Forcing and applications on bounding, splitting and almost disjointness

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Abstract

In this bachelor's thesis we develop the method of forcing for proving relative consistencies of *ZFC*. We will apply forcing (mostly Cohen forcing) to prove independence results about the real line. In particular we will consider the classical cardinal invariants of bounding, splitting and almost disjointness. Also we are going to prove similar results for the respective generalized invariants.

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Introduction

In the 1870's Georg Cantor discovered that very ordinary set theoretic principles inevitably lead to different sizes of infinities by means of bijective correspondence. In particular he showed that the set of real numbers cannot be put in a one to one correspondence with the natural numbers, that is to say they are uncountable or, they have size strictly bigger than the naturals. Another observation is that any line segment has already the same size as the whole real number line. Further analysing sizes of various infinite sets of reals, Cantor conjectured that **every** infinite subset of $\mathbb{R}$ is either in bijective correspondence with the natural numbers or with the real line itself. This is the so called Continuum Hypothesis (CH). The problem of whether the Continuum Hypothesis is true, or false, remained unsolved for nearly a century and was a major driving force behind the development of set theory.

In the 1920's Ernst Zermelo and Abraham Fraenkel succeeded to capture the set theoretical principles into a general framework of axioms, the axioms of *ZFC*, which was accepted by the most part of the mathematical community. In this framework the Continuum Hypothesis could be stated formally and the rather vague question of "is *CH* **true**?", was turned into "is *CH* **provable** from *ZFC*?".

In 1938, Gödel ([4]) gave a partial answer by showing that *ZFC* cannot prove the negation of *CH* (unless *ZFC* is inconsistent). But it was only in 1963 that the problem was fully solved by Paul Cohen ([3]). He showed that *ZFC* cannot prove *CH* (unless *ZFC* is inconsistent) and could therefore conclude that *CH* is independent of the axioms of *ZFC*. For this he invented a method which was very different from the ones used before and it was soon realized that it could be applied to solve many open problems from set theory, analysis, general topology, measure theory and even algebra. This is the method of forcing. It is now one of the major tools in set theory.

The general goal of forcing is to prove relative consistency results about the theory of sets, for us this will be *ZFC*. These relative consistency results are meta-theoretical statements about the axiomatic system *ZFC*, that are usually of the form " Con(*ZFC*) $\rightarrow$ Con(*ZFC*+ Δ) " where Δ is some set of statements in the language of set theory. By Gödel's second incompleteness theorem (see [6, IV.5]) we know that *ZFC*, and in particular any weaker system (e.g. Peano Arithmetic), is not strong enough to prove the consistency of *ZFC* unless it is inconsistent itself. That is why the word "relative" is important. We will argue that if *ZFC*+ Δ is inconsistent then already *ZFC* must be inconsistent; and in the end it will be a completely "finitary" argument in the sense that it can actually be carried out in a "low" system as Peano Arithmetic, that is strong enough to talk about concepts as " *ZFC*", "proof" etc... If, assuming *ZFC* to be consistent, a sentence is neither provable nor disprovable we say that it is independent (or more accurately relatively independent).

This thesis is a continuation of the author's first bachelor's thesis [8]. In this paper, we are first going to develop the general method of forcing, but only sketching the most important proofs. Our first goal will be to show the consistency of the negation of *CH* by using Cohen forcing.

The main focus will then lie on the four cardinal characteristics introduced in [8]. These are the bounding number $\mathfrak{b}$, the dominating number $\mathfrak{d}$, the splitting number $\mathfrak{s}$ and the almost-disjointness number $\mathfrak{a}$. We say that f dominates g, written as $g <^* f$, for $f, g \in \omega^\omega$ iff the set of $n \in \omega$ for which $g(n) \geq f(n)$ is finite. $\mathfrak{b}$ is then the least size of a family of functions, that is unbounded with respect to $<^*$. $\mathfrak{d}$ is the least size of a family of functions $\mathcal{D}$ that is dominating with respect to $<^*$, that is, for any $g \in \omega^\omega$, there is a $f \in \mathcal{D}$ so that $g <^* f$. S splits X, for $S, X \subseteq \omega$, iff $X \cap S$ and $X \setminus S$ are infinite. $\mathfrak{s}$ is the least size of a family of subsets of ω that contains a splitting set S for any X. A, B are almost disjoint, for $A, B \subseteq \omega$, iff $A \cap B$ is finite. A family of subsets of ω is called almost disjoint iff all elements are pairwise almost disjoint. $\mathfrak{a}$ is the least size of an infinite almost disjoint family which is maximal in the set of all such families with respect to inclusion (mad family). The most important inequalities between these cardinals characteristics are: $\aleph_1 \leq \mathfrak{b} = \mathrm{cf}(\mathfrak{b}) \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$, $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$, $\mathfrak{s} \leq \mathfrak{d}$. They were shown in [8] and will be very useful throughout this paper.

