\kappa} M_\alpha \in \mathcal{N}$. Fix $\{M_\alpha\}_{\alpha<\kappa}$.

FACT 2. A set $M \subseteq \mathbb{R}$ has Lebesgue measure zero, i.e. $\mu(M) = 0$ iff for every $\epsilon > 0$ there is an open $U \subseteq \mathbb{R}$ such that $M \subseteq U$ and $\mu(U) < \epsilon$.

Fix $\epsilon > 0$. Let $\mathbb{P}_\epsilon$ be the partial order of all open $U \subseteq \mathbb{R}$ such that $\mu(U) < \epsilon$ with extension relation superset, i.e. $p \le q$ iff $p \supseteq q$.

CLAIM 2.2. Let $p, q \in \mathbb{P}_\epsilon$. Then $p \not\perp q$ iff $\mu(p \cup q) < \epsilon$. In particula, if $p \not\perp q$ then $p \cup q \le p, q$.

CLAIM 2.3. Let $G \subseteq \mathbb{P}_\epsilon$ be a filter. Then $\mu(\mathcal{U}_G) \le \epsilon$, where $\mathcal{U}_G = \bigcup G = \bigcup\{p : p \in G\}$.

PROOF. If $p, q \in G$ then since $\exists r \in G(r \le p, q)$, we must have $r \le p \cup q$. However, G is closed with respect to weaker conditions and so $p \cup q \in G$. Therefore for every natural number n and every $\{p_j\}_{j\in n} \subseteq G$, we have $\bigcup_{j\in n} p_j \in G$. Let $\mathcal{B}$ be the base for the topology of $\mathbb{R}$ consisting of open intervals with rational endpoints. If $x \in \bigcup G$, then there is $p \in G$ such that $x \in p$. Since $\mathcal{B}$ is a base, there is $B \in \mathcal{B}$ such that $x \in B \subseteq p$. Then in particular

$$\mu(B) \le \mu(p) < \epsilon.$$

Furthermore, if $\{p_1, \cdots, p_n\} \subseteq G \cap B$, then $\bigcup_{j=1}^n p_j \in G$ and so $\mu(\bigcup_{j=1}^n p_j) < \epsilon$. The base $\mathcal{B}$ is a countable set and so $G \cap \mathcal{B}$ is also countable. Therefore

$$\mu \bigcup (G \cap \mathcal{B}) \le \sum \{\mu(p) : p \in G \cap \mathcal{B}\} \le \epsilon,$$

where we used the fact that all partial sums are strictly smaller than ϵ. □

CLAIM 2.4. The partial order $\mathbb{P}_\epsilon$ has the countable chain condition.

PROOF. Suppose by contradiction, that $\mathbb{P}_\epsilon$ is not ccc. Let $A = \{p_\alpha\}_{\alpha<\omega_1} \subseteq \mathbb{P}_\epsilon$ be an antichain, i.e, for all $\alpha \neq \beta$, $p_\alpha \perp p_\beta$. We claim that there is $n \in \omega$ such that for $\delta = \frac{1}{n}$ we have $0 < \delta < \epsilon$ and $X = \{\alpha \in \omega_1 : \mu(p_\alpha) \leq \epsilon - 3\delta\}$ is uncountable. Well, again, suppose this is not the case. Then, for every natural number n, the set $X_n = \{\alpha \in \omega_1 : \mu(p_\alpha) \leq \epsilon - \frac{1}{n}\}$ is countable. However $\omega_1 = \bigcup_{n\in\omega} X_n$ and a countable union of countable sets if countable, which is a contradiction. We will make use of the following fact.

FACT. If V is an open set (or a measurable subset of $\mathbb{R}$) and $\delta > 0$, then there is a finite family $\mathcal{C}$ of basic open subsets from $\mathcal{B}$ such that $C \triangle V = C\backslash V \cup V\backslash C$ is of Lebesgue measure $\leq \delta$.

Then, for each $\alpha \in X$ there is $C_\alpha \in \mathcal{C} = \{\bigcup \mathcal{B}' : \mathcal{B}' \in [\mathcal{B}]^{<\omega}\}$ such that $\mu(p_\alpha \triangle C_\alpha) \leq \delta$. Since for each distinct α, β from X, the conditions p_α and p_β are incompatible, we must have $\mu(p_\alpha \cup p_\beta) \geq \epsilon$. On the other hand, for all $\alpha, \beta \in X$ we have that

$$\mu(p_\alpha \cap p_\beta) \leq \mu(p_\alpha) \leq \epsilon - 3\delta.$$

Note that $p_\alpha \cup p_\beta = p_\alpha \triangle p_\beta \cup p_\alpha \cap p_\beta$. Therefore

$$\epsilon \leq \mu(p_\alpha \cup p_\beta) = \mu(p_\alpha \triangle p_\beta) + \mu(p_\alpha \cap p_\beta)$$

and so we obtain that $\mu(p_\alpha \triangle p_\beta) \geq 3\delta$. This implies that $\mu(C_\alpha \triangle C_\beta) \geq \delta$ and so in particular $C_\alpha \neq C_\beta$. Therefore $\{C_\alpha\}_{\alpha\in X}$ is an uncountable subset of $\mathcal{C}$ which is a contradiction, since $\mathcal{C}$ is countable. Therefore $\mathbb{P}_\epsilon$ is indeed ccc. □

Since $\mathbb{P}_\epsilon$ is ccc, we can apply $\mathsf{MA}(\kappa)$. Now for each $\alpha \in \kappa$, consider the set

$$D_\alpha = \{p \in \mathbb{P}_\epsilon : M_\alpha \subseteq p\}.$$

CLAIM 2.5. For all $\alpha \in \kappa$, D_α is dense.

PROOF. Fix $\alpha \in \kappa$. Let $q \in \mathbb{P}$ and let $\epsilon_q = \mu(q)$. Then $\epsilon_q < \epsilon$. By Fact 2 there is an open set V such that $M_\alpha \subseteq V$ and $\mu(V) < \epsilon - \epsilon_q$. Take $p = q \cup V$. Then p is open and

$$\mu(p) \leq \mu(q) + \mu(V) < \epsilon_q + \epsilon - \epsilon_q = \epsilon.$$

Thus $p \in \mathbb{P}_\epsilon$ and $p \in D_\alpha$. □

By $\mathsf{MA}(\kappa)$, there is a filter $G^\epsilon \subseteq \mathbb{P}_\epsilon$ such that $G^\epsilon \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. This implies that for all $\alpha < \kappa$,

$$M_\alpha \subseteq \bigcup G^\epsilon = \mathcal{U}_{G^\epsilon}.$$

Let $\mathcal{U}^\epsilon = \bigcup G^\epsilon$. Thus, $\bigcup_{\alpha<\kappa} M_\alpha \subseteq \mathcal{U}^\epsilon$. However $\mu(\mathcal{U}^\epsilon) \leq \epsilon$ and the above can be done for each ϵ, we obtain

$$\mu(\bigcup_{\alpha<\kappa} M_\alpha) = 0.$$

□

2.2. Applications to Category. Recall that a set $X \subseteq \mathbb{R}$ is said to be meager if $X \subseteq \bigcup_{n\in\omega} F_n$ where for each n, F_n is closed nowhere dense. The collection of all meager subsets $\mathcal{M}$ of all meager subsets of the real line forms a σ-ideal. Note that every countable set of real numbers is naturally a meager set, while the real line itself is not. Thus, the following cardinal value becomes of interest:

$$\mathrm{add}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{M}, \bigcup \mathcal{F} \notin \mathcal{M}\}.$$

By the above observation, we have that $\aleph_0 < \mathrm{add}(\mathcal{M}) \leq 2^{\aleph_0} = \mathfrak{c}$. We will show that MA implies that $\mathrm{add}(\mathcal{M}) = 2^{\aleph_0}$. More precisely, we will show the following:

THEOREM 2.6. *$MA(\kappa)$ implies that $add(\mathcal{M}) > \kappa$.*

PROOF. We have to show, that whenever $\{M_\alpha\}_{\alpha<\kappa}$ is a family of meager subsets of $\mathbb{R}$, then $\bigcup_{\alpha<\kappa} M_\alpha$ is meager. That is, given $\{M_\alpha\}_{\alpha<\kappa}$, we have to show that there is a countable family $\{H_n\}_{n\in\omega}$ of closed nowhere dense sets, such that

$$\bigcup_{\alpha<\kappa} M_\alpha \subseteq \bigcup_{n\in\omega} H_n.$$

The above is equivalent to $\bigcap_{n\in\omega} \mathbb{R}\backslash H_n \subseteq \bigcap_{\alpha<\kappa} \mathbb{R}\backslash M_\alpha$. Note that the complement of a closed, nowhere dense set is an open dense subset of $\mathbb{R}$. Thus, it is sufficient to show that whenever we have a family $\{U_\alpha\}_{\alpha<\kappa}$ of dense open subsets of $\mathbb{R}$, then there is a countable family $\{V_n\}_{n\in\omega}$ of dense open subsets of $\mathbb{R}$ such that

$$\bigcap_{n\in\omega} V_n \subseteq \bigcap_{\alpha<\kappa} U_\alpha.$$

Fix $\{U_\alpha\}_{\alpha<\kappa}$ a family of dense open subsets of $\mathbb{R}$. Let $\mathcal{B} = \{B_i\}_{i\in\omega}$ be an enumeration of all non-empty open intervals with rational end-points, i.e. intervals of the form (p,q) where p,q are rational numbers. Then $\mathcal{B}$ is a base, i.e. for every open $W \subseteq \mathbb{R}$ we have $W = \bigcup\{B_i : B_i \subseteq W\}$. Now, for each $j \in \omega$ let

$$c_j = \{i \in \omega : B_i \subseteq B_j\}$$

and let $\mathcal{C} = \{c_j : j \in \omega\}$. Thus, c_j is a subset of ω, while $\mathcal{C} \subseteq \mathcal{P}(\omega)$. For each $\alpha < \kappa$, let

$$a_\alpha = \{i \in \omega : B_i \nsubseteq U_\alpha\}$$

and let $\mathcal{A} = \{a_\alpha\}_{\alpha<\kappa}$. Thus, $a_\alpha \subseteq \omega$ and $\mathcal{A} \subseteq \mathcal{P}(\omega)$. Next, we will show that the families $\mathcal{A}$, $\mathcal{C}$ satisfy the conditions of Solovay's Lemma. Indeed, let $c_j \in \mathcal{C}$ and let $\mathcal{F} \in [\mathcal{A}]^{<\omega}$. We need to verify that $|c_j \backslash \bigcup \mathcal{F}| = \omega$. Say, $\mathcal{F} = \{a_\alpha : \alpha \in F\}$ for some finite $F \in [\kappa]^{<\kappa}$. Then,

$$c_j \backslash \bigcup_{\alpha\in F} a_\alpha = \{i \in \omega : B_i \subseteq B_j, B_i \subseteq \bigcap_{\alpha\in F} U_\alpha\} = \{i \in \omega : B_i \subseteq B_j \cap \bigcap_{\alpha\in F} U_\alpha\}.$$

Using the fact that $\bigcap_{\alpha\in F} U_\alpha$ is dense open, we can show that $|c_j \backslash \bigcup_{\alpha\in F} a_\alpha| = \omega$.

Therefore, Solovay's Lemma applies and so there is $d \subseteq \omega$ such that

$$\forall \alpha \in \kappa(|d \cap a_\alpha| < \omega) \text{ and } \forall j \in \omega(|d \cap c_j| = \omega).$$

Now, for each $n \in \omega$, define $V_n = \bigcup\{B_i : i \in d, i > n\}$.

CLAIM 2.7. For each $n \in \omega$, the set V_n is dense open.

Proof. Fix n. Clearly, V_n is an open set. To show that V_n is dense, it is sufficient to show that $V_n \cap B_j \neq \varnothing$ for every basic open B_j. So, fix j. Since $|d \cap c_j| = \omega$, there is $i > n$ such that $i \in d$ and $i \in c_j$. That is, $B_i \subseteq B_j$. But $B_i \subseteq V_n$ (since $i \in d$, $i > n$) and so $B_i \subseteq B_j \cap V_n$. Thus, $B_j \cap V_n \neq \varnothing$. □

It remains to show that $\bigcap_{n\in\omega} V_n \subseteq \bigcap_{\alpha<\kappa} U_\alpha$. Fix $\alpha < \kappa$. Since $|d \cap a_\alpha| < \omega$, there is a natural number n such that $d \cap a_\alpha \subseteq n$. Therefore for every $i \in \omega \setminus (n+1)$ if $i \in d$ then $i \notin a_\alpha$ and so $B_i \subseteq U_\alpha$. Thus, in particular, $V_n = \bigcup\{B_i : i \in d \wedge i > n\} \subseteq U_\alpha$. Therefore

$$\bigcap_{m\in\omega} V_m \subseteq V_n \subseteq U_\alpha.$$

Since $\alpha < \kappa$ was arbitrary, we obtain $\bigcap_{m\in\omega} V_m \subseteq \bigcap_{\alpha<\kappa} U_\alpha$. □

CHAPTER 4

Forcing

1. Generic Extensions

Discussion 1.1. The method of forcing allows to establish the relative consistency of $\neg\mathsf{CH}$. More precisely, we will show that if Ω is a finite subset of ZF, then there is a larger set subset Λ of ZFC such that every countable transitive model $\mathcal{M}$ of Λ has an extension $\mathcal{N}$ such that $\mathcal{N} \vDash \Omega + \neg\mathsf{CH}$.

To prove Con(ZFC) $\to$ Con(ZFC $+\neg\mathsf{CH}$), proceed as follows: If ZFC $+\neg\mathsf{CH} \vDash \varphi + \neg\varphi$ for some sentence φ, then there is a finite $\Omega \subseteq$ ZFC such that $\Omega + \neg\mathsf{CH} \vDash \varphi \wedge \neg\varphi$. Therefore, in ZFC we can produce a model $\mathcal{N}$ of the inconsistent theory $\Omega + \neg\mathsf{CH}$, thus ZFC is inconsistent.

Remark 1.2. Throughout, by "$\mathcal{M}$ is a c.t.m. for ZFC" we understand, that $\mathcal{M}$ is a countable transitive model for a sufficiently large fragment of ZFC.

Notation. Let $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ be a partial order with designated maximal element $1_{\mathbb{P}}$ such that

$$\forall q \in \mathbb{P}(q \leq 1_{\mathbb{P}})$$

(with other words 1_P is largest). We consider $\mathbb{P} \in \mathcal{M}$ for a model $\mathcal{M}$, as an abbreviation to $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}}) \in \mathcal{M}$. We refer to such partial orders, also as *forcing notions* and to the elements of a given partial order as *conditions*. Note that if $q \leq p$ we say that q is *stronger than* p, also that p is *weaker than* q, and that q is *an extension* of p. If p, q do not have common extension, we say that they are *incompatible*.

Definition 1.3. Let $\mathcal{M}$ be a c.t.m. and $\mathbb{P} \in \mathcal{M}$ be a forcing notion. A filter $G \subseteq \mathbb{P}$ is said to be $(\mathcal{M}, \mathbb{P})$*-generic* (also $\mathbb{P}$*-generic over* $\mathcal{M}$) if $G \cap D \neq \varnothing$ for all dense $D \subseteq \mathbb{P}$ such that $D \in \mathcal{M}$.

Remark 1.4. The model which we want to obtain is of the form $M[G]$. i.e. we adjoin to the model $\mathcal{M}$ a filter G, which is $(\mathcal{M}, \mathbb{P})$-generic.

Lemma 1.5 (Generic Filter Existence Lemma). Let $\mathcal{M}$ be a c.t.m. for $\mathsf{ZF\text{-}P}$. Let $\mathbb{P} \in \mathcal{M}$ be a forcing notion and let $p \in \mathbb{P}$. Then, there is an $(\mathcal{M}, \mathbb{P})$-generic filter G such that $p \in G$.

Proof. Let $\{D_n\}_{n\in\omega}$ be an enumeration of all dense subsets of $\mathbb{P}$, which are elements of $\mathcal{M}$. Recursively, define a sequence $\{p_n\}_{n\in\omega} \subseteq \mathbb{P}$ such that $p_0 \in D_0$ with the property that $p_0 \leq p$ and for each n, $p_{n+1} \in D_{n+1}$ is such that $p_{n+1} \leq p_n$. Then the upwards closure G of $\{p_n\}_{n\in\omega}$ in $\mathbb{P}$ is the desired $(\mathcal{M}, \mathbb{P})$-generic filter. That is $G = \{q \in \mathbb{P} : \exists n(p_n \leq q)\}$. Note that the given condition p does not necessarily belong to $\{p_n\}_{n\in\omega}$, but $p \in G$. $\square$

DEFINITION 1.6. Let $\mathbb{P}$ be a partial order. We say that an element $r \in \mathbb{P}$ is an *atom* if there are no incompatible p, q extending r. Moreover $\mathbb{P}$ is said to be *atomless*, if there are no atoms in $\mathbb{P}$.

LEMMA 1.7. Suppose $\mathbb{P}$ is an atomless poset and G is an $(\mathcal{M}, \mathbb{P})$-generic filter. Then $G \notin \mathcal{M}$.

PROOF. Let $D = \mathbb{P} \backslash G$. Let $r \in \mathbb{P}$. Then, since $\mathbb{P}$ is atomless, there are $p, q \leq r$ such that $p \perp q$. But, then at most one of q, p is an element of G, which means that at least one of $\{q, p\}$ belongs to D. Therefore D is dense. If $G \in \mathcal{M}$, then $D \in \mathcal{M}$. However $G \cap D = \varnothing$, which is a contradiction to the hypothesis that G is generic over $\mathcal{M}$. Thus $G \notin \mathcal{M}$. $\square$

DEFINITION 1.8 ($\mathbb{P}$-names). Let $\mathbb{P}$ be a partial order.

(1) A relation τ is a $\mathbb{P}$-name iff for every $\langle \sigma, p \rangle \in \tau$ we have that σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.
(2) With $V^{\mathbb{P}}$ we denote the collection of all $\mathbb{P}$-names. Note that $V^{\mathbb{P}}$ is a proper class.