We are going to see how Cohen forcing affects these four cardinal characteristics. In particular we will see that in a Cohen extension, when starting with a model of *GCH*, $\mathfrak{a} = \mathfrak{b} = \mathfrak{s} = \aleph_1 < \mathfrak{d} = \mathfrak{c}$, where $\mathfrak{c}$ is the size of the continuum. In another section we will present the method of product forcing and we will outline how iterated forcing is used. Product forcing is then applied to control the spectrum of mad families and to get respective consistency results. In [8], we also studied the generalized versions of $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$ and $\mathfrak{a}$ for arbitrary infinite cardinals. We are going to introduce the generalized version of Cohen forcing that allows to show similar results for them. Then we will also show that the dominating numbers of different regular cardinals are independent. At the end we will add some *ZFC* results concerning the interplay of cardinal characteristics at cardinals of the same cofinality.

1 Forcing

1.1 Generic Extensions

In this section we briefly introduce forcing setting aside the proofs for many of the relevant statements. We follow the exposition in [7] where most of the proofs can be found.

Definition 1.1 (Forcing poset)**.** A forcing poset is a triple $(\mathbb{P}, \leq, \mathbb{1})$ where $(\mathbb{P}, \leq)$ is a partially ordered set with a largest element $\mathbb{1}$ $(\forall p \in \mathbb{P}(p \leq \mathbb{1}))$.
Elements of $\mathbb{P}$ are often called conditions; $p \leq q$ reads p extends q; we say that $p, q \in \mathbb{P}$ are compatible iff there is $r \in \mathbb{P}$ extending p and q $(r \leq p, q)$; p and q are incompatible iff they are not compatible, written as $p \perp q$.

Definition 1.2 (Filter)**.** Let $(\mathbb{P}, \leq, \mathbb{1})$ be a forcing poset. We call $F(\neq \emptyset) \subseteq \mathbb{P}$ a filter on $\mathbb{P}$ iff:

- Any $p, q \in F$ are compatible in F, meaning $\exists r \in F(r \leq p, q)$.
- $p \in F \wedge p \leq q \to q \in F$

Example 1.3.

- The trivial $\mathbb{P} := \{\mathbb{1}\}$ where $\leq := \{(\mathbb{1}, \mathbb{1})\}$ is a forcing poset. $\mathbb{P}$ itself is a filter on $\mathbb{P}$.
- Let I, J be sets. Then $\mathrm{Fn}(I, J) := \{p \in [I \times J]^{<\omega} : p \textit{ is a partial function from } I \text{ to } J\}$ together with $\supseteq$ as order relation is a forcing poset with largest element $\emptyset$.
- For any topological space $(X, \mathcal{O})$, $(\mathcal{O} \setminus \{\emptyset\}, \subseteq, X)$ is a forcing poset. A filter F on this poset then is the same as what a filter usually denotes in the context of topological spaces if we close F with supersets (one can then ask about convergence, etc...).
- Let T be a theory in some language $\mathcal{L}$ and consider the set $\mathcal{T}$ of all consistent extensions of T. Then $(\mathcal{T}, \supseteq, T)$ is a forcing poset. Given a filter F, $\bigcup F$ is a consistent extension of T (one can then ask about completeness, etc...).

The idea of forcing is to, working in *ZFC*, start with a countable transitive model (ctm) $(M, \in)$ of *ZFC*, where $\in$ is the "real" membership relation of the *ZFC* universe $\mathbf{V}$ (that is the universe we imagine to work in), and extend it to a new model $N \supseteq M$ which has desired properties by adjoining to M a new set G. The idea is very similar to what algebraists do when they extend fields by adjoining roots of polynomials. In the same way that $\mathbb{Q}[\sqrt{2}]$ is the smallest field extension (up to isomorphism) of $\mathbb{Q}$ that contains a root of $x^2 - 2$, $N = M[G]$ will be the smallest transitive extension of M that contains G and satisfies *ZFC*. Transitive means that if $y \in x \in M$ then $y \in M$.