DEFINITION 1.9 (Generic extension). Let $\mathcal{M}$ be a c.t.m. of ZF-P and let $\mathbb{P} \in \mathcal{M}$. Then

$$\mathcal{M}^{\mathbb{P}} = V^{\mathbb{P}} \cap \mathcal{M} = \{\tau \in \mathcal{M} : (\tau \text{ is a } \mathbb{P}\text{-name})^{\mathcal{M}}\}.$$

DEFINITION 1.10 (Evaluation of $\mathbb{P}$-names). Let τ be a $\mathbb{P}$-name and let $G \subseteq \mathbb{P}$ be a filter. Then, the *evaluation of τ with respect to G*, denoted $\mathrm{val}(\tau, G)$, also τ^G, is the recursively defined set

$$\mathrm{val}(\tau, G) = \tau_G = \{\mathrm{val}(\sigma, G) : \exists p \in G(\langle \sigma, p \rangle \in \tau)\}.$$

DEFINITION 1.11 (Generic extension). Let $\mathcal{M}$ be a c.t.m. of ZF-P, $\mathbb{P} \in \mathcal{M}$ be a partial order. Let G be a $(\mathcal{M}, \mathbb{P})$-generic filter. Then the *generic extension of $\mathcal{M}$ via G* is the set

$$M[G] = \{\tau_G : \tau \in \mathcal{M}^{\mathbb{P}}\}.$$

REMARK 1.12. We will prove that $\mathcal{M}[G]$ is a model of a sufficiently large fragment of ZF-P.

EXAMPLE 1.13.

(1) $\varnothing$ is vacuously a $\mathbb{P}$-name and $\varnothing_G = \varnothing$.
(2) If σ^1, σ^2, σ^3 are $\mathbb{P}$-names and $\tau = \{\langle \sigma^1, 1_{\mathbb{P}} \rangle, \langle \sigma^2, 1_{\mathbb{P}} \rangle, \langle \sigma^3, 1_{\mathbb{P}} \rangle\}$, then τ is a $\mathbb{P}$-name and $\tau_G = \{\sigma^1_G, \sigma^2_G, \sigma^3_G\}$. Note that for each $\{p_i\}_{i=1}^3 \subseteq \mathbb{P}$, the set $\tau' = \{\langle \sigma^i, p_i \rangle\}_{i=1}^3$ is also a $\mathbb{P}$-name. However, the evaluation τ'_G depends on $G \cap \{p_i\}_{i=1}^3$.

DEFINITION 1.14 (Check names). For a forcing notion $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ and a set x, let

$$\check{x} = \{\langle \check{y}, 1_{\mathbb{P}} \rangle : y \in x\}.$$

We refer to the set $\check{x}$ as a *check name*.

LEMMA 1.15. If $\mathcal{M}$ is a transitive model of ZF-P, $\mathbb{P} \in \mathcal{M}$ and G is a $(\mathcal{M}, \mathbb{P})$-generic filter, then:

(1) for all $x \in \mathcal{M}$, we have that $\check{x} \in \mathcal{M}^{\mathbb{P}}$ and $\mathrm{val}(\check{x}, G) = x$;
(2) $\mathcal{M} \subseteq \mathcal{M}[G]$.

PROOF. To see item (1) note that recursive definitions are absolute and so $\check{x} \in \mathcal{M}$ for each $x \in \mathcal{M}$. Then inductively one can show that $\mathrm{val}(\check{x}, G) = x$. Item (2) follows directly from the Definition of $\mathcal{M}[G]$. $\square$

DEFINITION 1.16 (Canonical name for a filter). Let $\mathbb{P}$ be a forcing notion. Then $\Gamma = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$ is a canonical name for a generic filter.

REMARK 1.17. Indeed. If $\mathbb{P} \in \mathcal{M}$ then $\Gamma \in \mathcal{M}$ and if G is $(\mathcal{M}, \mathbb{P})$-generic, then $\Gamma_G = G$.

LEMMA 1.18 (Minimality of Generic Extensions). Let $\mathcal{M} \subseteq \mathcal{N}$ be transitive models of ZF-P, $\mathbb{P} \in \mathcal{M}$ a forcing notion and let G be an $(\mathcal{M}, \mathbb{P})$-generic filter such that $G \in \mathcal{N}$. Then $\mathcal{M}[G] \subseteq \mathcal{N}$.

PROOF. Recall that $\mathcal{M}[G] = \{\tau_G : \tau \in \mathcal{M}^{\mathbb{P}}\}$. For every $\tau \in \mathcal{M}^{\mathbb{P}}$, clearly $\tau \in \mathcal{N}$. The set τ_G is recursively defined from τ and G and so by absoluteness of evaluation of names, we have $\mathrm{val}(\tau, G) = \tau_G \in \mathcal{N}$. Thus $\mathcal{M}[G] \subseteq \mathcal{N}$. $\square$

2. The Forcing Language

DEFINITION 2.1 (The forcing language). Let $\mathbb{P}$ be a partial order. Then the forcing language $\mathcal{FL}_{\mathbb{P}}$ consists of all first order formulas which are obtained from the binary relation symbol $\in$ and all the names in $V^{\mathbb{P}}$, treated as constant symbols.

REMARK 2.2. $V^{\mathbb{P}}$ is a proper class. For a transitive model $\mathcal{M}$, $\mathcal{M} \cap \mathcal{FL}_{\mathbb{P}}$ is the set of all first order formulas obtained in the usual way from the binary relation $\in$ and all the names in $\mathcal{M}^{\mathbb{P}}$ used as constant symbols.

DEFINITION 2.3. For a closed formula ψ in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ define the satisfaction relation $\mathcal{M}[G] \vDash \psi$ as usual, by interpreting $\in$ as membership and each name τ as τ_G.

DEFINITION 2.4 (The forcing relation). Let $\mathcal{M}$ be a c.t.m. for ZF-P, let $\mathbb{P} \in \mathcal{M}$ be a forcing notion and let ψ be a closed formula in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$. Then, we say that *$p$ forces ψ over $\mathcal{M}$* or just *p forces ψ*, denoted

$$p \Vdash_{\mathbb{P},\mathcal{M}} \psi,$$

also denoted simply $p \Vdash \psi$ whenever $\mathbb{P}$, $\mathcal{M}$ are clear from the context, if for every $(\mathcal{M}, \mathbb{P})$-generic filter G such that $p \in G$, we have $\mathcal{M}[G] \vDash \psi$.

LEMMA 2.5 (Truth Lemma). Let $\mathcal{M}$ be a c.t.m. for ZF-P, $\mathbb{P} \in \mathcal{M}$ a forcing notion, ψ a sentence of $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ and let G be an $(\mathcal{M}, \mathbb{P})$-generic filter. Then

$$\mathcal{M}[G] \vDash \psi \text{ iff } \exists p \in G (p \Vdash \psi).$$

REMARK 2.6. Note that the implication from right to left in the above theorem follows from the definition of the forcing relation. On the other hand the implication from left to right is non-trivial. Let $\mathbb{P}$ be a forcing notion and let ψ be the formula $\check{p}_1 \in \Gamma \wedge \check{p}_2 \in \Gamma$. Suppose $r \in \mathbb{P}$ and r is a common extension of p_1 and p_2. Then $r \Vdash \psi$, since for every $(\mathcal{M}, \mathbb{P})$-generic filter G such that $r \in G$ we have $\mathcal{M}[G] \vDash \psi$. On the other hand if G' is an $(\mathcal{M}, \mathbb{P})$-generic filter such that $p_1 \in G'$, but $p_2 \notin G'$ then clearly $\mathcal{M}[G'] \nvDash \psi$.

LEMMA 2.7 (The Definability Lemma). Let $\mathcal{M}$ be a ctm for ZF-P, let $\varphi(x_1, \cdots, x_n)$ be a formula in $\mathcal{L}_\in$, with all free variables shown. Then, the set of all finite tuples $(p, \mathbb{P}, \leq_\mathbb{P}, 1_\mathbb{P}, \nu_1, \cdots, \nu_n)$ where $(\mathbb{P}, \leq, 1_\mathbb{P})$ is a forcing notion, $p \in \mathbb{P}$, $(\mathbb{P}, \leq_\mathbb{P}, 1_\mathbb{P}) \in \mathcal{M}$, $\nu_1, \cdots, \nu_n$ are elements of $\mathcal{M}^\mathbb{P}$ and $p \Vdash_{\mathbb{P},\mathcal{M}} \varphi(\nu_1, \cdots, \nu_n)$ is definable over $\mathcal{M}$ without parameters.

EXAMPLE 2.8. Let $\tau = \{\langle \check{n}, p\rangle : n \in \omega, p \in \mathbb{P}, p \Vdash \varphi(\check{n}, \sigma)\}$ where $\varphi(x, y)$ is a formula and $\tau \in \mathcal{M}$. Then

$$\tau_G = \{n \in \omega : \exists p \in G(p \Vdash \varphi(\check{n}, \sigma))\}.$$

By definition of the forcing relation $\Vdash$, we have

$$\tau_G \subseteq \{n \in \omega : \mathcal{M}[G] \vDash \varphi(n, \sigma_G)\}.$$

Denote the latter set S. We will show that $S \subseteq \tau_G$. Let $n \in S$. Then $\mathcal{M}[G] \vDash \varphi(n, \sigma_G)$. By the Truth Lemma, applied to the sentence $\varphi(\check{n}, \sigma) \in \mathcal{FL}_\mathbb{P} \cap \mathcal{M}$, there is $p \in G$ such that $p \Vdash \varphi(\check{n}, \sigma)$. Then $\langle \check{n}, p\rangle \in \tau$ and so $n \in \tau_G$.

3. ZFC and generic extensions

LEMMA 3.1. Let $\mathcal{M}$ be a transitive model for ZF-P, $\mathbb{P} \in \mathcal{M}$, G a filter on $\mathbb{P}$. Then:

(1) $\operatorname{rank}(\tau_G) \leq \operatorname{rank}(\tau)$ for all $\tau \in \mathcal{M}$.
(2) $o(\mathcal{M}[G]) = o(\mathcal{M})$.
(3) $|\mathcal{M}[G]| = |\mathcal{M}|$.

PROOF. Exercise. □

We will make use of the following:

DEFINITION 3.2 (Names for unordered and ordered pairs). Let σ and τ are $\mathbb{P}$-names. Then let

(1) $\operatorname{up}(\sigma, \tau) = \{\langle \sigma, 1_\mathbb{P}\rangle, \langle \tau, 1_\mathbb{P}\rangle\}$ and
(2) $\operatorname{op}(\sigma, \tau) = \operatorname{up}(\operatorname{up}(\sigma, \sigma), \operatorname{up}(\sigma, \tau))$.

LEMMA 3.3. Let $\mathcal{M}$ be a ctm for ZF-P, $\mathbb{P}$ a forcing notion in $\mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic filter. Then $\mathcal{M}[G]$ is a transitive model for ZF-P$\setminus$\{Replacement\}.

PROOF. The fact that $\mathcal{M}[G]$ is transitive is straightforward from the definition of $\mathcal{M}[G]$. Indeed, suppose $x \in \tau_G$. We have to show that $x \in \mathcal{M}[G]$. But by definition if $x \in \tau_G$, then $x = \sigma_G$ for some $\sigma \in \mathcal{M}^\mathbb{P}$ and so $x = \sigma_G \in \mathcal{M}[G]$. Thus $\mathcal{M}[G]$ is transitive. Extensionality and Foundation are also straightforward. Pairing holds, as given $\sigma, \tau \in \mathcal{M}^\mathbb{P}$, we have that

$$(\operatorname{up}(\sigma, \tau))^G = \{\sigma_G, \tau_G\} \in \mathcal{M}[G].$$

To prove the union axiom, we need to show that if $a \in \mathcal{M}[G]$ then there is $b \in \mathcal{M}[G]$ such that $\bigcup a \subseteq b$. Let $\tau \in \mathcal{M}^\mathbb{P}$ be such that $a = \tau_G$. Note that $\bigcup \operatorname{dom}(\tau)$ is a name (since every element of τ is of the form $\langle \sigma, p\rangle$ for $\sigma \in \mathcal{M}^\mathbb{P}$, $p \in \mathbb{P}$) and moreover $\bigcup \operatorname{dom}(\tau) \in \mathcal{M}$ be absoluteness of the union operation (in $\mathcal{M}$). Thus, take $\pi = \bigcup \operatorname{dom}(\tau)$. Then $\pi \in \mathcal{M}^\mathbb{P}$ and $b = \pi_G \in \mathcal{M}[G]$. We still need to show that $\bigcup a \subseteq b$. Let $c \in a$. Then $c = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$, i.e. $\sigma \subseteq \pi$. But, then $\sigma_G \subseteq \pi_G$ and so $\bigcup a \subseteq b$.

To prove the Axiom of Comprehension, consider a formula φ in the language of set theory, $\varphi(x, z, v_0, \cdots, v_{n-1})$ with all free variable shown. We will show that

$$\forall z, v_0, \cdots, v_{n-1} \in \mathcal{M}[G] \exists y \in \mathcal{M}[G] \forall x \in \mathcal{M}[G](x \in y \leftrightarrow x \in z \wedge \varphi^{\mathcal{M}[G]}(x, z, v_0, \cdots, v_{n-1})).$$

Fix elements $\pi_G, \sigma_G^0, \cdots, \sigma_G^{n-1}$ in $\mathcal{M}[G]$ corresponding to the variables $z, v_0, \cdots, v_{n-1}$ and names $\{\pi, \sigma^i\}_{i=0}^n \subseteq \mathcal{M}^{\mathbb{P}}$. Let

$$S = \{x \in \pi_G : \varphi^{\mathcal{M}[G]}(x, \pi_G, \sigma_G^0, \cdots, \sigma_G^{n-1})\}.$$

It is sufficient to show that $S \in \mathcal{M}[G]$. Consider the $\mathcal{FL}_{\mathbb{P}}$-formula $\varphi(x, \pi, \sigma^0, \cdots, \sigma^{n-1}) = \tilde{\varphi}(x)$ and note that $\tilde{\varphi}(x) \in \mathcal{M}^{\mathbb{P}}$. Note that

$$S = \{\nu_G : \nu \in \mathrm{dom}(\pi) \wedge \mathcal{M}[G] \models \nu_G \in \pi_G \wedge \tilde{\varphi}(\nu_G)\}.$$

Let $\tau = \{\langle \nu, p\rangle : \nu \in \mathrm{dom}(\pi) \wedge p \in \mathbb{P} \wedge p \Vdash (\nu \in \pi \wedge \tilde{\varphi}(\nu))\}$. By the *Definability Lemma* $\tau \in \mathcal{M}^{\mathbb{P}}$ and so $\tau_G \in \mathcal{M}[G]$. Moreover

$$\tau_G = \{\nu_G : \nu \in \mathrm{dom}(\pi) \wedge \exists p \in G \text{ s.t. } p \Vdash (\mu \in \pi \wedge \tilde{\varphi}(\nu))\}.$$

Now, by the definition of the forcing relation $\tau_G \subseteq S$. To see that $S \subseteq \tau_G$, take any $\nu_G \in S$. Thus, $\nu \in \mathrm{dom}(\pi)$ and $\mathcal{M}[G] \models \nu_G \in \pi_G \wedge \tilde{\varphi}(\nu_G)$. Then $(\nu, p) \in \tau$ and so $\nu_G \in \tau_G$.

The Axiom of Infinity holds in $\mathcal{M}[G]$, since $\omega \in \mathcal{M}[G]$. □

THEOREM 3.4. *Let $\mathcal{M}$ be a ctm for* ZFC*, let $\mathbb{P} \in \mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic. Then $\mathcal{M}[G]$ is a model for* ZFC.

PROOF. We continue with the Power Set Axioms, Replacement and Choice.

Power set axiom: We have to show that if $a \in \mathcal{M}[G]$, then there is $b \in \mathcal{M}[G]$ such that $\mathcal{P}(a) \cap \mathcal{M}[G] \subseteq b$. Consider a set $a \in \mathcal{M}[G]$ and fix a name $\tau \in \mathcal{M}^{\mathbb{P}}$ such that $\tau_G = a$. Let $Q = \{\nu \in \mathcal{M}^{\mathbb{P}} : \mathrm{dom}(\nu) \subseteq \mathrm{dom}(\tau)\}$. By Comprehension $Q \in \mathcal{M}$ and so $\pi = Q \times \{1_{\mathbb{P}}\} \in \mathcal{M}^{\mathbb{P}}$. We claim that $b = \pi_G$ is as desired.

Let $c \in \mathcal{P}(a) \cap \mathcal{M}[G]$ and let $\chi \in \mathcal{M}^{\mathbb{P}}$ be such that $\chi_G = c$. Consider the name

$$\nu = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \chi\}.$$

By the *Definability Lemma* $\nu \in \mathcal{M}^{\mathbb{P}}$. Clearly $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(\tau)$ and so $\nu \in Q$. Thus $\nu_G \in \pi_G$. It remains to show that $\nu_G = c$. Note that

$$\nu_G = \{\sigma_G : \langle \sigma, p\rangle \in \nu \wedge p \in G\}.$$

If $\sigma_G \in \nu_G$, then there is $p \in G$ such that $p \Vdash \sigma_G \in \chi_G$ and so $\mathcal{M}[G] \models \sigma_G \in c$. Therefore $\nu_G \subseteq c$. On the other hand, if $d \in c$, then $d = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. Now $\sigma_G \in c = \chi_G$ and by the *Truth Lemma* there is $p \in G$ such that $p \Vdash \sigma \in \chi$. Then, by definition of ν, we get $\langle \sigma, p\rangle \in \nu$. Therefore $\sigma_G \in \nu_G$, as desired.