Notice that saying "M is a model of *ZFC*" is actually ambiguous and can be understood in two different ways, which are important to be distinguished. One way to read it, is to view "M is a model of *ZFC*" as a single sentence in the language of set theory that says that the model M satisfies the axioms of *ZFC*. What we then mean, is that internally $\mathbf{V}$ "thinks" M is a model of *ZFC*. In particular it would mean that for $\mathbf{V}$, *ZFC* is consistent. Another way of reading it, is to see "M is a model of *ZFC*" as a scheme that contains all axioms of *ZFC* relativized to M. That is, we assert that for any axiom φ we (or our "metatheory") know about, $M \models \varphi$ holds true in $\mathbf{V}$.

These two ways of reading are really different and we will always chose the latter one.

The main reason we use transitive models is because the elements of M can then be seen as the same as they are in $\mathbf{V}$. They have the same internal set structure they have in $\mathbf{V}$ and many properties are true of them in M if and only if they are true of them in $\mathbf{V}$. This is generally referred to as "absoluteness". A big class of properties that is absolute between transitive models are the Δ_0 formulas, also called bounded formulas. These are formulas that have only bounded quantifiers (of the form $\forall x \in y$, $\exists x \in y$) so they can only "talk" about the internal properties of sets $x_1, \dots, x_n$ and not ones related to the whole universe. For example x being an ordinal is a property which only depends on what sets x consists of and it is a property expressed by a bounded formula. Also many elementary set theoretic notions as $\cup$, $\cap$, S, $\setminus$, $\emptyset$ are absolute between transitive models. Concretely this means for example that, if $a = b \cap c$ in M where a, b, c are sets in M, then also $a = b \cap c$ in N.

The construction of N involves a forcing poset which lies in M and that we can freely chose (depending on what we try to "force"). The G that will be adjoined will then be a special filter on this poset. But more about that will follow. The first step of the construction is done by defining so called "names" for the sets in N.

Definition 1.4. Let M be a ctm and $(\mathbb{P}, \leq, \mathbb{1}) \in M$ a forcing poset. Then we define the set $M^{\mathbb{P}}$ of names recursively on the ordinals $o(M)$ in M as follows:

- $M_0^{\mathbb{P}} := \emptyset$
- $M_{\alpha+1}^{\mathbb{P}} := \mathcal{P}(M_\alpha^{\mathbb{P}} \times \mathbb{P})$
- $M_\eta^{\mathbb{P}} := \bigcup_{\alpha<\eta} M_\alpha^{\mathbb{P}}$
- $M^{\mathbb{P}} := \bigcup_{\alpha<o(M)} M_\alpha^{\mathbb{P}}$

Here, $\mathcal{P}(X)$ denotes the powerset of X inside M. We can view $\bigcup$ as the union inside M or outside without any difference because $\bigcup$ is absolute for transitive models; $\mathcal{P}$ is not. By this definition a name is set of pairs of names and conditions in $\mathbb{P}$. An example of a name would be $\emptyset$, or $\{(\emptyset, \mathbb{1}), (\{(\emptyset, \mathbb{1})\}, \mathbb{1})\}$.
Note that each $M_\alpha^{\mathbb{P}}$ is in M and so every name $\sigma \in M^{\mathbb{P}}$ is also in M. $M^{\mathbb{P}}$ is only a class from point of view of M, but it is a set in $\mathbf{V}$.

From now on M will always denote a ctm for *ZFC* and $(\mathbb{P}, \leq, \mathbb{1})$ a forcing poset in M. Intuitively, a name is a basic construction plan of how to build a set that says something like: "Under condition p, i will contain a set that itself contains, under condition q, the empty set. Also, under condition r, i contain the empty set". This description would correspond to the name $\{(\{(\emptyset, q)\}, p), (\emptyset, r)\}$. The construction plans are available to M, but to really build a set from it we need to know what the conditions are.

Definition 1.5. For any set $G \subseteq \mathbb{P}$ we define a function $\mathrm{val}(., G)$ recursively on $M^{\mathbb{P}}$ as follows:

- for $M_0^{\mathbb{P}}$ there is nothing to define as $M_0^{\mathbb{P}}$ is empty
- if $\sigma \in M_\alpha^{\mathbb{P}}$ and $\mathrm{val}(., G)$ was already defined on $M_\beta^{\mathbb{P}}$ for $\beta < \alpha$ then $\mathrm{val}(\sigma, G) := \{\mathrm{val}(\pi, G) : \exists p \in G[(\pi, p) \in \sigma]\}$

We often write σ_G for $\mathrm{val}(\sigma, G)$ for convenience.

Definition 1.6. Let $G \subseteq \mathbb{P}$. Then we define

$$M[G] := \mathrm{ran}(\mathrm{val}(., G)) = \{\sigma_G : \sigma \in M^{\mathbb{P}}\}$$

Definition 1.7. We define inductively a function $\check{.} : M \to M^{\mathbb{P}}$ as follows:

- $\check{\emptyset} := \emptyset$
- $\check{x} := \{(\check{y}, \mathbb{1}) : y \in x\}$

Lemma 1.8. *For any filter $G \subseteq \mathbb{P}$ we have that:*

1. $\forall x \in M(\check{x}_G = x)$

2. $M \subseteq M[G]$
3. $M[G]$ *is transitive.*
4. $|M| = |M[G]|$

Proof. Note that $\mathbb{1} \in G$, then (1) is an easy induction. (2) follows directly from (1). (3): If $x \in \sigma_G \in M[G]$ then by definition of σ_G, x is of the form π_G for some name π. So there is a name for x in M and $x \in M[G]$. (4): $|M| \leq |M[G]|$ is clear by (2). For $|M[G]| \leq |M|$ note that $|M[G]| \leq |M^{\mathbb{P}}|$ by the surjection $\mathrm{val}(., G)$ and $M^{\mathbb{P}} \subseteq M$. $\square$

Definition 1.7 thus gives us a name for any set in M. We can also find a name for G, which yields that $G \in M[G]$.

Lemma 1.9. *Let* $\Gamma := \{(\check{p}, p) : p \in \mathbb{P}\}$. *For any set* G *we have that* $\Gamma_G := G$.

Proof. $\Gamma_G := \{\sigma_G : \exists p \in G[(\sigma, p) \in \Gamma]\} = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$ $\square$

Remember the analogy of names with construction plans. The last few lemmata suggest that our set of conditions that we use to build our sets is a filter. In this metaphor $\mathbb{1}$ would be a condition that always holds true. Also, we can only have conditions that are compatible and do not contradict each other. And moreover our set of conditions is closed under implications, so $p \leq q$ can be read as p implies q. Note how we translated precisely the definition of filter to our analogy.

If G is an arbitrary filter, $M[G]$ will not necessarily satisfy *ZFC*. In order to make our construction work we need to take a special kind of filter on $\mathbb{P}$. This filter will usually not exist in M but we have a name Γ for it, so that it will be an element of $M[G]$. Actually if (and only if) the filter is in M our new model $M[G]$ would be equal to M.

Definition 1.10. Let $(\mathbb{P}, \leq, \mathbb{1})$ be a forcing poset. A set $D \subseteq \mathbb{P}$ is called dense iff $\forall p \in \mathbb{P} \exists q \in D(q \leq p)$.

Definition 1.11. Let M be a ctm of *ZFC*, $(\mathbb{P}, \leq, \mathbb{1})$ a forcing poset. A filter $G \subseteq \mathbb{P}$ is called $\mathbb{P}$-generic over M iff it meets all dense sets in M that is $G \cap D \neq \emptyset$ for all dense sets $D \in M$ of $\mathbb{P}$.

Note that “dense” is an absolute notion, so that $D \in M$ is dense in M iff it is dense in $\mathbf{V}$.

To give again an intuition about the construction, you can consider dense sets as sets of conditions that should not be omitted. You can view a dense set as a complete set of answers to a question and one of those answers must be a right one. For each condition there is one with more information (one that extends it) that has an answer. A generic filter is then a consistent ($\widehat{=}$ filter) set of conditions that is also complete ($\widehat{=}$ generic) and provides an answer to every question M could ask for.

Lemma 1.12. *For any* $p \in \mathbb{P}$ *there is a* $\mathbb{P}$*-generic filter* G *over* M *containing* p.