Replacement Let $\tilde{\varphi}(x, y)$ be $\mathcal{FL}_{\mathbb{P}}$-formula in $\mathcal{M}$ and let $a \in \mathcal{M}[G]$ so that

$$\mathcal{M}[G] \models \forall x \in a \exists y \tilde{\varphi}(x, y).$$

To show Replacement, we will find $b \in \mathcal{M}[G]$ so that $\mathcal{M}[G] \vDash \forall x \in a \exists y \in b \tilde{\varphi}(x,y)$. Fix a $\mathbb{P}$-name $\tau \in \mathcal{M}^{\mathbb{P}}$ for a, i.e. such that $\tau_G = a$. Consider the function $f_\tau : \mathrm{dom}(\tau) \times \mathbb{P} \to \mathcal{M}^{\mathbb{P}}$ defined by

$$f_\tau(\sigma, p) = \begin{cases} \nu & \text{if } \exists \nu \in \mathcal{M}^{\mathbb{P}} \text{ such that } p \Vdash \tilde{\varphi}(\sigma, \nu) \\ \varnothing & \text{otherwise} \end{cases}$$

Note that there is $\alpha < o(\mathcal{M})$ such that $\mathrm{range}(f_\tau) \subseteq \mathcal{M}^{\mathbb{P}} \cap (R(\alpha))^{\mathcal{M}}$. Take $Q = \mathcal{M}^{\mathbb{P}} \cap (R(\alpha))^{\mathcal{M}}$. Then $Q \in \mathcal{M}$ and so $\pi = Q \times \{1_{\mathbb{P}}\} \in \mathcal{M}^{\mathbb{P}}$. It remains to show that $b = \pi_G$ as desired. For this, consider $x \in a$. Thus $x = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$ and by hypothesis $\mathcal{M}[G] \vDash \exists y \tilde{\varphi}(x,y)$. By the *Truth Lemma*, we can find $p \in G$ and $\nu \in \mathcal{M}^{\mathbb{P}}$ such that $p \Vdash \tilde{\varphi}(\sigma, \nu)$. But then $f(\sigma, p)$ is defined and $f(\sigma, p) = \nu'$ for some $\nu' \in \mathcal{M}^{\mathbb{P}}$ such that $p \Vdash \tilde{\varphi}(\sigma, \nu')$. By definition of Q, $\nu' \in Q$ and we can take $y' := \nu'_G$. Then $\mathcal{M}[G] \vDash y' \in b \wedge \tilde{\varphi}(x, y')$.

Axiom of Choice It is sufficient to show that every set in $\mathcal{M}[G]$ can be well-ordered in $\mathcal{M}[G]$. Fix $a = \tau_G \in \mathcal{M}[G]$ and using the Axiom of Choice in $\mathcal{M}$ to well-order $\mathrm{dom}(\tau)$ in order type α, i.e. $\mathrm{dom}(\tau) = \{\sigma_\xi : \xi < \alpha\}$. Let

$$\dot{f} = \{\langle \mathrm{op}(\check{\xi}, \sigma_\xi), 1_{\mathbb{P}} \rangle : \xi < \alpha\}.$$

In $\mathcal{M}[G]$, take $f = \dot{f}_G$. Then $\dot{f}_G = \{\langle \xi, (\sigma_\xi)_G \rangle : \xi < \alpha\}$ and so in $\mathcal{M}[G]$, $\mathrm{dom}(f) = \alpha$ and $a \subseteq \mathrm{ran}(f)$. For $x, y (x \neq y)$ elements of a define

$$x \lhd y \text{ iff } \min\{\xi : f(\xi) = x\} < \min\{\xi : f(\xi) = y\}.$$

Then $\lhd$ is a well-order on a (in $\mathcal{M}[G]$). □

4. Some Properties of the Forcing Relation

EXAMPLE 4.1.

(1) If $p \leq q$ then $p \Vdash \check{q} \in \dot{G}$, by upwards closure of G. Here $\dot{G} = \Gamma$ is the canonical name for the generic filter.
(2) $1_{\mathbb{P}} \Vdash \psi$ iff $\mathcal{M}[G] \vDash \psi$ for all $(\mathcal{M}, \mathbb{P})$-generic filters G.
(3) If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$. Indeed, let G be an $(\mathcal{M}, \mathbb{P})$-generic filter such that $q \in G$. Then, by the upwards closure of G, $p \in G$. But, then by definition $\mathcal{M}[G] \vDash \varphi$. Therefore applying the definition once again, we obtain $q \Vdash \varphi$.

LEMMA 4.2. Let G be a $\mathbb{P}$-generic filter over $\mathcal{M}$. Assume $D \subseteq \mathbb{P}$, $D \in \mathcal{M}$ and D is dense below $p \in \mathbb{P}$. If $p \in G$, then $G \cap D \neq \varnothing$.

PROOF. Let $D^+ = D \cup \{q \in \mathbb{P} : p \perp q\}$. Then D^+ is dense. Not that $\{q \in \mathbb{P} : p \perp q\} = \{q \in \mathbb{P} : \neg(\exists r \leq q \text{ s.t. } r \in D)\}$ is definable from $\mathbb{P}$ and D, and so is in $\mathcal{M}$, which implies that $D^+ \in \mathcal{M}$. Therefore, $G \cap D \neq \varnothing$, because $p \in G$ and G is a filter. □

LEMMA 4.3. For any forcing notion $\mathbb{P} \in \mathcal{M}$ and sentences φ, ψ in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ the following hold:

(1) No p can force both φ and $\neg\varphi$.
(2) If φ, ψ are logically equivalent, then $p \Vdash \varphi$ iff $p \Vdash \psi$.
(3) If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.

(4) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.
(5) $p \Vdash \neg\varphi$ iff $\neg\exists q \leq p(q \Vdash \varphi)$; and $p \Vdash \varphi$ iff $\neg\exists q \leq p(q \Vdash \neg\varphi)$.
(6) $p \Vdash \varphi \to \psi$ iff $\neg\exists q \leq p(q \Vdash \varphi \wedge q \Vdash \neg\psi)$.
(7) $p \Vdash \varphi \vee \psi$ iff $\{q \leq p : q \Vdash \varphi \vee q \Vdash \psi\}$ is dense below p.
(8) $p \Vdash \varphi \leftrightarrow \psi$ iff $\neg\exists q \leq p(q \Vdash \varphi \wedge q \Vdash \neg\psi)$ and $\neg\exists q \leq p(q \Vdash \psi \wedge q \Vdash \neg\varphi)$.

Proof. Items (1) – (4) are direct corollaries to the definition of forcing.

(5) From left to right follows from (3)&(1). To prove the implication from right to left, suppose $p \nVdash \neg\varphi$. Then, there is a generic filter G such that $p \in G$ and $\mathcal{M}[G] \nvDash \neg\varphi$. That is $\mathcal{M}[G] \vDash \varphi$. But, then by the Truth Lemma there is $q' \in G$ such that $q' \Vdash \varphi$. Since $p, q' \in G$ there is $q \in G(q \leq p, q')$. So, $q \Vdash \varphi$. That is a contradiction to the assumption that $\neg\exists q \leq p(q \Vdash \varphi)$. Therefore $p \Vdash \neg\varphi$.

To see item (6), note that

$$\begin{array}{llll} p \Vdash \varphi \to \psi & \text{iff} & \neg\exists q \leq p(q \Vdash \neg(\varphi \to \psi)) & \text{by item (5)} \\ & \text{iff} & \neg\exists q \leq p(q \Vdash \varphi \wedge \neg\psi) & \text{by item (2)} \\ & \text{iff} & \neg\exists q \leq p(q \Vdash \varphi \wedge q \Vdash \neg\psi) & \text{by item (4).} \end{array}$$

To see item (7), observe

$$\begin{array}{llll} p \Vdash \varphi \vee \psi & \text{iff} & p \Vdash \neg\varphi \to \psi & \text{by item (2)} \\ & \text{iff} & \neg\exists r \leq p((r \Vdash \neg\varphi) \wedge (r \Vdash \neg\psi)) & \text{by item (6)} \\ & \text{iff} & \neg\exists r \leq p \forall q \leq r((q \nVdash \varphi) \wedge (q \nVdash \psi)) & \text{by item (5).} \end{array}$$

So, $p \Vdash \varphi \vee \psi$ iff $\forall r \leq p \exists q \leq r(q \Vdash \varphi \vee q \Vdash \psi)$.

(8) Follows from (6)&(2) since $\varphi \leftrightarrow \psi$ is logically equivalent to $(\varphi \to \psi) \wedge (\psi \to \varphi)$. $\square$

Remark 4.4. Note that if G is $(\mathcal{M}, \mathbb{P})$-generic, $\mathcal{M}[G] \vDash \varphi$ and $p \in G$, then by the Truth Lemma there is $q \in G$ such that $q \Vdash \varphi$. But any two conditions in G are compatible and so there is $r \in G$ such that $r \leq p, q$. Thus, $r \Vdash \varphi$. In particular, we proved that $\exists r \leq p(r \Vdash \varphi)$.

Lemma 4.5. For any forcing poset $\mathbb{P} \in \mathcal{M}$ and formula $\varphi(x) \in \mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ with all free variable shown:

(1) $p \Vdash \forall x\varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\tau \in \mathcal{M}^{\mathbb{P}}$.
(2) $p \Vdash \exists x\varphi(x)$ iff $\{q \leq p : \exists \tau \in \mathcal{M}^{\mathbb{P}}(q \Vdash \varphi(\tau))\}$ is dense below p.

Proof. To see item (1) note that $p \Vdash \forall x\varphi(x)$ iff for every $(\mathcal{M}, \mathbb{P})$-generic filter G such that $p \in G$, we have that $\mathcal{M}[G] \vDash \forall x\varphi(x)$. However, the latter is equivalent to the statement that for all $(\mathcal{M}, \mathbb{P})$-generic filters G such that $p \in G$ and every $\tau \in \mathcal{M}^{\mathbb{P}}$ we have that $\mathcal{M}[G] \vDash \varphi(\tau_G)$, which itself is equivalent to the statement that for every $\tau \in \mathcal{M}^{\mathbb{P}}$, $p \Vdash \varphi(\tau)$.

To see item (2), note that $\exists x\varphi(x)$ is equivlaent to $\neg\forall x\neg\varphi(x)$. Now, $p \Vdash \neg\forall x\neg\varphi(x)$ iff $\forall r \leq pr \nVdash \forall x\neg\varphi(x)$. However by (1), $r \nVdash \forall x\neg\varphi(x)$ iff $\exists \tau \in \mathcal{M}^{\mathbb{P}}$ such that $r \nVdash \neg\varphi(\tau)$. But $r \nVdash \neg\varphi(\tau)$ iff $\exists r' \leq r(r' \Vdash \varphi(\tau))$. So $r \nVdash \forall x\neg\varphi(x)$ iff $\exists \tau \in \mathcal{M}^{\mathbb{P}} \exists r' \leq r(r' \Vdash \varphi(\tau))$. Thus, $p \Vdash \neg\forall x\neg\varphi(x)$ iff $\forall r \leq p \exists \tau \in \mathcal{M}^{\mathbb{P}} \exists r' \leq r(r' \Vdash \varphi(\tau))$. $\square$

5. Cardinal evaluation in generic extensions

EXAMPLE 5.1. Let $\mathcal{M}$ be a ctm and let $I, J \in \mathcal{M}$ be infinite sets. Let $\mathbb{P} = \mathrm{Fn}(I, J)$ be the partial order of all finite partial functions from I to J with extension relation superset. Suppose G is $(\mathcal{M}, \mathbb{P})$-generic and let $f = \bigcup G$.

Consider, the special case that $I = \omega$ and $J = \omega_5^{\mathcal{M}}$. By absoluteness $\omega = \omega^{\mathcal{M}}$, however ω_5^M is just a countable ordinal (in V), while according to $\mathcal{M}$ it is the fifth uncountable cardinal. In $\mathcal{M}[G]$ however f is an onto mapping from ω onto $\omega_5^{\mathcal{M}}$ and so

$$\mathcal{M}[G] \vDash (\omega_5 \text{ is countable}).$$

DEFINITION 5.2. Let $\mathcal{M}$ be a ctm and let $\mathbb{P} \in \mathcal{M}$. Then we say that the forcing notion $\mathbb{P}$ preserves:

(1) cardinals, if for all generic filters G and all $\beta < o(\mathcal{M})$:

$$(\beta \text{ is a cardinal})^{\mathcal{M}} \text{ iff } (\beta \text{is a cardinal})^{\mathcal{M}[G]}.$$

(2) cofinalities if for all generic filters G and all limit $\gamma < o(\mathcal{M})$ we have that

$$\mathrm{cf}^{\mathcal{M}}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\gamma) \text{ for all } \gamma < o(\mathcal{M}).$$

REMARK 5.3.

(1) Suppose β is a cardinal in $\mathcal{M}[G]$. Then

$$\forall \alpha < \beta \forall f \neg (f \text{ is an onto function from } \alpha \text{ onto } \beta),$$

which is Π_1 in absolute notions. However Π_1 properties are downwards absolute and so β is a cardinal in $\mathcal{M}$.

(2) Regarding the notion of cofinality, note that $\mathrm{cf}^{\mathcal{M}}(\gamma) \geq \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$. If $\gamma = \omega_1^{\mathcal{M}}$ and $\gamma^{\mathcal{M}[G]}$ is countable, then $\gamma = \mathrm{cf}^{\mathcal{M}}(\gamma) > \omega = \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$.

LEMMA 5.4. Let $\mathbb{P}$ be a forcing notion in $\mathcal{M}$. Then

(1) $\mathbb{P}$ preserves cofinalities iff for every $(\mathcal{M}, \mathbb{P})$-generic filter G and all limit β such that $\omega < \beta < o(\mathcal{M})$:

$$(*) \text{ if } (\beta \text{ is regular})^{\mathcal{M}} \text{ then } (\beta \text{ is regular })^{\mathcal{M}[G]}.$$

(2) If $\mathbb{P}$ preserves cofinalities then $\mathbb{P}$ preserves cardinals.

PROOF. To see item (1) note that if $\mathbb{P}$ preserves cofinalities, then the statement $(*)$ holds by definition. Now, suppose $(*)$ holds for all γ such that $\omega < \gamma < o(\mathcal{M})$. Let $\beta = \mathrm{cf}^{\mathcal{M}}(\gamma)$. We need to show that $\mathrm{cf}^{\mathcal{M}[G]}(\gamma) = \beta$. In $\mathcal{M}$ let X be a subset of γ such that X is unbounded in γ and the order type of X is β. Then in particular $(\beta \text{ is regular})^{\mathcal{M}}$ and so by property $(*)$ we have that $(\beta \text{ is regular})^{\mathcal{M}[G]}$. Therefore

$$(\mathrm{cf}(\gamma))^{\mathcal{M}[G]} = (\mathrm{cf}(\beta))^{\mathcal{M}[G]} = \beta.$$

To see item (2) note that by item (1) the forcing notion $\mathbb{P}$ preserves regular cardinals and so $\mathcal{M}$ and $\mathcal{M}[G]$ have the same regular cardinals. However,every limit cardinal is a supremum of regular, successor cardinals and so $\mathbb{P}$ does preserve all cardinals. $\square$

EXAMPLE 5.5. There are partial orders which preserve cardinals, but not cofinalities, however we will not be working with such.

LEMMA 5.6 (Approximation Lemma). Let $\mathbb{P} \in \mathcal{M}$, $(\mathbb{P}\text{ is ccc})^{\mathcal{M}}$ and $A, B \in \mathcal{M}$. Let G be $(\mathcal{M}, \mathbb{P})$-generic and in $\mathcal{M}[G]$ let $f : A \to B$. Then, there is $F : A \to \mathcal{P}(B)$ in $\mathcal{M}$ such that for all $a \in A$, $f(a) \in F(a)$ and $(|F(a)| \leq \aleph_0)^{\mathcal{M}}$.

PROOF. Thus $\mathcal{M}[G] \models f : A \to B$. By the Truth Lemma there are a $\mathbb{P}$-name $\dot{f}$ in $\mathcal{M}$ and $p \in G$ such that $\dot{f}_G = f$ and $p \Vdash \dot{f} : \check{A} \to \check{B}$. Now, define the function $F : A \to \mathcal{P}(B)$ by

$$F(a) = \{b \in B : \exists q \leq p(q \Vdash \dot{f}(\check{a}) = \check{b})\}.$$

Note that by the Definability Lemma $F \in \mathcal{M}$. Suppose $\mathcal{M}[G] \models f(a) = b$. Then by the Truth Lemma, there is $q \leq p$ such that $q \Vdash f(a) = b$. Then clearly, $b \in F(a)$ and so $\mathcal{M}[G] \models f(a) \in F(a)$.

It remains to verify that $(|F(a)| \leq \aleph_0)^{\mathcal{M}}$ for all $a \in A$. For this we will use the countable chain condition of $\mathbb{P}$. Indeed, for each $b \in F(a)$ we can chose $q_b \leq p$ such that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$. Since forcing is inherited by stronger conditions, if $c \neq b$ then q_c and q_b must be incompatible. However $(\mathbb{P}\text{ is ccc})^{\mathcal{M}}$ and so there are only countable many incompatible conditions below p, i.e. $|F(a)| \leq \aleph_0$. ☐

THEOREM 5.7. *If* $\mathbb{P} \in \mathcal{M}$ *and* $(\mathbb{P}\text{ is ccc})^{\mathcal{M}}$, *then* $\mathbb{P}$ *preserves cofinalities and hence preserves cardinals.*

PROOF. Let $\beta \in o(\mathcal{M})$, $(\beta\text{ regular})^{\mathcal{M}}$. Suppose $(\beta\text{ is not regular })^{\mathcal{M}[G]}$. Thus, there is $X \subseteq \beta$ such that $X \in \mathcal{M}[G]$ such that $\sup(X) = \beta$ and $\text{type}(X) = \alpha < \beta$. Then, in $\mathcal{M}[G]$ let $f : \alpha \to X$ be the unique order preserving bijection. In particular $f : \alpha \to \beta$ and by the Approximation Lemma, there is $F \in \mathcal{M}$ such that $F : \alpha \to \mathcal{P}(\beta)$, such that $\forall \xi \in \alpha(|F(\xi)| \leq \aleph_0)$ and $\mathcal{M}[G] \models \forall \xi \in \alpha(f(\xi) \in F(\xi))$. Now, in $\mathcal{M}$ consider the set $Y = \bigcup_{\xi < \alpha} F(\xi)$. Then, $Y \subseteq \beta$ and $\sup Y = \beta$. However $|Y| \leq \aleph_0 \cdot \alpha = \alpha$ and so

$$\mathcal{M} \models |Y| < \beta \wedge \sup(Y) = \beta,$$

i.e. $(\beta\text{ is not regular})^{\mathcal{M}}$, which is a contradiction. Therefore $(\beta\text{ is regular})^{\mathcal{M}[G]}$. ☐

THEOREM 5.8 (A model of $\neg$CH). *Fix* $\alpha < o(\mathcal{M})$ *and let* $\kappa = (\aleph_\alpha)^{\mathcal{M}}$. *Let* $\mathbb{P} = Fn(\kappa \times \omega, 2)$ *and let* G *be a* $\mathbb{P}$*-generic filter over* $\mathcal{M}$. *Then* $(2^{\aleph_0} \geq \aleph_\alpha)^{\mathcal{M}[G]}$.

PROOF. By the Δ-system Lemma $(\mathbb{P}\text{ is ccc})^{\mathcal{M}}$. Thus, $\mathbb{P}$ preserves cofinalities and hence cardinals. Therefore $\kappa = (\aleph_\alpha)^{\mathcal{M}} = (\aleph_\alpha)^{\mathcal{M}[G]}$. For each $\beta < \kappa$ define

$$h_\beta = \bigcup\{p(\beta, n) : p \in G, n \in \omega \text{ s.t. } (\beta, n) \in \text{dom}(p)\}.$$

Then $h_\beta : \omega \to 2$ for each $\beta < \kappa$ and furthermore if $\beta_1 \neq \beta_2$ then $h_{\beta_1} \neq h_{\beta_2}$. Therefore

$$\mathcal{M}[G] \models (2^{\aleph_0} \geq \kappa = \aleph_\alpha).$$

☐

REMARK 5.9. Our next goal is to show that in $(2^{\aleph_0} = \aleph_\alpha)^{\mathcal{M}[G]}$.

DEFINITION 5.10. For $\tau \in V^{\mathbb{P}}$, a nice name for a subset of τ is a name of the form

$$\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau)\}$$

where for all $\sigma \in \operatorname{dom}(\tau)$, the set A_σ is an antichain.

LEMMA 5.11 (Counting nice names). Let $\tau \in V^{\mathbb{P}}$, $\kappa = |\mathbb{P}|$, $\lambda = |\operatorname{dom}(\tau)|$. Assume $\mathbb{P}$ is ccc, κ, λ are infinite. Then, there are no more than κ^λ nice names for subsets of κ.

PROOF. Note that $|[\mathbb{P}]^{\aleph_0}| \leq \kappa^{\aleph_0}$ and so the number of antichains does not exceed $\kappa^{\aleph_0}$. Each nice name for a subset of τ is determined by λ-many antichains and so there are no more than

$$(\kappa^{\aleph_0})^\lambda = \kappa^{\aleph_0 \cdot \lambda} = \kappa^\lambda$$

nice names. □

LEMMA 5.12 (Every subset of a given set has a nice name). Let $\mathbb{P} \in \mathcal{M}$, τ, μ be elements of $\mathcal{M}^{\mathbb{P}}$. Then, there is a nice name $\nu \in \mathcal{M}^{\mathbb{P}}$ for a subset of τ such that

$$1_{\mathbb{P}} \Vdash (\text{if } \mu \subseteq \tau \text{ then } \mu = \nu).$$

PROOF. Consider τ and $\operatorname{dom}(\tau)$. For each $\sigma \in \operatorname{dom}(\tau)$ if there is $p \in \mathbb{P}$ such that $p \Vdash \sigma \in \mu$, fix a maximal antichain of such conditions. Otherwise, take $A_\sigma = \varnothing$. Let

$$\nu = \{\{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau)\}.$$

Fix an $(\mathcal{M}, \mathbb{P})$-generic filter and suppose $\mathcal{M}[G] \vDash \mu_G \subseteq \tau_G$. We will show that $\mathcal{M}[G] \vDash \mu_G = \nu_G$.

First, we show that $\underline{\nu_G \subseteq \mu_G}$: Let $a \in \nu_G$. Then, $a = \sigma_G$, where $\langle \sigma, p\rangle \in \nu$ and $p \in G$. However, $p \Vdash \sigma \in \mu$ (by definition of A_σ) and so $a \in \mu_G$.

Second, we show that $\underline{\mu_G \subseteq \nu_G}$: Suppose $a \in \mu_G \backslash \nu_G$. Then, $a \in \mu_G \subseteq \tau_G$ and so $a = \sigma_G$ for some $\sigma \in \operatorname{dom}(\tau)$. Furthermore, by hypothesis

$$\mathcal{M}[G] \vDash \sigma \in \mu \wedge \sigma \notin \nu.$$

Then, by the Truth Lemma, there is $q \in G$ such that

$$q \Vdash (\sigma \in \mu \wedge \sigma \notin \nu).$$

Thus, $q \Vdash \sigma \in \mu$ and since $q \Vdash \sigma \notin \nu$, we must have that q is incomaptible with every $p \in A_\sigma$ (otherwise, for $r \leq q, p$, we get $r \Vdash \sigma \in \nu$ which is a contradiction). Thus, we reached a contradiction to the hypothesis that A_σ is maximal. □

LEMMA 5.13 (Upper bound). Fix $\mathbb{P} \in \mathcal{M}$ and assume that in $\mathcal{M}$ the forcing notion $\mathbb{P}$ is ccc, κ, λ and δ are infinite cardinals, $\kappa = |\mathbb{P}|$, $\delta = \kappa^\lambda$. Let G be $(\mathcal{M}, \mathbb{P})$-generic. Then

$$(2^\lambda \leq \delta)^{\mathcal{M}[G]}.$$

PROOF. The name $\check{\lambda} = \{\langle \check{\xi}, 1_{\mathbb{P}}\rangle : \xi \in \lambda\}$, $|\check{\lambda}| = \lambda$. By the previous Lemma, there are no more than κ^λ many nice names for subsets of λ and so we can list them as $\langle \nu_\zeta : \zeta < \delta\rangle$. Let $\dot{f}$ be the following name:

$$\dot{f} = \{\langle \operatorname{op}(\check{\zeta}, \nu_\zeta), 1_{\mathbb{P}}\rangle : \zeta < \delta\},$$

where $\mathrm{op}(\check{\zeta}, \nu_\zeta) = \mathrm{up}(\mathrm{up}(\check{\zeta}, \check{\zeta}), \mathrm{up}(\check{\zeta}, \nu_\zeta))$. In $\mathcal{M}[G]$, $\mathrm{dom}(\dot{f}_G) = \delta$ and $\dot{f}_G(\zeta) = (\nu_\zeta)_G$. If $\mathcal{M}[G] \models s \subseteq \lambda$, then $s = \mu_G$ for some μ and so there is $\zeta < \delta$ such that

$$1_{\mathbb{P}} \Vdash (\mu \subseteq \check{\lambda} \to \mu = \nu_\zeta).$$

Therefore $\mathcal{M}[G] \models \dot{f}_G(\zeta) = s$ and so $\mathcal{M}[G] \models \mathcal{P}(\lambda) \subseteq \mathrm{ran}(\dot{f}_G)$. Therefore $\mathcal{M}[G] \models 2^\lambda \leq \delta$. Since $\mathbb{P}$ is ccc, $\mathbb{P}$ preserves cardinals ans so $(\delta \text{ is a cardinal})^{\mathcal{M}[G]}$. □

THEOREM 5.14. *Let $\alpha < o(\mathcal{M})$ and let $(\kappa = \aleph_\alpha)^{\mathcal{M}}$. Let $\mathbb{P} = Fn(\kappa \times \omega, 2)$ and let G be $\mathbb{P}$-generic over $\mathcal{M}$. Then $(2^{\aleph_0} = \aleph_\alpha = \kappa)^{\mathcal{M}[G]}$.*

PROOF. By a previous result $(2^{\aleph_0} \geq \aleph_\alpha = \kappa)^{\mathcal{M}[G]}$ and by the previous Lemma, $(2^{\aleph_0} \leq \kappa)^{\mathcal{M}[G]}$. □

6. The Forcing Star Relation: Truth and Definability

Our goal in this section is to prove the Truth and Definability Lemmas. To do this, we will introduce a relation between the elements of a given partial order $\mathbb{P}$ and the formulas in $\mathcal{FL}_{\mathbb{P}}$ which will be definable and is in a very strong sense equivalent to the forcing relation. We will refer to this definable relation as the *forcing star relation* and will denote it $\Vdash^*$. First we will introduce the forcing star relation between the elements of $\mathbb{P}$ and the atomic formulas of $\mathcal{FL}_{\mathbb{P}}$ by recursion on a well-founded and set-like relation $\mathcal{R}$ on the class $\mathbb{P} \times \mathcal{FL}_{\mathbb{P}}$. After we establish some basic properties of the so defined (fragment of the) forcing star relation, we will extend its definition to all formulas of the forcing language, by induction on complexity of the formulas.

We start with paying a special attention to the atomic formulas of $\mathcal{FL}_{\mathbb{P}}$.

DEFINITION 6.1. Let $\mathcal{AL}_{\mathbb{P}}$ denote the class of all atomic sentences in $\mathcal{FL}_{\mathbb{P}}$. That is, $\mathcal{AL}_{\mathbb{P}}$ consists of all formulas of the form $\tau = \nu$ and $\pi \in \tau$ for τ, π, ν in $V^{\mathbb{P}}$.

Now, we give the definition of the forcing star relation for atomic formulas.

DEFINITION 6.2. For a partial order $\mathbb{P}$ and τ, ν, π in $V^{\mathbb{P}}$ define:

(1) $p \Vdash^* \tau = \nu$ iff for all $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\nu)$ and all $q \leq p$ we have
$$q \Vdash^* \sigma \in \tau \text{ iff } q \Vdash^* \sigma \in \nu.$$
(2) $p \Vdash^* \pi \in \nu$ iff the set
$$\{q \leq p : \exists \langle \sigma, r \rangle \in \nu (q \leq r \text{ and } q \Vdash^* \pi = \sigma)\}$$
is dense below p.

To justify that the above notion is well-defined, we will make use of the following relations $\mathcal{R}$.

REMARK 6.3. The definition of the forcing star relation above is done by recursion on $\mathcal{R}$, where $\mathcal{R}$ is a relation on $\mathbb{P} \times \mathcal{AL}_{\mathbb{P}}$ defined as follows. Fix $\sigma_1, \tau_1, \sigma_2, \tau_2$ in $V^{\mathbb{P}}$ and p_1, p_2 in $\mathbb{P}$ and define:

(1) $(p_1, \sigma_1 \in \tau_1) \mathcal{R} (p_2, \sigma_2 = \tau_2)$ iff $\big(\sigma_1 \in \mathrm{trcl}(\sigma_2) \text{ or } \sigma_1 \in \mathrm{trcl}(\tau_2)\big)$ and $\big(\tau_1 = \sigma_2 \text{ or } \tau_1 = \tau_2\big)$.
(2) $(p_1, \sigma_1 = \tau_1) \mathcal{R} (p_2, \sigma_2 \in \tau_2)$ iff $\sigma_1 = \sigma_2$ and $\tau_1 \in \mathrm{trcl}(\tau_2)$.

(3) Neither $(p_1, \sigma_1 \in \tau_1)\mathcal{R}(p_2, \sigma_2 \in \tau_2)$, nor $(p_1, \sigma_1 = \tau_1)\mathcal{R}(p_2, \sigma_2 = \tau_2)$.

Note that $\mathcal{R}$ is set-like, because $\mathbb{P}$ is a set. Moreover, we will show that $\mathcal{R}$ is well-founded. Proceed as follows. Define

$$\rho(p, \sigma = \tau) = \rho(p, \sigma \in \tau) = \max\{\operatorname{rank}(\sigma), \operatorname{rank}(\tau)\}$$

and observe that if $(p_1, \varphi_1)\mathcal{R}(p_2, \varphi_2)$ then $\rho(p_1, \varphi_1) \leq \rho(p_2, \varphi_2)$. Furthermore, let

$$\chi : \mathbb{P} \times \mathcal{AL}_{\mathbb{P}} \to \{0, 1, 2\}$$

be defined via

$$\chi(p, \varphi) = \begin{cases} 1 & \text{if } \varphi \text{ is of the form } \sigma = \tau \\ 0 & \text{if } \varphi \text{ is of the form } \sigma \in \tau \text{ and } \operatorname{rank}(\sigma) < \operatorname{rank}(\tau) \\ 2 & \text{if } \varphi \text{ is of the form } \sigma \in \tau \text{ and } \operatorname{rank}(\sigma) \geq \operatorname{rank}(\tau). \end{cases}$$

Now, define $\Phi : \mathbb{P} \times \mathcal{AL}_{\mathbb{P}} \to \mathbb{ON}$ as follows:

$$\Phi(p, \varphi) = 3 \cdot \rho(p, \varphi) + \chi(p, \varphi).$$

It remains to observe that if $(p_1, \varphi_1)\mathcal{R}(p_2, \varphi_2)$ then $\Phi(p_1, \varphi_1) < \Phi(p_2, \varphi_2)$. Thus, by an earlier Lemma, the relation $\mathcal{R}$ is indeed well-founded.

We continue by establishing some basic properties of $\Vdash^*$.

Lemma 6.4 (Properties of $\Vdash^*$ for Atomic Sentences). For $\varphi \in \mathcal{AL}_{\mathbb{P}}$:

(1) If $p \Vdash^* \varphi$ and $p_1 \leq p$, then $p_1 \Vdash^* \varphi$.
(2) $p \Vdash^* \varphi$ iff $\{p_1 \leq p : p_1 \Vdash^* \varphi\}$ is dense below p.

Proof. Note that item (1) holds by definition. The direction ($\Rightarrow$) of item (2) holds by (1), since $\{p_1 \leq p : p_1 \Vdash^* \varphi\} = \{p_1 \in \mathbb{P} : p_1 \leq p\}$.

To see ($\Leftarrow$) of item (2), consider an arbitrary formula φ of the form $\pi \in \tau$. Let

$$\Delta(t, \pi \in \tau) = \{t' \leq t : \exists \langle \sigma, t'' \rangle \in \tau (t' \leq t'' \wedge t' \Vdash^* \pi = \sigma)\}.$$

Then, by definition $p \Vdash^* \pi \in \tau$ iff $\Delta(p, \pi \in \tau)$ is dense below p. Suppose

$$\{p_1 \leq p : \Delta(p_1, \pi \in \tau) \text{ is dense below } p_1\}$$

is dense below p. That is, for every $q \leq p$ there is $q' \leq q$ such that $\Delta(q', \pi \in \tau)$ is dense below q'. Therefore $\Delta(p, \pi \in \tau)$ is dense below p and so $p \Vdash^* \pi \in \tau$. □

Next, we extend the definition of the forcing star relation to the class of all negations of atomic formulas, and so we obtain the relation for all basic formulas of the language.

Definition 6.5 (Forcing Star for all Basic Formulas). For $\varphi \in \mathcal{AL}_{\mathbb{P}}$, $p \in \mathbb{P}$ define

$$p \Vdash^* \neg\varphi \text{ iff } \neg\exists q \leq p (q \Vdash^* \varphi).$$

As an immediate corollary of Lemma 6.4 and the above definition we obtain:

Corollary 6.6. For $\varphi \in \mathcal{AL}_{\mathbb{P}}$, $p \in \mathbb{P}$ we have

$$p \Vdash^* \varphi \text{ iff } \neg\exists q \leq p (q \Vdash^* \neg\varphi).$$

PROOF. ($\Rightarrow$) Take p such that $p \Vdash^* \varphi$. Suppose $q \leq p$ and $q \Vdash^* \neg\varphi$. Then, by definition

$$\neg\exists q' \leq q(q' \Vdash^* \varphi).$$

However for every extension q'' of q we have $q'' \leq p$ and so by by Lemma 6.4.(1), $q'' \Vdash^* \varphi$, which is a contradiction.

($\Leftarrow$) By definition of $p \Vdash^* \neg\neg\varphi$, since $\neg\neg\varphi$ is equivalent to φ (indeed, by definition $p \Vdash^* \neg\neg\varphi$ iff $\neg\exists q \leq p(q \Vdash^* \neg\varphi)$). $\square$

LEMMA 6.7 (Forcing Star Lemma for Atomic Sentences). Let $\mathcal{M}$ be a ctm of ZF-P, $\mathbb{P} \in \mathcal{M}$. Let $\varphi \in \mathcal{AL}_{\mathbb{P}} \cap \mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic filter. Then:

(1) If $p \in G$ and $(p \Vdash^* \varphi)^{\mathcal{M}}$ then $\mathcal{M}[G] \vDash \varphi$.
(2) If $\mathcal{M}[G] \vDash \varphi$ then there is $p \in G$ such that $(p \Vdash^* \varphi)^{\mathcal{M}}$.

PROOF. We proceed by induction on $\mathrm{rank}_{\mathcal{R},\mathbb{P}\times\mathcal{AL}_{\mathbb{P}}}$.

(1) Let $p \in G$. We have two cases to consider: φ is $\pi \in \tau$ and φ is $\tau = \nu$.

Suppose φ is $\pi \in \tau$. That is $p \Vdash^* \pi \in \tau$. Consider the set

$$\Delta(p, \pi \in \tau) = \{q \leq p : \exists\langle\sigma, r\rangle \in \tau(q \leq r \wedge q \Vdash^* \pi = \sigma)\}.$$

Then $\Delta(p, \pi \in \tau) \in \mathcal{M}$ and $\Delta(p, \pi \in \tau)$ is dense below p by definition of $\Vdash^*$. Since G is $(\mathcal{M}, \mathbb{P})$-generic, we can fix $q \in G \cap \Delta(p, \pi \in \tau)$. So, there is $\langle\sigma, r\rangle \in \tau$ such that $q \leq r$ and $q \Vdash^* \pi = \sigma$. Note that $(q, \pi = \sigma)\mathcal{R}(p, \pi \in \tau)$ and so we can apply the Inductive Hypothesis to $q \Vdash^* \pi = \sigma$. Thus, $\mathcal{M}[G] \vDash \pi = \sigma$ and so $\pi_G = \sigma_G$. On the other hand $r \in G$ (as G is upwards closed) and so by definition of evaluation of names $\sigma_G \in \tau_G$. Therefore $\pi_G \in \tau_G$, i.e. $M[G] \vDash \pi \in \tau$, as we wanted.