Proof. As M is countable, the dense sets of $\mathbb{P}$ in M can be enumerated by a sequence $\langle D_n \rangle_{n\in\omega}$ (in the case that there are only finitely many dense sets, they can be repeated). Define a sequence $\langle p_n \rangle_{n\in\omega}$ as follows: $p_0 := p$, p_n being defined chose for p_{n+1} an element of D_n with $p_{n+1} \leq p_n$. Let $G := \{q \in \mathbb{P} : \exists n \in \omega(q \geq p_n)\}$. G is then a generic filter containing p. $\square$

Until now we have seen, that for a countable transitive model $(M, \in)$ of *ZFC* (or *ZF*), an arbitrary forcing poset $\mathbb{P}$ in M and a filter $G \subseteq \mathbb{P}$ we can get a new countable transitive model $(M[G], \in)$. We will now show that when G is generic over M, then $M[G]$ will also be a model of *ZFC* (respectively *ZF*). First we introduce a notion that will become very important:

Definition 1.13. Let $p \in \mathbb{P}$ and φ be a sentence in the language $\{\in\} \cup M^{\mathbb{P}}$ (called the forcing language) where names in $M^{\mathbb{P}}$ are constants. Then $p \Vdash \varphi$ (reads "p forces φ") iff for all M-generic filters $G \subseteq \mathbb{P}$ that contain p, $M[G] \models \varphi$, where a constant $\sigma \in M^{\mathbb{P}}$ is interpreted as σ_G in $M[G]$.

So a condition p forces some sentence means that it forces that it is true in the generic model $M[G]$. We now understand why we call p a "condition". We might think at this point that the truth of some sentence φ in $M[G]$ will not be likely to depend on just one condition p that lies in G, but rather on much more complex aspects of G. So it is intuitive to think that a single condition will not force much to hold (except logically valid sentences). But the (maybe) astonishing truth is that any sentence is true in $M[G]$ because of one single condition p that is in G, and this makes forcing extremely strong and very handy to work with (very often this p will be just $\mathbb{1}$ so that a sentence is true for any generic G). We make this formal with the following Lemma:

Lemma 1.14 (Truth Lemma)**.** *Let G be generic over M, φ a sentence in the language $\{\in\}$, $\sigma_1, \dots, \sigma_n \in M^{\mathbb{P}}$. Then*

$$M[G] \models \varphi(\sigma_1, \dots, \sigma_n) \leftrightarrow \exists p \in G[p \Vdash \varphi(\sigma_1, \dots, \sigma_n)]$$

The notion of forcing is a notion that we defined inside $\mathbf{V}$. Actually it also depends on M and $\mathbb{P}$ but they are usually clear from context so that we content ourself with $p \Vdash \varphi$. The next very useful Lemma tells us that in fact the notion of forcing for some fixed sentence φ is definable within M, so that M can already "talk" about if something will be true in its generic extension depending on some condition p.

Lemma 1.15 (Definability Lemma)**.** *For any formula $\varphi(x_1, \dots, x_n)$ of the language $\{\in\}$, there is a sentence $\psi(q, y_1, \dots, y_n)$ of the language $\{\in\}$ so that for any names $\sigma_1, \dots, \sigma_n$ and $p \in \mathbb{P}$:*

$$M \models \psi(p, \sigma_1, \dots, \sigma_n) \leftrightarrow p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$$

We will usually write $p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n)$ for that sentence $\psi(p, \sigma_1, \dots, \sigma_1)$.

Again, the formula $p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n)$ depends in fact on the additional parameter $\mathbb{P}$ ($\mathbb{P}$ containing the whole information on the poset as $\mathbb{1}$ and $\leq$).
Lemma 1.14 and Lemma 1.15 are proved simultaneously. We give a sketch of how the proof is done:

Proof. (Sketch) We prove Lemma 1.14 and Lemma 1.15 simultaneously by induction on the complexity of φ. When φ is atomic (this is in fact the most difficult part), it is of the form $x_1 = x_2$ or $x_1 \in x_2$. One can define relations recursively on $M^{\mathbb{P}}$ by formulas $\psi_=(p, x_1, x_2)$ and $\psi_\in(p, x_1, x_2)$ that hold iff $p \Vdash x_1 = x_2$, respectively $p \Vdash x_1 \in x_2$. Within this same recursion you can also prove the Truth Lemma for atomic formulas. The induction step for $\wedge$ is very easy: If $p \Vdash^* \varphi_1$ and $p \Vdash^* \varphi_2$ are defined, then we can define $p \Vdash^* \varphi_1 \wedge \varphi_2$ as $p \Vdash^* \varphi_1 \wedge p \Vdash^* \varphi_2$. If the Truth Lemma is known for φ_1 and φ_2, then the one for $\varphi_1 \wedge \varphi_2$ is trivial (use that G is a filter). For $\neg$ we define $p \Vdash^* \neg\varphi$ as