Suppose φ is $\tau = \nu$ and $p \Vdash^* \tau = \nu$. We will prove that $\mathcal{M}[G] \vDash (\tau_G \subseteq \nu_G$ and $\nu_G \subseteq \tau_G)$. We will show that $\tau_G \subseteq \nu_G$. Take any $\sigma_G \in \tau_G$. Thus, there is $r \in G$ such that $\langle\sigma, r\rangle \in \tau$. Let $q \in G(q \leq p, r)$. Then since $q \leq r$, we obtain that $\Delta(q, \sigma \in \tau)$ is dense below q and so by definition $q \Vdash^* \sigma \in \tau$. Moreover, by Lemma 6.4.(1) we have that $q \Vdash^* \tau = \nu$. Again by definition of $\Vdash^*$ we obtain $q \Vdash^* \sigma \in \nu$. Since $(q, \sigma \in \mu)\mathcal{R}(p, \tau = \nu)$, we can apply the inductive hypothesis and obtain $\mathcal{M}[G] \vDash \sigma \in \nu$. Therefore $\sigma_G \in \nu_G$. The proof of $\nu_G \subseteq \tau_G$ is similar.

(2) Suppose $\mathcal{M}[G] \vDash \pi_G \in \tau_G$. We need to find $p \in G$ such that

$$\Delta(p, \pi \in \tau) = \{q \leq p : \exists\langle\sigma, r\rangle \in \tau(q \leq r \wedge q \Vdash^* \pi = \sigma)\}$$

is dense below p. By definition of the evaluation of names, we can find $r \in G$ and $\langle\sigma, r\rangle \in \tau$ such that $\pi_G = \sigma_G$. By the inductive hypothesis there is $p \in G$ such that $p \Vdash^* \pi = \sigma$. Without loss of generality $p \leq r$. But then for every $q \leq p$ we have that the pair $\langle\sigma, r\rangle$ is a witness to $q \in \Delta(p, \pi \in \tau)$ and so $\Delta(p, \pi \in \tau)$ is dense below p. Thus $p \Vdash^* \pi \in \tau$.

Suppose $\mathcal{M}[G] \vDash \tau_G = \nu_G$. Recall that $p \Vdash^* \tau = \nu$ iff for every $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\nu)$ and for every $q \leq p$ ($q \Vdash^* \sigma \in \tau$ iff $q \Vdash^* \sigma \in \nu$). Consider, the set $D \subseteq \mathbb{P}$ of all $p \in \mathbb{P}$ such that

- either $p \Vdash^* \tau = \nu$,
- or there is $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\nu)$ such that $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma \notin \nu$,
- or there is $\sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\nu)$ such that $p \Vdash^* \sigma \notin \tau$ and $p \Vdash^* \sigma \in \nu$.

Then $D \in \mathcal{M}$ and D is dense. Let $p \in G \cap D$. If $p \Vdash^* \tau = \nu$, we are done. Suppose there is $\sigma \in \operatorname{dom}(\tau) \cup \operatorname{dom}(\nu)$ such that $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma \notin \nu$. By Part (1), $\mathcal{M}[G] \vDash \sigma \in \tau$ and so by definition of evaluation of names $\sigma_G \in \tau_G$ and since $\tau_G = \nu_G$ we also get $\sigma_G \in \nu_G$. By the Inductive Hypothesis of item (2), there is $q \in G$ such that $q \Vdash^* \sigma \in \nu$. Since $q, p \in G$ there is $r \in G$ such that $r \leq p, q$. But, then $r \Vdash^* \sigma \in \nu$, contradicting that $p \Vdash^* \neg(\sigma \in \nu)$, which by Corollary 6.6 is equivalent to $\neg \exists q' \leq p(q' \Vdash^* \sigma \in \nu)$. $\square$

LEMMA 6.8 (Equivalence of $\Vdash$ and $\Vdash^*$ for Atomic Sentences). Let $\mathcal{M}$ be a ctm for ZF-P with $\mathbb{P} \in \mathcal{M}$. For $p \in \mathcal{M}$, $\varphi \in \mathcal{AL}_{\mathbb{P}} \cap \mathcal{M}$,

$$p \Vdash \varphi \text{ iff } (p \Vdash^* \varphi)^{\mathcal{M}}.$$

PROOF. ($\Leftarrow$) If $(p \Vdash^* \varphi)^{\mathcal{M}}$. Then by Lemma 6.7 for every $(\mathcal{M}, \mathbb{P})$-generic filter G such that $p \in G$, we have $\mathcal{M}[G] \vDash \varphi$. However, by definition this is exactly $p \Vdash \varphi$.

($\Rightarrow$) Suppose by way of contradiction that $p \Vdash \varphi$ and $(p \nVdash^* \varphi)^{\mathcal{M}}$. Then by Corollary 6.6 there is $q \leq p$ such that $q \Vdash^* \neg\varphi$ and so by definition of $\Vdash^*$, $\neg \exists r \leq q(r \Vdash^* \varphi)$. Take an $(\mathcal{M}, \mathbb{P})$-generic filter G such that $q \in G$. Then $p \in G$ and so $\mathcal{M}[G] \vDash \varphi$. By the previous Lemma, there is $s \in G$ such that $(s \Vdash^* \varphi)^{\mathcal{M}}$. Since $q, s \in G$ there is $r \in G$ such that $r \leq q$ and $r \leq s$. However $\Vdash^*$ is inherited by stronger conditions and so $(r \Vdash^* \varphi)^{\mathcal{M}}$. Since φ is atomic, we have $(r \Vdash^* \varphi)^{\mathcal{M}}$ iff $r \Vdash^* \varphi$ and so $r \Vdash^* \varphi$. Thus, since $r \leq q$, we reached a contradiction. $\square$

Next, we extend the forcing star relation to the class of all formulas of the forcing language.

DEFINITION 6.9. Let $\mathbb{P}$ be a forcing notion, $\varphi \in \mathcal{FL}_{\mathbb{P}}$. Then

(1) $p \Vdash^* \neg\varphi$ iff $\neg \exists q \leq p(q \Vdash^* \varphi)$.
(2) $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
(3) $p \Vdash^* \varphi \to \psi$ iff $\neg \exists q \leq p(q \Vdash^* \varphi$ and $q \Vdash^* \neg\psi)$.
(4) $p \Vdash^* \varphi \vee \psi$ iff $\{q \in \mathbb{P} : (q \Vdash^* \varphi)$ or $(q \Vdash^* \psi)\}$ is dense below p.
(5) $p \Vdash^* \varphi \leftrightarrow \psi$ iff $\neg \exists q \leq p(q \Vdash^* \varphi$ and $q \Vdash^* \neg\psi)$, and $\neg \exists q \leq p(q \Vdash^* \psi$ and $q \Vdash^* \neg\varphi)$.
(6) $p \Vdash^* \forall x \varphi(x)$ iff $p \Vdash^* \varphi(x)$ for all $\tau \in V^{\mathbb{P}}$.
(7) $p \Vdash^* \exists x \varphi(x)$ iff $\{q : \exists \tau \in V^{\mathbb{P}}(q \Vdash^* \varphi(\tau))\}$ is dense below p.

We extend the properties we observed in Lemma 6.4 and Corollary 6.6 to all of $\mathcal{FL}_{\mathbb{P}}$.

LEMMA 6.10. (Properties of $\Vdash^*$) For $\varphi \in \mathcal{FL}_{\mathbb{P}}$:

(1) If $p \Vdash^* \varphi$ and $p_1 \leq p$, then $p_1 \Vdash^* \varphi$.
(2) $p \Vdash^* \varphi$ iff $\{p_1 \leq p : p_1 \Vdash^* \varphi\}$ is dense below p.
(3) $p \Vdash^* \varphi$ iff $\neg \exists q \leq p(q \Vdash^* \neg\varphi)$.

PROOF. By induction on the formulas. $\square$

LEMMA 6.11 (Forcing Star Lemma). Let $\mathcal{M}$ be a ctm for ZF-P, $\mathbb{P} \in \mathcal{M}$, $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic filter. Then:

(a) If $p \in G$ and $(p \Vdash^* \varphi)^{\mathcal{M}}$ then $\mathcal{M}[G] \vDash \varphi$.
(b) If $\mathcal{M}[G] \vDash \varphi$ then there is $p \in G$ such that $(p \Vdash^* \varphi)^{\mathcal{M}}$.

PROOF. By induction on the formula φ, we will prove the statement $\mathcal{L}(\varphi) = (a)_\varphi \wedge (b)_\varphi$, where

$(a)_\varphi$ If $p \in G$ and $(p \Vdash^* \varphi)^{\mathcal{M}}$ then $\mathcal{M}[G] \vDash \varphi$.
$(b)_\varphi$ If $\mathcal{M}[G] \vDash \varphi$ then there is $p \in G$ such that $(p \Vdash^* \varphi)^{\mathcal{M}}$.

Suppose $L(\varphi)$ holds. We will show $L(\neg\varphi)$. To show $(a)_{\neg\varphi}$, take any $p \in G$ such that $(p \Vdash^* \neg\varphi)^{\mathcal{M}}$ and suppose by way of contradiction that $\mathcal{M}[G] \vDash \varphi$. Then by $(b)_\varphi$, we have that there is $r \in G$ such that $(r \Vdash^* \varphi)^{\mathcal{M}}$. Since $p, r \in G$ there is $q \in G$ such that $q \leq p, r$. Then $q \Vdash^* \varphi$ (since $q \leq r$). Since $q \leq p$, we get a contradiction to $(p \Vdash^* \neg\varphi)^{\mathcal{M}}$. Next we will show $(b)_{\neg\varphi}$. Since $\Vdash^*$ is a definable relation, the set

$$D = \{p \in \mathbb{P} : (p \Vdash^* \varphi)^{\mathcal{M}} \text{ of } (p \Vdash^* \neg\varphi)^{\mathcal{M}}\} \in \mathcal{M}.$$

By definition of $\Vdash^* \neg\varphi$, D is dense. Now, suppose $\mathcal{M}[G] \vDash \neg\varphi$, and let $p \in G \cap D$. If $(p \Vdash^* \neg\varphi)^{\mathcal{M}}$, we are done. Otherwise, $(p \Vdash^* \varphi)^{\mathcal{M}}$ and so by $(a)_\varphi$, $\mathcal{M}[G] \vDash \varphi$, which is a contradiction to the hypothesis $\mathcal{M}[G] \vDash \neg\varphi$.

Suppose, we have show $\forall \tau \in \mathcal{M}^{\mathbb{P}}(L(\varphi(\tau)))$. We will prove $L(\exists x \varphi(x))$. To see $(a)_{\exists x \varphi(x)}$, suppose $p \in G$ and $(p \Vdash^* \exists x \varphi(x))^{\mathcal{M}}$. Then, by definition

$$D = \{q : \exists \tau \in \mathcal{M}^{\mathbb{P}}(q \Vdash^* \varphi(\tau))^{\mathcal{M}}\}$$

is dense below p. Thus $G \cap D \neq \varnothing$ and so $\exists q \in G \cap D$ such that $q \Vdash^* \varphi(\tau))^{\mathcal{M}}$ for some $\tau \in \mathcal{M}^{\mathbb{P}}$. By hypothesis, $\mathcal{M}[G] \vDash \varphi(\tau)$ and so $\mathcal{M}[G] \vDash \exists x \varphi(x)$. To see $(b)_{\exists x \varphi(x)}$, note that if $\mathcal{M}[G] \vDash \exists x \varphi(x)$, then there is $\tau \in \mathcal{M}^{\mathbb{P}}$ such that $\mathcal{M}[G] \vDash \varphi(\tau)$. By part (b) for $\varphi(\tau)$ in the inductive hypothesis, there is $p \in G$ such that $(p \Vdash^* \varphi(\tau))^{\mathcal{M}}$. Then $(p \Vdash^* \exists x \varphi(x))^{\mathcal{M}}$, because for all $q \leq p$, $q \Vdash^* \varphi(\tau)$. $\square$

LEMMA 6.12 (Equivalence of $\Vdash$ and $\Vdash^*$). Let $\mathcal{M}$ be a ctm of ZF-P, $\mathbb{P} \in \mathcal{M}$, $\mathbb{P} \in \mathcal{M}$, $p \in \mathbb{P}$, $\varphi \in \mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$. Then

$$p \Vdash \varphi \text{ iff } (p \Vdash^* \varphi)^{\mathcal{M}}.$$

PROOF. Analogously to the case for atomic formulas. $\square$

On the basis of Lemma 6.11 and Lemma 6.12, we can complete the proofs of the Truth and Definability Lemmas.

LEMMA (Truth Lemma). Let $\mathcal{M}$ be a c.t.m. for ZF-P, $\mathbb{P} \in \mathcal{M}$ a forcing notion, ψ a sentence of $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ and let G be an $(\mathcal{M}, \mathbb{P})$-generic filter. Then

$$\mathcal{M}[G] \vDash \psi \text{ iff } \exists p \in G(p \Vdash \psi).$$

PROOF. By Lemma 6.11, $\mathcal{M}[G] \vDash \psi$ iff there is $p \in G$ such that $p \Vdash^* \psi$. By Lemma 6.12,

$$p \Vdash^* \psi \text{ iff } p \Vdash \psi$$

and so $\mathcal{M}[G] \vDash \psi$ iff $\exists p \in G(p \Vdash \psi)$. $\square$

LEMMA (The Definability Lemma). Let $\mathcal{M}$ be a ctm for ZF-P, let $\varphi(x_1,\cdots,x_n)$ be a formula in $\mathcal{L}_\in$, with all free variables shown. Then, the set of all finite tuples $(p,\mathbb{P},\leq_\mathbb{P},1_\mathbb{P},\nu_1,\cdots,\nu_n)$ where $(\mathbb{P},\leq,1_\mathbb{P})$ is a forcing notion, $p\in\mathbb{P}$, $(\mathbb{P},\leq_\mathbb{P},1_\mathbb{P})\in\mathcal{M}$, $\nu_1,\cdots,\nu_n$ are elements of $\mathcal{M}^\mathbb{P}$ and $p\Vdash_{\mathbb{P},\mathcal{M}}\varphi(\nu_1,\cdots,\nu_n)$ is definable over $\mathcal{M}$ without parameters.

PROOF. By Lemma 6.12,

$$p\Vdash_{\mathbb{P},\mathcal{M}}\varphi(\nu_1,\cdots,\nu_n) \text{ iff } (p\Vdash^*\varphi(\nu_1,\cdots,\nu_n))^\mathcal{M}.$$

However $(p\Vdash^*\varphi(\nu_1,\cdots,\nu_n))^\mathcal{M}$ is definable over $\mathcal{M}$. □

As a corollary, we obtain:

COROLLARY 6.13. For any forcing notion $\mathbb{P}\in\mathcal{M}$, names $\tau,\nu,\pi\in\mathcal{M}^\mathbb{P}$:

(1) $p\Vdash\tau=\nu$ iff $\forall\sigma\in\mathrm{dom}(\tau)\cup\mathrm{dom}(\nu)\forall q\leq p(q\Vdash\sigma\in\tau \text{ iff } q\Vdash\sigma\in\mu)$.
(2) $p\Vdash\pi\in\tau$ iff $\{q\leq p:\exists\langle\sigma,r\rangle\in\tau(q\leq r \text{ and } q\Vdash\pi=\sigma)\}$ is dense below p.

PROOF. By equivalence of the relations, $\Vdash$ and $\Vdash^*$. □

7. Complete and Dense Embeddings

DEFINITION 7.1 (Complete Embedding). Let $(\mathbb{Q},\leq_\mathbb{Q},\mathbb{1}_\mathbb{Q})$ and $(\mathbb{P},\leq_\mathbb{P},\mathbb{1}_\mathbb{P})$ be forcing posets and $i:\mathbb{Q}\to\mathbb{P}$. Then i is a complete embedding iff

(1) $i(\mathbb{1}_\mathbb{Q})=\mathbb{1}_\mathbb{P}$
(2) for all q_1,q_2 in $\mathbb{Q}$, we have that if $q_1\leq_\mathbb{Q}q_2$ then $i(q_1)\leq_\mathbb{P}i(q_2)$.
(3) for all q_1,q_2 we have that $(q_1\perp_\mathbb{Q}q_2$ iff $i(q_1)\perp_\mathbb{P}i(q_2))$.
(4) If $A\subseteq\mathbb{Q}$ is a maximal antichain in $\mathbb{Q}$, then the image of A under i, i.e. the set $\{i(a): a\in A\}$ is a maximal antichain in $\mathbb{P}$.

DEFINITION 7.2 (Dense Embedding). We say that i is a dense embedding, if items (1) – (3) above hold and $i(\mathbb{Q})$ is a dense subset of $\mathbb{P}$.

DEFINITION 7.3. The partial order $\mathbb{Q}$ is a complete suborder of $\mathbb{P}$, denoted $\mathbb{Q}\lessdot\mathbb{P}$, if

(1) for all $n\in\omega$ and all $q_1,\cdots,q_n$ in $\mathbb{Q}$ if there is $p\in\mathbb{P}$ such that $p\leq q_i$ for all i, then there is $q\in\mathbb{Q}$ such that $q\leq q_i$ for all i.
(2) if $A\subseteq\mathbb{Q}$ is a maximal antichain in $\mathbb{Q}$, then A is a maximal antichain in $\mathbb{P}$.

DEFINITION 7.4. Let $\mathbb{P}$, $\mathbb{Q}$, i satisfy items (1) – (3) from Definition 7.1. Let $p\in\mathbb{P}$. A condition $p^*\in\mathbb{Q}$ is said to be a reduction of p to $\mathbb{Q}$ if for all $q\in\mathbb{Q}$ we have that

$$\text{if } i(q)\perp_\mathbb{P}p \text{ then } q\perp_\mathbb{Q}p^*.$$

REMARK 7.5. We will use the following notation. Let $\mathbb{P}$ be a partial order and let $A\subseteq\mathbb{P}$, $A\neq\varnothing$. Then $p\perp A$ iff for all $a\in A$ we have that $p\perp a$.

LEMMA 7.6 (Characterization of Complete Embeddings). If $\mathbb{Q}$, $\mathbb{P}$, i satisfy items (1) – (3) of Definition 7.1, then i is a complete embedding iff for all $p\in\mathbb{P}$ there is $p^*\in\mathbb{Q}$ which is a reduction of p to $\mathbb{Q}$.