$\neg\exists q \leq p(q \Vdash^* \varphi)$, for $p \Vdash^* \forall x\varphi(x)$ we take $\forall\sigma \in M^{\mathbb{P}}[p \Vdash^* \varphi(\sigma)]$. $M^{\mathbb{P}}$ is of course a proper class in M but "$\sigma \in M^{\mathbb{P}}$" is definable in M (namely by $\exists\alpha(\sigma \in M^{\mathbb{P}}_\alpha)$). In this proof we make often use of the genericity of G. $\square$

A key point in the proof is that forcing allows us in some sense to imitate first order logic. Remember our analogy filter $\widehat{=}$ consistent and generic $\widehat{=}$ complete. Now that we have the truth and definability lemma we can suggest to identify a condition with all the formulas that it forces. The generic filter then corresponds to the complete theory of the new model. This idea of taking a structure (poset) that is able to imitate first order logic and makes it possible to prove the truth and definability lemma is probably one of the main ideas and intuitions in the development of forcing. There are actually many other kinds of structures for which it is possible to develop forcing and they all have this key feature of imitating logic. You then have similar analogies as: implication $\widehat{=} \leq$, contradiction $\widehat{=} \perp$ and you can define something like dense sets, generic filters, etc. ... Such structures can be boolean algebras, latices, topological spaces, ... Partial orders are a nice choice because they are in some sense the minimal structure you need for forcing.

We are now ready to prove that $M[G]$ satisfies the axioms of *ZFC* (or *ZF*).

Theorem 1.16. *If $M \models$ ZF then for any generic G, $M[G] \models$ ZF. If $M \models$ ZFC then for any generic G, $M[G] \models$ ZFC.*

Proof. (Sketch) We just check every axiom: Extensionality holds because $M[G]$ is transitive by Lemma 1.8. Foundation holds in $M[G]$ because it already holds in $\mathbf{V}$. All other axioms state that some specific sets of some form exist. They are proven by finding names in M for them. For example given $\sigma_G \in M[G]$, $\bigcup \mathrm{dom}(\sigma)$ is a name for a superset of $\bigcup \sigma_G$ proving the axiom of Union in $M[G]$. Similarly one can find a name for the Powerset and Pairing axioms. For Comprehension and Replacement you need the Definability and the Truth Lemma for being able to define a suitable name in M. For example for Comprehension you might take the name: $\{(\tau, p) : \tau \in \mathrm{dom}(\sigma) : p \Vdash^* \tau \in \sigma \wedge \varphi(\tau)\}$. Infinity is clear because $\omega \in M[G]$ and being ω is an absolute notion. The Axiom of Choice is proved by finding for every set a name for a well order, using an existing well order in M (note that "well order" is also absolute). $\square$

Lemma 1.17. *Let G be generic, then $M[G]$ is the smallest transitive model of* ZFC *extending M and containing G. That is, for any transitive $N \ni G$ extending M and satisfying* ZFC*, $M[G] \subseteq N$.*

Proof. Let N be such a model. Then as $M^{\mathbb{P}}_\alpha \in N$ for any $\alpha \in M$ and $G \in N$, also $\mathrm{val}(\sigma, G) \in N$ for any $\sigma \in M^{\mathbb{P}}$. This is because *ZFC* proves that there is a function $\mathrm{val}(., G)$ on each $M^{\mathbb{P}}_\alpha$ and $\mathrm{val}(., G)$ is Δ_0 ($y = \mathrm{val}(., G)$ iff y is a function with domain $M^{\mathbb{P}}_\alpha$ and $\forall\sigma \in M^{\mathbb{P}}_\alpha[y(\sigma) = \{y(\tau) : \exists p \in G[(\tau, p) \in \sigma]\}]$), thus absolute for transitive models. $\square$

Note that Lemma 1.17 actually tells us that the notion $M[G]$ does not depend on $\mathbb{P}$, but really only on M and G.