PROOF. ($\Leftarrow$) Suppose for all $p \in \mathbb{P}$ there is $p^* \in \mathbb{Q}$ which is a reduction of p to $\mathbb{Q}$. We will show that whenever A is a maximal antichain in $\mathbb{Q}$ then $\{i(a) : a \in A\}$ is a maximal antichain in $\mathbb{P}$. Suppose by way of contradiction that there is a maximal antichain A in $\mathbb{Q}$ such that the image of A under i is not a maximal antichain in $\mathbb{P}$. Then there is $p \in \mathbb{P}$ such that $p \perp i(a)$ for all $a \in A$. Let $p^* \in \mathbb{Q}$ be a reduction of p. Since A is a maximal antichain in $\mathbb{Q}$, there is $a \in A$ such that $p^* \not\perp a$. However, by hypothesis $i(a) \perp p$ and since p^* is a reduction of p, we must have that $a \perp p^*$, which is a contradiction.

($\Rightarrow$) Suppose $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding and let $p \in \mathbb{P}$. We will find a reduction p^* of p. Consider the collection $\mathcal{P}$ of all $A \subseteq \mathbb{Q}$ such that A is an antichain and $i''A \perp p$. Then $\varnothing \in \mathcal{P}$ and by Zorn's Lemma, there is $A \in \mathcal{P}$ which is maximal under inclusion. Then $i''A \perp p$ and since i is a complete embedding, A is not a maximal antichain in $\mathbb{Q}$ (otherwise, we get a contradiction to property (4)). So, let $p^* \in \mathbb{Q}$ be such that $p^* \perp A$.

We claim that p^* is a reduction of p. Let $q \in \mathbb{Q}$ and suppose $i(q) \perp p$, but $q \not\perp p^*$. Let $q_1 \leq q, p^*$. Then $q_1 \leq p^*$ and since $p^* \perp A$, we must have that $q_1 \perp A$. That is $A \cup \{q_1\}$ is an antichain. On the other hand $i(q_1) \leq i(q)$ and since $i(q) \perp p$, we must have $i(q_1) \perp p$. Therefore $A \cup \{q_1\} \in \mathcal{P}$, which is a contradiction to the maximality of A in $\mathcal{P}$. Thus, for all $q \in \mathbb{Q}$ if $i(q) \perp p$ then $q \perp p^*$ and so p^* is a reduction of p to $\mathbb{Q}$. □

LEMMA 7.7. Let $\mathcal{M}$ be a transitive model of ZFC, $\mathbb{Q}$, $\mathbb{P}$ forcing posets, elements of $\mathcal{M}$. Let $i : \mathbb{Q} \to \mathbb{P}$ be a complete embedding, let $i \in \mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic filter. Then $i^{-1}(G)$ is $(\mathcal{M}, \mathbb{Q})$-generic.

PROOF. Let $D \subseteq \mathbb{Q}$ be a dense subset of $\mathbb{Q}$, $D \in \mathcal{M}$. Fix a maximal antichain $A \subseteq D$ such that $A \in \mathcal{M}$. Then $i''A \subseteq \mathbb{P}$ is a maximal antichain of $\mathbb{P}$ and since i is a complete embedding, we have that

$$D_{i(A)} = \{d \in \mathbb{P} : \exists a' \in i(A)(d \leq a')\}$$

is dense in $\mathbb{P}$. Then $G \cap D_{i(A)} \neq \varnothing$ and so there is $a' \in i(A)$ and there is $d \in G$ such that $d \leq a'$. However G is upwards closed and so $a' \in G$. Then, since $a' = i(a)$ for some $a \in A$, we get $a \in i^{-1}(G) \cap A$ and so $i^{-1}(G) \cap D \neq \varnothing$. Thus, $i^{-1}(G)$ is $(\mathcal{M}, \mathbb{Q})$-generic. □

LEMMA 7.8. If G_1, G_2 are both $(\mathcal{M}, \mathbb{P})$-generic and $G_1 \subseteq G_2$ then $G_1 = G_2$.

PROOF. Fix $p \in G_2$ and let $D = \{r \in \mathbb{P} : r \leq p \vee r \perp p\}$. The set D is dense and $D \in \mathcal{M}$. Since G_1 is $(\mathcal{M}, \mathbb{P})$-generic, there is $r \in G_1 \cap D$. If $r \perp p$, we get a contradiction to the elements of G being pairwise compatible. Then $r \leq p$. Then $p \in G_1$ and so $G_1 \subseteq G_2 \subseteq G_1$. □

REMARK 7.9.

(1) Recall that if $\mathcal{M}$ is a ctm of ZF-P, $\mathbb{P} \in \mathcal{M}$ and G is a filter on $\mathbb{P}$, then $\mathcal{M} \subseteq \mathcal{M}[G]$ and if $\mathcal{N}$ is a transitive ZF-P model with $\mathcal{M} \subseteq \mathcal{N}$, $G \in \mathcal{N}$, then $\mathcal{M}[G] \subseteq \mathcal{N}$.
(2) Now, suppose i is as in Lemma 7.7 and let $H = i^{-1}(G)$, $\mathcal{N} = \mathcal{M}[G]$. Then $i, G \in \mathcal{N}$ and so $i^{-1}(G) = H \in \mathcal{N}$. Therefore $\mathcal{M}[H] \subseteq \mathcal{N} = \mathcal{M}[G]$.

Furthermore, there is a natural inclusion induced by the following correspondence of names.

DEFINITION 7.10. Let $\mathbb{P}$, $\mathbb{Q}$ be forcing posets, $i : \mathbb{Q} \to \mathbb{P}$. Define $i_* : V^{\mathbb{Q}} \to V^{\mathbb{P}}$ by

$$i_*(\tau) = \{\langle i_*(\sigma), i(q)\rangle : \langle \sigma, q\rangle \in \tau\}.$$

REMARK 7.11. Note that i_* is absolute for transitive models.

LEMMA 7.12. Let $\mathcal{M}$ be a transitive model of ZFC, with $\mathbb{P}$, $\mathbb{Q}$ forcing notions in $\mathcal{M}$. Assume $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, $i \in \mathcal{M}$. Let G be $(\mathcal{M}, \mathbb{P})$-generic and let $H = i^{-1}(G)$. Then

(1) For each $\tau \in \mathcal{M}^{\mathbb{Q}}$, $i_*(\tau) \in \mathcal{M}^{\mathbb{P}}$ and $i_*(\tau)_G = \tau_H$.
(2) $\mathcal{M}[H] \subseteq \mathcal{M}[G]$.

PROOF. Straightforward. □

DEFINITION 7.13. If $i : \mathbb{Q} \to \mathbb{P}$ and $H \subseteq \mathbb{Q}$, let

$$\tilde{i}(H) = \{p \in \mathbb{P} : \exists q \in H i(q) \leq p\}.$$

That is, $\tilde{i}(H)$ is the upwards closure of the pointwise image of H under i.

LEMMA 7.14 (Characterisation of Dense Embeddings). Let $\mathcal{M}$ be a transitive model of ZFC, $\mathbb{Q}$, $\mathbb{P}$, i in $\mathcal{M}$. Assume $i : \mathbb{Q} \to \mathbb{P}$ is a dense embedding. Then:

(1) If $H \subseteq \mathbb{Q}$ is $(\mathcal{M}, \mathbb{Q})$-generic and $G = \tilde{i}(H)$, then G is $\mathbb{P}$-generic over $\mathcal{M}$ and $H = i^{-1}(G)$.
(2) If $G \subseteq \mathbb{P}$ is $(\mathcal{M}, \mathbb{P})$-generic and $H = i^{-1}(G)$, then H is $(\mathcal{M}, \mathbb{Q})$-generic and $G = \tilde{i}(H)$.
(3) If items (1) and (2) hold, then $\mathcal{M}[H] = \mathcal{M}[G]$.
(4) $q \Vdash_{\mathbb{Q}} \varphi(\tau_1, \cdots, \tau_n)$ iff $i(q) \Vdash_{\mathbb{P}} \varphi(i_*(\tau_1), \cdots, i_*(\tau_n))$, where $\varphi(x_1, \cdots, x_n)$ is a formula of $\mathcal{L}_\in$, $q \in \mathbb{Q}$ and $\tau_1, \cdots, \tau_n$ are in $\mathcal{M}^{\mathbb{Q}}$.

PROOF. (1) It is easy to see that G is a filter.

CLAIM 7.15. G is $(\mathcal{M}, \mathbb{P})$-generic.

PROOF. Let D be a dense subset of $\mathbb{P}$, $D \in \mathcal{M}$ and let $\tilde{D} = \{q \in \mathbb{P} : \exists d \in D(q \leq d)\}$. That is, $\tilde{D}$ is the closure of D with respect to stronger conditions (we say that $\tilde{D}$ is dense open). Then $\tilde{D} \in \mathcal{M}$. Now, note that $i^{-1}(\tilde{D})$ is dense in $\mathbb{Q}$ and so there is $q \in H \cap i^{-1}(\tilde{D})$, where H is $(\mathcal{M}, \mathbb{Q})$-generic. Then $i(q) \in i(H) \cap \tilde{D}$. But, then there is $d \in D$ such that $i(q) \leq d$ and since $\tilde{i}(H)$ is the upwards closure of $i(H)$ we have that $d \in \tilde{i}(H) \cap D$. Therefore $G = \tilde{i}(H)$ is $(\mathcal{M}, \mathbb{P})$-generic. □

CLAIM 7.16. $H = i^{-1}(G)$.

PROOF. Now $H \subseteq i^{-1}(G)$ and since $i^{-1}(G)$ is also $(\mathcal{M}, \mathbb{Q})$-generic, we must have $H = i^{-1}(G)$. □

(2) $H = i^{-1}(G)$ is $(\mathcal{M}, \mathbb{Q})$-generic. Then by item (1), $\tilde{i}(H)$ is $(\mathcal{M}, \mathbb{P})$-generic. However $G \subseteq \tilde{i}(H)$ (indeed $G = i(i^{-1}(G)) = i(H) \subseteq \tilde{i}(H)$) and so $G = \tilde{i}(H)$.

(3) Since i is a complete embedding, by part (2), $\mathcal{M}[H] \subseteq \mathcal{M}[G]$. Since H, i are in $\mathcal{M}[H]$, we have that $G = \tilde{i}(H) \in \mathcal{M}[H]$. Therefore by the minimality of the forcing extension $\mathcal{M}[G] \subseteq \mathcal{M}[H]$. Thus, $\mathcal{M}[H] = \mathcal{M}[G]$.

(4) Let H, G be as in (1) and (2). That is $G = \tilde{i}(H)$ and $H = i^{-1}(G)$. Then, we have that $q \in H$ if and only if $i(q) \in G$. For each $\mathcal{M}^{\mathbb{Q}}$-name τ, we have $(\tau)_H = (i_*(\tau))_G$ and so $\mathcal{M}[H] = \mathcal{M}[G] \vDash \varphi[(\tau_1)_H, \cdots, (\tau_n)_H]$ if and only if $\mathcal{M}[H] = \mathcal{M}[G] \vDash \varphi[i_*(\tau_1)_G, \cdots, i_*(\tau_N)_G]$. Therefore

$$q \Vdash_{\mathbb{Q}} \varphi(\tau_1, \cdots, \tau_n) \text{ iff } i(q) \Vdash_{\mathbb{P}} \varphi(i_*(\tau_1), \cdots, i_*(\tau_n)).$$

□

8. Maximality Principle

LEMMA 8.1. In $\mathcal{M}$ let $A \subseteq \mathbb{P}$ be an antichain such that for every $q \in A$ there is a $\mathbb{P}$-name σ_q. Then there is a $\mathbb{P}$-name τ such that for all $q \in A$, $q \Vdash \tau = \sigma_q$.

PROOF. Let $q \downarrow = \{p \in \mathbb{P} : p \leq q\}$. In $\mathcal{M}$ define

$$\tau = \bigcup_{q \in A} \{\langle \pi, r \rangle \in \mathrm{dom}(\sigma_q) \times q \downarrow : r \Vdash \pi \in \sigma_q\}.$$

Let $q \in A$ and let G be $(\mathcal{M}, \mathbb{P})$-generic such that $q \in G$. Let

$$\tau_G = \{\pi_G : \pi \in \mathrm{dom}(\sigma_G) \wedge \exists r \in G \cap q \downarrow \text{ s.t. } r \Vdash \pi \in \sigma_q\}.$$

Clearly $\tau_G \subseteq (\sigma_q)_G$. Indeed: If $\pi_G \in \tau_G$ then there is $r \in G$ such that $r \Vdash \pi \in \sigma_q$ and so $\pi_G \in (\sigma_q)_G$. Thus, $\tau_G \subseteq (\sigma_q)_G$. To verify $(\sigma_q)_G \subseteq \tau_G$ consider any $\pi_G \in (\sigma_q)_G$, where $\pi \in \mathrm{dom}(\sigma_q)$. Then by the Truth Lemma there is $r \in G$ such that $r \Vdash \pi \in \sigma_q$ and without loss of generality $r \leq q$ (since $q \in G$). Then $\langle \pi, r \rangle \in \tau$ and so $\pi_G \in \tau_G$. Thus $(\sigma_q)_G \subseteq \tau_G$. □

REMARK 8.2. Recall that $p \Vdash \exists x \varphi(x)$ iff $\{q \leq p : \exists \tau \in \mathcal{M}^{\mathbb{P}}(q \Vdash \varphi(\tau))\}$ is dense below p.

THEOREM 8.3 (Maximality Principle). *Let $\mathcal{M}$ be a ctm and let $\mathbb{P} \in \mathcal{M}$ be a forcing notion, $\varphi(x) \in \mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ with a single free variable x. Then*

$$p \Vdash \exists x \varphi(x) \textit{ iff } \exists \tau \in \mathcal{M}^{\mathbb{P}} p \Vdash \varphi(\tau).$$

PROOF. Note that $(\Leftarrow)$ is clear from the definition of the forcing relation. To show $(\Rightarrow)$ assume $p \Vdash \exists x \varphi(x)$. By the above Remark 8.2, we can find an antichain A which is maximal below p such that

$$\forall q \in A \exists \sigma \in \mathcal{M}^{\mathbb{P}}(q \Vdash \varphi(\sigma)).$$

Now, for all $q \in A$ pick $\sigma_q \in \mathcal{M}^{\mathbb{P}}$ such that $q \Vdash \varphi(\sigma_q)$ and using the above Lemma find $\tau \in \mathcal{M}^{\mathbb{P}}$ such that for all $q \in A$, $q \Vdash \tau = \sigma_q$. Then, in particular, for all $q \in A$, $q \Vdash \varphi(\tau)$.

We claim that $p \Vdash \varphi(\tau)$. Suppose, this is not the case. Then there is $r \leq p$ such that $r \Vdash \neg\varphi(\tau)$. On the other hand $p \Vdash \exists x \varphi(x)$ and since $r \leq p$ we must have $r \Vdash \exists x \varphi(x)$. By Remark 8.2, there is $s \leq r$ and there is $\sigma \in \mathcal{M}^{\mathbb{P}}$ such that $s \Vdash \varphi(\sigma)$. Again, since $s \leq r$, $s \Vdash \neg\varphi(\tau)$ and so $s \perp A$. Then $A \cup \{s\}$ contradicts the maximality of A. □

9. Models where GCH fails first above $\aleph_0$

DEFINITION 9.1. Let I, J be sets, λ a cardinal. Let $\mathrm{Fn}_\lambda(I,J)$ be the partial order of all $p \in [I \times J]^{<\lambda}$ such htat p is a graph of a function with extesnion relation $q \le p$ iff $q \supseteq p$ and $\mathbb{1} = \varnothing$.

EXAMPLE 9.2.

- $\mathrm{Fn}(I,J) = \mathrm{Fn}_\omega(I,J)$ is the poset of finite partial functions from I to J.
- $\mathrm{Fn}_{\aleph_1}(I,J)$ is the poset of countable partial functions from I to J.

REMARK 9.3. For $\lambda > \omega$, the partial order $\mathrm{Fn}_\lambda(I,J)$ is not absolute: take I, J in $\mathcal{M}$ and $(\mathrm{Fn}_\lambda(I,J))^{\mathcal{M}}$.

DEFINITION 9.4 (θ-cc). Let θ be a cardinal. The p.o. $\mathbb{P}$ is said to have the θ-chain condition (shortly θ-*cc*) if in $\mathbb{P}$ every antichain $A \subseteq \mathbb{P}$ is of cardinality strictly smaller than θ.

REMARK 9.5. Thus, in particular, ccc is $\aleph_1$-cc.

LEMMA 9.6. Let $\lambda \ge \omega$. Then $\mathrm{Fn}_\lambda(I,J)$ has the $(|J|^{<\lambda})^{*}$-cc. Thus, whenever $|J| \le 2^{<\lambda}$, $\mathrm{Fn}_\lambda(I,J)$ has the $(2^{<\lambda})^{+}$-cc.

PROOF. Let $\kappa = (|J|^{<\lambda})^{+}$. Without loss of generality $|J| \ge 2$ and so κ is regular, $\kappa > \lambda$. Let $W \subseteq \mathrm{Fn}_\lambda(I,J)$ be an antichain, $|W| = \kappa$. We have to reach a contradiction.

Note that, we can assume that λ is regular. Indeed, if not, then for all $\gamma \in W$, $|p| < \lambda$ and since κ is regular, there is $\sigma < \lambda$ such that $W' = \{p \in W : |p| \le \sigma\}$ is of cardinality κ. Then, taking $\lambda = \sigma^+$ and $W = W'$, it is sufficient to reach a contradiction from the assumption that λ is regular.