Lemma 1.18. *For any generic G, $o(M) = o(M[G])$.*

Proof. Being an ordinal is absolute for transitive models so $o(M) \subseteq o(M[G])$. It is easy to see by induction that for any name σ, $\mathrm{rank}(\sigma) \geq \mathrm{rank}(\sigma_G)$ (rank is also absolute so we don't have to relativize it to some model). Now assume σ is a name for some ordinal in $M[G]$. Then $\mathrm{rank}(\sigma) \geq \mathrm{rank}(\sigma_G) = \sigma_G$. But $\mathrm{rank}(\sigma) \in M$ and thus by transitivity $\sigma_G \in M$. $\square$

Yet we don't know much about our new model $M[G]$. We know that it has the same ordinals as M, but what about the cardinals? Being an ordinal is absolute, but in the definition of cardinal we make a quantification over the whole model (for any ordinal below κ, there is no bijection from κ to that ordinal). The only cardinal that always stays one is ω, because being the first limit ordinal is absolute as already mentioned. The next lemma provides a nice example of a simple forcing poset that collapses an arbitrary uncountable cardinal κ. A poset collapses a cardinal κ means that κ won't be a cardinal in the generic model any more.

Lemma 1.19. *Let κ be an infinite cardinal in M, $\mathbb{Q} := \mathrm{Fn}(\omega, \kappa)$ as in Example 1.3, $G \subseteq \mathbb{Q}$ generic over M. Then $M[G] \models |\check{\kappa}| = \check{\omega}$.*

This is a nice first density argument to begin with. Note that we could have omitted the $\check{}$ in $\check{\omega}$ above, then ω would just have meant "the first limit ordinal", which is named by $\check{\omega}$. Very often the $\check{}$'s are actually not used when it should be clear what is meant.

Proof. We will show that $\bigcup G$ is a function that maps ω onto κ so that κ will be countable in the new model $M[G]$. The first easy observation is that $\bigcup G \subseteq \omega \times \kappa$. Next we see that $\forall n \in \omega[n \in \mathrm{dom}(\bigcup G)]$. This is because for each n, $D_n := \{p \in \mathbb{Q} : n \in \mathrm{dom}(p)\}$ is dense (for any $q \in \mathbb{Q}$, if $n \in \mathrm{dom}(q)$, then $q \in D_n$, and else $q \geq q \cup \{(n, 0)\} \in D_n$) and so G intersects each D_n.
Now assume that $(n, \alpha) \neq (n, \beta) \in \bigcup G$; this is not possible because G was a filter and conditions p and q containing those pairs respectively would be incompatible.
The last step is to show that $\bigcup G$ is onto. For that define $E_\alpha := \{p \in \mathbb{Q} : \alpha \in \mathrm{ran}(p)\}$ for each $\alpha \in \kappa$. Then each E_α is dense (for any $q \in \mathbb{Q}$, if $\alpha \in \mathrm{ran}(q)$, then $q \in E_\alpha$, and else $q \geq q \cup \{(\max \mathrm{ran}(q) + 1, \alpha)\} \in E_\alpha$) and so G intersects all of them and $\mathrm{ran}(\bigcup G) = \kappa$. $\square$

Remember that M is countable, so that in fact, seen from $\mathbf{V}$, all ordinals in M are countable. So the possibility of collapsing a cardinal is not at all counter-intuitive.

We can generalize Lemma 1.19 to the following:

Lemma 1.20. *Let I be an infinite set, J another set, $\mathbb{Q} := \mathrm{Fn}(I, J)$, $G \subseteq \mathbb{Q}$ generic over M. Then $M[G] \models \bigcup \mathring{G}$ is a function from I onto J.*

For a set x in $M[G]$, we denote with $\mathring{x}$ a name for x in M.

Very often we don't want cardinals to be collapsed, because then the sizes of sets in M stay the same in $M[G]$, which helps us working over $M[G]$ and its combinatorial properties. We will now define a property of $\mathbb{P}$ that is sufficient for not collapsing any cardinals.

Definition 1.21. $A \subseteq \mathbb{P}$ is called an antichain iff $\forall p, q \in A[p \perp q]$. We say $\mathbb{P}$ has the ccc (countable chain condition) iff every antichain in $\mathbb{P}$ is at most countable.

Lemma 1.22. *If $\mathbb{P}$ has the ccc than $\mathbb{P}$ does not collapse cardinals, i.e. it preserves cardinals. Furthermore $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$ for any α and any generic G.*

In general if t is a term in our (extended) language of set theory which contains all the well-defined notions as $\emptyset, \mathcal{P}, \mathbb{R}, \aleph_0, \omega_1, \ldots$ then $(t)^M$ denotes the unique object in M that interprets t.
We are almost ready for forcing $2^{\aleph_0} \neq \aleph_1$.