Enumerate W as $\{p_\alpha : \alpha < \kappa\}$ and let $S_\alpha = \mathrm{dom}(p_\alpha)$. Then, we can apply the Δ-system Lemma to $\{S_\alpha : \alpha < \kappa\}$ to find $B \subseteq \kappa$, $|B| = \kappa$ such that for all α, β in B, $S_\alpha \cap S_\beta = R$ for some $R \subseteq I$, $|R| < \lambda$. However $|J^R| < \kappa$ and so there are $\alpha \ne \beta$ in B such that $p_\alpha \restriction R = p_\beta \restriction R$. Then $p_\alpha \cup p_\beta \le p_\alpha, p_\beta$, which is a contradiction to $p_\alpha \perp p_\beta$. □

DEFINITION 9.7. Let $\mathbb{P} \in \mathcal{M}$, $\mathcal{M}$ be a ctm, $(\theta \text{ is a cardinal})^{\mathcal{M}}$. We say that:

(1) $\mathbb{P}$ preserves cardinals $\ge \theta$ iff whenever $\theta \le \beta < o(\mathcal{M})$:

$$(\beta \text{ is a cardinal})^{\mathcal{M}} \text{ iff } (\beta \text{ is a cardinal})^{\mathcal{M}[G]}.$$

(2) $\mathbb{P}$ preserves cofinalities $\ge \theta$ iff for all limit $\gamma < o(\mathcal{M})$ such that $\mathrm{cf}^{\mathcal{M}}(\gamma) \ge \theta$:

$$\mathrm{cf}^{\mathcal{M}}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\gamma).$$

REMARK 9.8. We saw the above for $\theta = (\omega_1)^{\mathcal{M}}$. Note also that $\mathrm{cf}^{\mathcal{M}}(\gamma) \ge \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$.

LEMMA 9.9. Let $\mathbb{P} \in \mathcal{M}$ be a forcing notion, $(\theta \text{ is a regular cardinal})^{\mathcal{M}}$.

(1) $\mathbb{P}$ preserves cofinalities $\ge \theta$ if and only if

($*$) for all limit β with $\theta \le \beta < o(\mathcal{M})$, if $(\beta \text{ is regular})^{\mathcal{M}}$ then $(\beta \text{ is regular })^{\mathcal{M}[G]}$.

(2) If $\mathbb{P}$ preserves cofinalities $\ge \theta$, then $\mathbb{P}$ preserves cardinals $\ge \theta$.

REMARK 9.10. The proof is very similar to the case $\theta = (\omega_1)^{\mathcal{M}}$. To conclude (2) from (1) observe that if $\beta > \theta$ is singular, then $\beta = \sup\{r_\gamma : \gamma < \lambda\}$, where $r_\gamma \ge \theta$ is regular for all γ.

LEMMA 9.11. Let $\mathbb{P} \in \mathcal{M}$, ($\theta$ an uncountable cardinal)$^{\mathcal{M}}$, ($\mathbb{P}$ is θ-cc)$^{\mathcal{M}}$. Fix $A, B \in \mathcal{M}$ and let G be $(\mathcal{M}, \mathbb{P})$-generic. Let $f \in \mathcal{M}[G]$, $f : A \to B$. Then, there is $F : A \to \mathcal{P}(B)$ with $F \in \mathcal{M}$ such that for all $a \in A$, $f(a) \in F(a)$ and $(|F(a)| < \theta)^{\mathcal{M}}$.

Using the above Lemma and arguing similarly to the case $\theta = (\omega_1)^{\mathcal{M}}$ one can show:

THEOREM 9.12. *If $\mathbb{P} \in \mathcal{M}$ and (θ is regular)$^{\mathcal{M}}$, ($\mathbb{P}$ is θ-cc)$^{\mathcal{M}}$, then $\mathbb{P}$ preserves cofinalities $\geq \theta$ and hence preserves cardinals $\geq \theta$.*

DEFINITION 9.13. A forcing notion $\mathbb{P}$ is λ-closed iff whenever $\delta < \lambda$ and $\langle p_\xi : \xi < \delta \rangle$ is a sequence in $\mathbb{P}$ such that for all $\xi_1 < \xi_2 < \delta$, $p_{\xi_2} \leq p_{\xi_1}$, then there is $q \in \mathbb{P}$ such that for all $\xi < \delta$, $q \leq p_\xi$. We say that $\mathbb{P}$ is countably closed, if it is ω_1-closed.

LEMMA 9.14. If λ is regular, then $\mathrm{Fn}_\lambda(I, J)$ is λ-closed.

PROOF. Let $\langle p_\xi : \xi < \delta \rangle$, $\delta < \lambda$ be as in the above definition. Then $q = \bigcup p_\xi$ is a common extension. $\square$

THEOREM 9.15. *Let $\mathcal{M}$ be a ctm, $A, B \in \mathcal{M}$, ($\mathbb{P}$ is λ-closed)$^{\mathcal{M}}$, $(|A| < \lambda)^{\mathcal{M}}$. Let G be $(\mathcal{M}, \mathbb{P})$-generic, $f \in \mathcal{M}[G]$, $f : A \to B$. Then $f \in \mathcal{M}$.*

PROOF. It is sufficient to show that $f \in \mathcal{M}$ when $A = \alpha < \lambda$. In the general case, fix $j \in \mathcal{M}$ such that $j : \alpha \to A$ is a bijection and apply the particular case of A being an ordinal to $f \circ j : \alpha \to B$ to show that $f \circ j \in \mathcal{M}$ and so $f \in \mathcal{M}$.

So, without loss of generality $A = \alpha < \lambda$. Let $K := ({}^\alpha B)^{\mathcal{M}} = {}^\alpha B \cap \mathcal{M}$ and $f \in {}^\alpha B \cap \mathcal{M}[G]$. We want to show that $f \in K$. Suppose not. Then there is $\tau \in \mathcal{M}^{\mathbb{P}}$ such that $f = \tau_G$ and $p \in G$ such that that

$$p \Vdash \tau : \check{\alpha} \to \check{B} \wedge \tau \notin \check{K}.$$

Recursively (in $\mathcal{M}$) define sequences $\{p_\eta : \eta \leq \alpha\} \subseteq \mathbb{P}$, $\{z_\eta : \eta < \alpha\} \subseteq B$ such that: $p_0 = p$, $p_\eta \leq p_\xi$ for all $\xi \leq \eta$ and

$$p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta.$$

<u>Successor steps</u> Suppose p_η has been defined. Then $p_\eta \leq p$ and so $p_\eta \Vdash \tau : \check{\alpha} \to \check{B}$. Then, in particular $p_\eta \Vdash \exists x \in \check{B}(\tau(\check{\eta}) = x)$. Then there is $z_\eta \in B$ and $p_{\eta+1} \leq p_\eta$ such that $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$.

<u>Limit steps</u> For η limit, use the fact that $\mathbb{P}$ is λ-closed, to find $p_\eta \leq p_\xi$ for all $\xi < \eta$. Let $g = \langle z_\eta : \eta < \alpha \rangle$, i.e. $g : \alpha \to B$, $g(\eta) = z_\eta$. Note that $g \in \mathcal{M}$ and so $g \in K$. Now, let H be $(\mathcal{M}, \mathbb{P})$-generic such that $p_\alpha \in H$. Then $p \in H$. However $\mathcal{M}[H] \models \tau = \check{g} \in \check{K}$, which is a contradiction. $\square$

THEOREM 9.16. *In $\mathcal{M}$, let $\mathbb{P} = Fn_\lambda(I, J)$ where $\lambda \geq \aleph_0$ is regular, $2^{<\lambda} = \lambda$, $|J| \leq \lambda$. Then $\mathbb{P}$ preserves cofinalities and (hence) cardinals.*

PROOF. Sufficient to show that if (β is regular)$^{\mathcal{M}}$ then (β is regular)$^{\mathcal{M}[G]}$ for all limit β such that $\omega < \beta < o(\mathcal{M})$.

If $\delta \leq \lambda$, then ${}^\delta\lambda \cap \mathcal{M} = {}^\delta\lambda \cap \mathcal{M}[G]$ for all $\delta < \lambda$ and so $\mathrm{cf}^{\mathcal{M}}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$ for all limit $\gamma \leq \lambda$. If $\delta > \lambda$, then $\mathbb{P}$ is λ^+-cc and so $\mathbb{P}$ preserves all cardinals and cofinalities $\geq \lambda^+$. $\square$

Theorem 9.17. *In $\mathcal{M}$, assume $\mathbb{P} = Fn_\lambda(\kappa\times\lambda, 2)$ where κ, λ are cardinals such that $\kappa > \lambda \geq \aleph_0$, λ is regular, $\kappa^\lambda = \kappa$, $2^{<\lambda} = \lambda$. Then $\mathbb{P}$ preserves cofinalities and so cardinals, and $\mathcal{M}[G] \vDash 2^\lambda = \kappa$ where G is $(\mathcal{M},\mathbb{P})$-generic.*

Proof. By the previous theorem, cofinalities and cardinalities are preserved. Let G be $(\mathcal{M},\mathbb{P})$-generic. Then $\bigcup G : \kappa\times\lambda \to 2$ encodes a κ-sequence of pairwise distinct functions in ${}^\lambda 2$. Therefore $\mathcal{M}[G] \vDash 2^\lambda \geq \kappa$. On the other hand, if $A \subseteq \mathbb{P}$ is an antichain, then $|A| \leq \lambda$ and since $|\mathbb{P}| = \kappa^{\leq\lambda} = \kappa$, there are no more than $|[\mathbb{P}]^{\leq\lambda}| = \kappa^\lambda = \kappa$ many antichains in $\mathbb{P}$ and so no more than $\kappa^\lambda = \kappa$ many nice names for subsets of λ. Since every subset of λ in $\mathcal{M}[G]$ has a nice name, we obtain $\mathcal{M}[G] \vDash 2^\lambda \leq \kappa$. Thus, $\mathcal{M}[G] \vDash 2^\lambda = \kappa$.

To see that $(2^\lambda \leq \kappa)^{\mathcal{M}[G]}$ we proceed by counting names. If $A \subseteq \mathbb{P}$ is an antichain, then $|A| \leq \lambda$ and $|\mathbb{P}| = \kappa^{<\lambda} = \kappa$. Therefore, there are no more than $|[\mathbb{P}]^{\leq\lambda}| = \kappa^\lambda = \kappa$ antichains. Therefore there are no more than $\kappa^\lambda = \kappa$ many nice names for subsets of λ. $\square$

Corollary 9.18 (Top Down Approach). Assume there is a countable transitive model for ZFC. Then, there is a ZFC model such that CH holds, $2^{\aleph_1} = \aleph_5$, $2^{\aleph_2} = \aleph_{\omega+1}$ and for all $\theta \geq \aleph_2$, $2^\theta = \max\{\theta^+, \aleph_{\omega+1}\}$.

Proof. (Outline) Assume $\mathcal{M} \vDash$ GCH. Let $\mathbb{P} = \mathrm{Fn}_{\omega_2}(\omega_{\omega+1}\times\omega_2, 2)^{\mathcal{M}}$ and let G be $(\mathcal{M},\mathbb{P})$-generic. Consider $\mathcal{N} = \mathcal{M}[G]$. Then in $\mathcal{N}$, CH holds and $2^{\aleph_1} = \aleph_2$ by the ω_2-closure of $\mathbb{P}$. Furthermore $2^{\aleph_2} = \aleph_{\omega+1}$ (the same analysis as in the general case) and counting names $\forall\theta \geq \aleph_2(2^\theta = \max\{\theta^+, \aleph_{\omega+1}\})$. Let $\mathbb{Q} = (\mathrm{Fn}_{\omega_1}(\omega_5\times\omega_1, 2))^{\mathcal{N}}$ and let H be $(\mathcal{M}[G],\mathbb{Q})$-generic. Note that $\mathbb{Q}$ preserves cofinalities and cardinalities, and $(2^{\aleph_1} = \aleph_5)^{\mathcal{N}[H]}$. Moreover since $\mathbb{Q}$ is ω_1-closed in $\mathcal{N}$, $({}^\omega 2)^{\mathcal{N}[H]} = ({}^\omega 2)^{\mathcal{N}}$ and so $\mathcal{N}[H] \vDash$ CH; Since $\mathcal{N} \vDash 2^{\aleph_2} = \aleph_{\omega+1}$ and $\mathcal{N} \subseteq \mathcal{N}[H]$, and cardinals are preserved, we must have $\mathcal{N}[H] \vDash 2^{\aleph_2} \geq \aleph_{\omega+1}$; To show that $\mathcal{N}[H] \vDash \forall\theta \geq \aleph_2(2^\theta = \max\{\theta^+, \aleph_{\omega+1}\})$ count nice names in $\mathcal{N}$. $\square$

CHAPTER 5

Forcing combinatorics

1. Cohen Forcing

In the following we will consider some properties of Cohen forcing.

DEFINITION 1.1 (Cohen Forcing). Let $\mathbb{C}$ be the partial order consisting of all finite partial functions $p : \omega \to \omega$ with extension relation $q \leq p$ superset. That is q is an extension of p if $q \supseteq p$.

Since $\mathbb{C}$ is a countable partial order, it trivially has the countable chain condition.

1.1. The Cohen generic real in unbounded.

DEFINITION 1.2.

(1) Let ${}^\omega\omega$ be the set of all functions from ω to ω. For f, g in ${}^\omega\omega$ define $f \leq^* g$ if there is a natural number n such that for all $m \geq n$, $f(m) \leq g(m)$. We say that g *eventually dominates* f.
(2) A family $\mathcal{F} \subseteq {}^\omega\omega$ is said to be *dominating* if $\forall g \in {}^\omega\omega \exists f \in \mathcal{G}$ such that $g \leq^* f$.
(3) We let $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq {}^\omega\omega, \mathcal{D}$ is dominating$\}$ and refer to this cardinal value as the *dominating number*.

LEMMA 1.3. $\aleph_0 < \mathfrak{d} \leq \mathfrak{c}$.

PROOF. Easy diagonalization. □

LEMMA 1.4. Assume MA. Let $\mathcal{D} \subseteq {}^\omega\omega$ be such that $|\mathcal{D}| < \mathfrak{c}$. Then $\mathcal{D}$ is not dominating.

PROOF. Consider the partial order $\mathbb{C}$. If $G \subseteq \mathbb{C}$ is a filter, then $f_G = \bigcup G = \bigcup\{p : p \in G\}$ is a partial functions, since the elements of a filter are pairwise compatible. Note that to guarantee that f_G has a full domain, i.e. is a function from ω to ω is is sufficient to guarantee that for each $n \in \omega$ there is $p \in G$ such that $n \in \operatorname{dom}(p)$. Moreover, we have the following:

CLAIM. For each $n \in \omega$ the set $D_n = \{p \in \mathbb{C} : n \in \operatorname{dom}(p)\}$ is dense.

PROOF. Take any $p \in \mathbb{C}$. If $n \in \operatorname{dom}(p)$ then $p \in D_n$. Otherwise, take $q = p \cup \{(n, m)\}$ is in D_n and extends p, where $m \in \omega$ was arbitrary. □

Now, given an arbitrary function $f \in {}^\omega\omega$ in order to guarantee that $f_G \nleq^* f$ it is sufficient to provide that there are infinitely many $m \in \omega$ such that $f(m) < f_G(m)$.

CLAIM. Let $f \in {}^\omega\omega$. Then the set

$$D_{f,n} = \{p \in \mathbb{C} : \exists m > n(p(m) > f(m))\}$$

is dense.

PROOF. Take any $p \in \mathbb{C}$ and let m be a natural number such that $m > n$ and $m \notin \mathrm{dom}(p)$. Then $q = p \cup \{(m, f(m)+1)\} \in D_{f,n}$ and $q \leq p$. □

Consider, the family of $\Delta = \{D_{f,n} : f \in \mathcal{D}, n \in \omega\} \cup \{D_n : n \in \omega\}$. Then $|\Delta| < \mathfrak{c}$ and so by MA there is a filter $G \subseteq \mathbb{C}$ which meets every element of Δ on a non-empty set. Thus, $f_G = \bigcup G$ is function with domain ω which is not dominated by any element of $\mathcal{D}$. □

COROLLARY 1.5. MA implies that $\mathfrak{d} = \mathfrak{c}$.

LEMMA 1.6. Let $\mathcal{M}$ be a ctm, $\mathbb{C} \in \mathcal{M}$ and let G be a $\mathbb{C}$-generic filter over $\mathcal{M}$. Let $f_G = \bigcup G$. Then for every $f \in {}^\omega\omega \cap \mathcal{M}$ we have

$$\mathcal{M}[G] \vDash f_G \nleq^* f.$$

With other words for each $f \in \mathcal{M} \cap {}^\omega\omega$, $1_{\mathbb{C}} \Vdash \dot{f}_G \nleq^* \check{f}$, where $\dot{f}_G$ is a $\mathbb{C}$-name for f_G and $\dot{f}_G \nleq^* \check{f}$ is an abbreviation for a formula of the forcing language. We say that the Cohen real is unbounded.

PROOF. Since for each $n \in \omega$ and each $f \in {}^\omega\omega \cap \mathcal{M}$, the sets D_n and $D_{f,n}$ from Lemma 1.4 are not only dense in $\mathbb{C}$ but also elements of $\mathbb{M}$, by genericity of G we have that G has a non-empty intersection with each of those sets. But, then just as in Lemma 1.4 it is straightforward to show that in $\mathcal{M}[G]$, the function f_G is not eventually dominated by any ground model function $f \in \mathcal{M} \cap {}^\omega\omega$. □

1.2. The Cohen generic real is splitting. Consider the partial order $\mathrm{Fn}(\omega, 2)$ consisting of all finite partial functions from ω to $2 = \{0, 1\}$ with extension relation superset. That is $q \leq p$ iff $q \supseteq p$. If G is a filter in $\mathrm{Fn}(\omega, 2)$ then $f_G : \omega \to 2$ is a (possibly partial) function. If $\mathrm{dom}(f_G) = \omega$, then we f_G is in particular the characteristic function of $a_G = f_G^{-1}(1)$.

DEFINITION 1.7.

(1) Let $a, b \in [\omega]^\omega$. We say that a splits b if both $b \cap a$ and $b \backslash a$ are infinite.
(2) We say that a set $a \subseteq \omega$ is infinite, co-infinite if both a and its complement $\omega \backslash a$ are infinite. Note that if b splits a, then a is infinite co-infinite.
(3) A family $\mathcal{A} \subseteq [\omega]^\omega$ is said to be un-split, if no infinite subset of ω simultaneously splits every element of $\mathcal{A}$.
(4) The least cardinality of an un-split family is denoted $\mathfrak{r}$ and is called *the shattering number*.

LEMMA 1.8. Assume MA. Let $\mathcal{D} \subseteq [\omega]^\omega$ be a family of cardinality strictly smaller than $\mathfrak{c}$. Then $\mathcal{D}$ is not un-split.

PROOF. Consider $\mathrm{Fn}(\omega, 2)$. Let $b \in [\omega]^\omega$ and $n \in \omega$. We will show that the set

$$D_{b,n} = \{p \in \mathrm{Fn}(\omega, 2) : \exists m_1 > n(m_1 \in b \cap p^{-1}(1)) \text{ and } \exists m_2 > n(m_2 \in b \cap p^{-1}(0))\}$$

is dense. Fix any $p \in \mathrm{Fn}(\omega, 2)$. Since $\mathrm{dom}(p)$ is finite and b is infinite, there are $m_1 \neq m_2$ such that m_1, m_2 are not in $\mathrm{dom}(p)$, they are both greater than n and they both belong to b. Take $q = p \cup \{(m_1, 1)\} \cup \{(m_2, 0)\}$. Then $m_1 \in q^{-1}(1) \cap b$ and $m_2 \in q^{-1}(0) \cap b$.

Suppose $G \subseteq \mathrm{Fn}(\omega, 2)$ is a filter such that $G \cap D_n \neq \emptyset$ for each $n \in \omega$, where $D_n = \{p \in \mathrm{Fn}(\omega, 2) : n \in \mathrm{dom}(p)\}$. Thus, $f_G : \omega \to \{0, 1\}$ is a function with $\mathrm{dom}(f_G) = \omega$. Now, suppose in addition that $G \cap D_{b,n} \neq \emptyset$ for all $n \in \omega$. Then in particular, for each $n \in \omega$ there are $m_1, m_2 > n$ such that $m_1 \in f_G^{-1}(1) \cap b$ and $m_2 \in f_G^{-1}(0) \cap b$. Thus each of $f_G^{-1}(1) \cap b$ and $f_G^{-1}(0) \cap b$ contains arbitrarily large natural numbers, which means that they are both infinite. Take $a_G = f_G^{-1}(1)$. Then $\omega \backslash a_G = f_G^{-1}(0)$ and so we showed that a_G splits b.

To complete the proof of the Lemma, consider the family of dense sets

$$\Delta = \{D_{b,n} : b \in \mathcal{D}, n \in \omega\} \cup \{D_n : n \in \omega\}.$$

Since $|\Delta| < \mathfrak{c}$, by MA there is a filter G having a non-empty intersection with each element of Δ. But then $a_G = f_G^{-1}(1)$ splits every element of $\mathcal{D}$ and so $\mathcal{D}$ is not un-split. □

Corollary 1.9. MA implies that $\mathfrak{r} = \mathfrak{c}$.

Lemma 1.10. Let $\mathcal{M}$ be a ctm and let G be $\mathrm{Fn}(\omega, 2)$-generic over $\mathcal{M}$. Then for every $b \in [\omega]^\omega \cap \mathcal{M}$ we have that

$$\mathcal{M}[G] \vDash |b \cap a_G| = |b \cap (\omega \backslash a_G)| = \omega,$$

where $a_G = f_G^{-1}(1)$ for $f_G = \bigcup G$. With other words for each $b \in \mathcal{M} \cap [\omega]^\omega$, we have $1_{\mathrm{Fn}(\omega,2)} \Vdash \dot{a}_G$ splits $\check{b}$, where $\dot{a}_G$ is a $\mathrm{Fn}(\omega, 2)$-name for a_G and $|\check{b} \cap \dot{a}_G| = |\check{b} \cap (\omega \backslash \dot{a}_G)| = \omega$ abbreviates a formula of the forcing language. We say that the Cohen real adds a splitting real.

Proof. Take any $b \in [\omega]^\omega \cap \mathcal{M}$. Then, for every natural number m, the set $D_{b,n}$ is not only dense in $\mathrm{Fn}(\omega, 2)$, but also belongs to $\mathcal{M}$ since it is definable from parameters in $\mathcal{M}$. Since G is generic over $\mathcal{M}$, $G \cap D_{b,n} \neq \emptyset$ for all $n \in \omega$. □

2. Hechler Forcing for Adding a Dominating Real

Definition 2.1.

(1) Let ${}^\omega\omega$ be the set of all functions from ω to ω. We say that g *eventually dominates* f, denoted $f \leq^* g$, if there is $n \in \omega$ such that for all $m \geq n$, $f(m) \leq g(m)$.

(2) A family $\mathcal{F} \subseteq {}^\omega\omega$ is said to be *unbounded* if it is not the case that there is $g \in {}^\omega\omega$ such that $\forall f \in \mathcal{F}(f \leq^* g)$. With other words, $\mathcal{F}$ is unbounded, if for all $g \in {}^\omega\omega \exists f \in \mathcal{F}(f \nleq^* g)$.

(3) Let $\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is unbounded}\}$. We say that $\mathfrak{b}$ is the *bounding number*.

Definition 2.2 (Hechler Forcing for adding a dominating real). Hechler forcing (known also as Hechler forcing for adding a dominating real) is the partial order consisting of all pairs (s, F) where $s \in \omega^{<\omega} = \bigcup_{n\in\omega} {}^n\omega$ and $F \in [{}^\omega\omega]^{<\omega}$ with extension relation $(t, H) \leq (s, F)$ defined as follows:

- t end-extends s (that is if $\mathrm{dom}(t) = m$ and $\mathrm{dom}(s) = n$ then $n \leq m$ and $t \restriction n = s$),
- $H \supseteq F$,
- for all $k \in \mathrm{dom}(t) \backslash \mathrm{dom}(s) \forall f \in F(t(k) > f(k))$,

In our application below we will consider a special variant of Hechler forcing, known as the relativization of Hechler forcing to a family of reals, or as restricted Hechler forcing.

LEMMA 2.3. MA implies that $\mathfrak{b} = 2^{\aleph_0}$.

PROOF. Consider a set $\mathcal{F} \subseteq {}^\omega\omega$ such that $|\mathcal{F}| < \mathfrak{c}$. We aim to show that under MA, $\mathcal{F}$ is not unbounded. Let $\mathbb{H}(\mathcal{F})$ be the restriction of 2.2 to the filter the family $\mathcal{F}$, that is $\mathbb{H}(\mathcal{F})$ consisting of all pairs $(s, F) \in \mathbb{H}$ for which $F \in [\mathcal{F}]^{<\omega}$ with extension relation just as in Definition 2.2. Note that if (s, F) and (t, H) are conditions in $\mathbb{H}(\mathcal{F})$ and $s = t$, then $(s, F \cup H)$ is their common extension. This implies that $\mathbb{H}(\mathcal{F})$ is σ-centered and so in particular ccc (also in fact, Knaster). For a filter G consider the set

$$f_G = \bigcup\{s : \exists F(s, F) \in G\}.$$

Now, if $G \cap D_n \neq \emptyset$ for each $n \in \omega$, where $D_n = \{(s, F) \in \mathbb{H}(\mathcal{F}) : n \in \mathrm{dom}(s)\}$ then f_G is a function with domain ω.

Fix an $f \in \mathcal{F}$ and note that $D_f = \{(s, F) : f \in F\}$ is dense. Indeed, if $(t, H) \in \mathbb{H}(\mathcal{F})$ and $f \notin H$ then $(t, H \cup \{f\})$ is an extension of (t, H) from D_f. Now, suppose $(s, F) \in G \cap D_f$ and f_G has a full domain. Take any $m \in \omega$ such that $m > \max\mathrm{dom}(s)$. Then $m \in \mathrm{dom}(f_G)$ and so by definition of f_G there is some $(t, H) \in G$ such that $m \in \mathrm{dom}(t)$. However (t, H) and (s, F) are compatible, as they belong to a filter. Take $(r, E) \in G$ which is their common extension. Note that $(r, E) \subseteq (s \cup t, H \cup F)$ and that $s \cup t$ is in fact just the set t. Since G is upwards closed $(t, H \cup F) \in G$. But then $f_G(m) = t(m) > f(m)$ by definition of the extension relation and the fact that $(t, H \cup F) \leq (s, F)$.

Now, it remains to find a filter $G \subseteq \mathbb{H}(\mathcal{F})$ which meets all sets $\{D_f\}_{f \in \mathcal{F}}$ and $\{D_n\}_{n \in \omega}$. Since $|\mathcal{F}| < \mathfrak{c}$ and $\mathbb{H}(\mathcal{F})$ is ccc the existence of this filter is guaranteed by Martin Axiom. $\square$

COROLLARY 2.4. Let $\mathcal{M}$ be a ctm and let $\mathcal{M}[G]$ be a $\mathbb{H}$ generic extension of $\mathcal{M}$. Then for every $f \in {}^\omega\omega \cap \mathcal{M}$, we have

$$\mathcal{M}[G] \vDash \check{f} \leq^* \dot{f}_G$$

where $\dot{f}_G$ is a $\mathbb{H}$-name for f_G from the above Lemma and $\check{f} \leq^* \dot{f}_G$ is in fact an abbreviation for a formula in the forcing language. With other words, for each $f \in \mathcal{M} \cap {}^\omega\omega$

$$1_{\mathbb{H}} \Vdash \check{f} \leq^* \dot{f}_G.$$

Thus in the Hechler generic extension the ground model reals are dominated. We also say that Hechler forcing adds a dominating real.

PROOF. Note that, the first paragraph in the above proof shows that $1_{\mathbb{H}} \Vdash \mathrm{dom}(\dot{f}_G) = \omega$, while the second paragraph shows that for each $f \in \mathcal{M} \cap {}^\omega\omega$, for each $(s, F) \in \mathbb{H}$ with $f \in F$ and each $m > \max\mathrm{dom}(s)$, we have $(s, F) \Vdash \check{f}(m) < \dot{f}_G$ and so

$$(s, F) \Vdash \check{f} \leq^* \dot{f}_G.$$

It remains to observe that for $f \in \mathcal{M}$, the set $D_f = \{(s, F) \in \mathbb{H} : f \in F\}$ is not only dense, but also an element of $\mathcal{M}$. $\square$

3. Mathias Forcing Relativized to a Filter

DEFINITION 3.1.

(1) A family $\mathcal{E} \subseteq [\omega]^\omega$ has the *Strong Finite Intersection Property* (abbreviated SFIP) if for every finite $\mathcal{F} \in [\mathcal{E}]^{<\omega}$ the set $\bigcap \mathcal{F}$ is infinite.

(2) Let A, B be in $[\omega]^\omega$. We say that A *is almost contained in* B, denoted $A \subseteq^* B$, if $A \backslash B$ is finite. A set $K \in [\omega]^\omega$ is a *pseudo-intersection* of a family $\mathcal{E} \subseteq [\omega]^\omega$ if for all $Z \in \mathcal{E}$, we have $K \subseteq^* Z$.

(3) The *pseudo-intersection number* $\mathfrak{p}$ is defined as the minimal cardinality of a family $\mathcal{E}$ which has SFIP but no pseudo-intersection.

REMARK 3.2. If $\mathcal{F}$ has SFIP then $\mathcal{F}$ **generates a filter** $\hat{\mathcal{F}}$ defined as the least subset of $[\omega]^\omega$ **containing** $\mathcal{F}$ which is **closed** with respect to **finite intersections** and with respect to **supersets**. That is, the filter generated by $\mathcal{F}$ is the least family $\hat{\mathcal{F}} \subseteq [\omega]^\omega$ such that

- $\mathcal{F} \subseteq \hat{\mathcal{F}}$,
- for all finite $\mathcal{H} \subseteq \hat{\mathcal{F}}$ the intersection $\bigcap \mathcal{H} \in \hat{\mathcal{F}}$,
- for all $A, B \in [\omega]^\omega$ if $A \in \hat{\mathcal{F}}$ and $A \subseteq B$ then $B \in \hat{\mathcal{F}}$.

DEFINITION 3.3 (Mathias Forcing). Let $\mathcal{F} \subseteq [\omega]^\omega$ be a filter and let $\mathbb{M}(\mathcal{F})$ be the partial order of all pairs (s, F) where $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$ and $\max s < \min F$ with extension relation $(t, H) \leq (s, F)$ defined as follows:

- t end-extends s (i.e. s is an initial segment of t) and $t \backslash s \subseteq F$,
- $H \subseteq F$.

LEMMA 3.4. MA implies that $\mathfrak{p} = 2^{\aleph_0}$.

PROOF. Consider a set $\mathcal{F}_0 \subseteq [\omega]^\omega$ such that $|\mathcal{F}_0| < \mathfrak{c}$, $\mathcal{F}_0$ has SFIP and let $\mathcal{F}$ be the filter generated by $\mathcal{F}_0$. We aim to show that MA implies that $\mathcal{F}$ has a pseudo-intersection and so in particular $\mathcal{F}_0$ has a pseudo-intersection. Consider the forcing notion $\mathbb{M}(\mathcal{F})$. Note that if (s, F) and (t, H) are elements of $\mathbb{M}(\mathcal{F})$ and $s = t$, then $(s, F \cap H)$ is their common extension since $F \cap H \in \mathcal{F}$. This implies that $\mathbb{M}(\mathcal{F})$ is σ-centered and so in particular ccc. Take any $F \in \mathcal{F}_0$. We will show that the set $D_F = \{(s, E) \in \mathbb{M}(\mathcal{F}) : E \subseteq F\}$ is dense. Well, take any $(t, A) \in \mathbb{M}(\mathcal{F})$. Then $A \cap F \in F$ and so $(t, A \cap F)$ is an extension of (t, A) from D_F.

Let $G \subseteq \mathbb{M}(\mathcal{F})$ be a filter and let $a_G = \bigcup\{s : \exists E (s, E) \in G\}$. Then for each $n \in \omega$, the set $D_n = \{(t, A) \in \mathbb{M}(\mathcal{F}) : \exists m > n (m \in t)\}$ is dense and so if $G \cap D_n \neq \varnothing$ for all $n \in \omega$, then a_G is an infinite subset of ω.

Consider and $F \in \mathcal{F}_0$ and suppose $(s, A) \in D_F \cap G$. Take any $m \in a_G$ and $m > maxs$. Then, by definition of a_G there is $(t, E) \in G$ such that $m \in t$. But (s, A) and (t, E) being elements of a filter are compatible and so there is $(r, H) \in G$ which is their common extension. Then $s \cup t \subseteq r$ and $(r, H) \leq (s, A)$. Thus $m \in r \backslash s$ and so by definition of the extension relation $m \in A$. But $A \subseteq F$ and so $m \in F$. Therefore $a_G \backslash (\max s + 1) \subseteq F$ and so $a_G \subseteq^* F$.

Thus, to obtain a pseudo-intersection of the family $\mathcal{F}_0$ it is sufficient to find a filter $G \subseteq \mathbb{M}(\mathcal{F})$ which meets every dense set D_n for $n \in \omega$ and every D_F for $F \in \mathcal{F}_0$. Since $|\mathcal{F}_0| < \mathfrak{c}$, the existence of such a filter is guaranteed by MA. ☐

Remark 3.5. One can ask: For which filters $\mathcal{F}$ does $\mathbb{M}(\mathcal{F})$ add a dominating real? This is a very interesting and deep question, which is in the hart of on-going research in set theory. Filters for which $\mathbb{M}(\mathcal{F})$ does not add a dominating real are known as Canjar filter and are subject of continuing research in combinatorial set theory. For a recent survey on the subject, see the Master thesis of my student Lukas Schembecker available at *<www.logic.univie.ac.at/~vfischer>*.

Remark 3.6. It is natural to ask: What if we drop the relativization to $\mathcal{F}$? Indeed, let $\mathbb{M}$ be the partial order of all pairs $(s, F) \in [\omega]^{<\omega} \times [\omega]^{\omega}$ such that $\max s < \min F$ and extension relation as in Definition 3.3. Then $\mathbb{M}$ is a forcing notion, known as Mathias forcing, which has broad applications. However the partial order is not ccc and will be discussed only next semester. Nevertheless, what we can state is the following: If G is $\mathbb{M}$-generic over $\mathcal{M}$ and $a_G = \bigcup\{s : \exists A(s, A) \in G\}$, then for every $a \in \mathcal{M} \cap [\omega]^{\omega}$

$$\mathcal{M}[G] \vDash a_G \subseteq^* a \text{ or } a \subseteq^* \omega \backslash a_G.$$

With other words, for every $A \in \mathcal{M} \cap [\omega]^{\omega}$,

$$1_{\mathbb{M}} \Vdash \dot{a}_G \subseteq^* A \text{ or } \dot{a}_G \subseteq^* \omega \backslash A$$

and we say that *Mathias forcing adds an unsplit real*. Can you express the latter property in terms of dense sets? If (s, A) is arbitrary and $B \in [\omega]^{\omega}$ then either $A \cap B$ or $A \cap (\omega \backslash B)$ is infinite. Thus, either $(s, A \cap B)$ or $(s, A \cap \omega \backslash B)$ is an extension of (s, A). This implies that for every $B \in [\omega]^{\omega}$, the set $D_B = \{(s, A) : A \subseteq B \text{ or } A \subseteq \omega \backslash B\}$ is dense, which completes the proof of the above claim.

Lemma 3.7. Let G be $\mathbb{M}$-generic over $\mathcal{M}$. Then in $\mathcal{M}[G]$ there is a real which eventually dominates every ground model real.

Proof. Let $f \in \mathcal{M} \cap {}^{\omega}\omega$. Without loss of generality f is strictly increasing. For an infinite subset x of ω, we identify x with its enumerating function, i.e. the function such that $x(0) = \min x$ and for each $n \geq 1$, $x(n+1) = \min\{m \in x : x(n) < m\}$. Note that the set $D_f = \{(t, E) \in \mathbb{M} : \forall n \geq |t|, n \in \omega(f(n) < E(n))\}$ is dense in $\mathbb{M}$. Indeed. Consider an arbitrary $(s, A) \in \mathbb{M}$. Since A is infinite, we can find $A_f \subseteq A$ such that for each $n \geq |s|$, $n \in \omega$, $A_f(n) > f(n)$. Then $(s, A_f) \in D_f$ and $(s, A_f) \leq (s, A)$.

Let G be $\mathbb{M}$-generic and let $a_G = \bigcup\{s : \exists A(s, A) \in G\}$. We identify a_G with its enumerating function. Consider any $f \in \mathcal{M} \cap {}^{\omega}\omega$. Then, there is $(s, A) \in D_f \cap G$ and so $a_G \backslash s \subseteq A$. But then for each $n \geq |s|$, $a_G(n) \geq A(n) > f(n)$. Thus $f \leq^* a_G$. $\square$