1.2 Cohen Forcing and the negation of CH

Lemma 1.23. $\mathbb{C}_\kappa := \mathrm{Fn}(\kappa \times \omega, \omega)$ *has the ccc.*

The forcing poset $\mathbb{C}_\kappa$ will give us a model in which $2^{\aleph_0} \geq \kappa$ yielding $\neg CH$ when $\kappa > \aleph_1$. The idea behind this is that a generic over $\mathbb{C}_\kappa$ will give a function from $\kappa \times \omega$ to ω which codes κ many different functions $\omega \to \omega$, raising the size of $2^{\aleph_0}$. As κ still stays the same cardinal greater than $\aleph_1$, ${}^\omega\omega$ will have size greater than $\aleph_1$.

Theorem 1.24. *Let* $(\kappa \geq \aleph_2)^M$*, then for* $\mathbb{C}_\kappa$ *we have that* $\mathbb{1} \Vdash \neg\mathrm{CH}$*.*

Proof. Let G be generic. Then by Lemma 1.20 $\bigcup G \in M[G]$ is a function from $\kappa \times \omega$ to ω. Furthermore we have that for all $\alpha, \beta \in \kappa$ there is a $n \in \omega$ so that $\bigcup G(\alpha, n) \neq \bigcup G(\beta, n)$, the funtions $f_\alpha(n) := \bigcup G(\alpha, n)$ are thus pairwise distinct and $\bigcup G$ is an injection. This is because the sets $E_{\alpha,\beta} := \{p \in \mathbb{C}_\kappa : \exists n \in \omega[p(\alpha, n) \neq p(\beta, n)]\}$ are dense. It is now clear that in $M[G]$, $2^{\aleph_0} \geq |\kappa| = \kappa > \aleph_1$. $\square$

Theorem 1.24 gives us a lower bound on $2^{\aleph_0}$ in $M[G]$, sufficient for having $\neg CH$, but it does not tell us what will be the exact value of it. We will now try to get an upper bound.

Definition 1.25. Let τ be a $\mathbb{P}$ name. We call σ a nice name for a subset of τ iff $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ and for all $(\vartheta, q) \neq (\vartheta, p) \in \sigma$, $q \perp p$.

A nice name is then a name where for any $\vartheta \in \mathrm{dom}(\sigma)$, $\{p \in \mathbb{P} : (\vartheta, p) \in \sigma\}$ is an antichain. The "nice" thing about nice names is that any subset $\sigma_G \subseteq \tau_G$ is named by a nice name and the number of these nice names for a subset of τ can be decided in M.

Lemma 1.26. *Let* τ *be a* $\mathbb{P}$ *name. Then for any name* σ*, there is a nice name* ϑ *for a subset of* τ*, so that* $\mathbb{1} \Vdash \sigma \subseteq \tau \to \sigma = \vartheta$*.*

Lemma 1.27. *If* $\mathbb{P}$ $(|\mathbb{P}| > 1)$ *has the ccc and* τ *is a* $\mathbb{P}$ *name,* G *is generic, then in* $M[G]$*,* $|\mathcal{P}(\tau_G)| \leq (|\mathbb{P}|^{\aleph_0 \cdot |\mathrm{dom}(\tau)|})^M$*.*

Proof. We estimate the number of nice names for a subset of τ in M. As $\mathbb{P}$ has the ccc, there are at most $|\mathbb{P}|^{\aleph_0}$ many antichains. Thinking of a nice name as a function that assigns to each σ in its domain an antichain of conditions, we get at most $\lambda := |\mathbb{P}|^{\aleph_0 \cdot |\mathrm{dom}(\tau)|}$ many nice names.
We then have a function $f : \lambda \to N(\tau)$, where $N(\tau)$ stays for the nice names of subsets of τ, that lists all nice names. Furthermore in $M[G]$ we can get a function g that assigns to each $\alpha \in \lambda$ $f(\alpha)_G$ as $G \in M[G]$. By Lemma 1.26 we get that $\mathrm{ran}(g) \supseteq \mathcal{P}(\tau_G)$. $\square$

Theorem 1.28. *Let* $(\kappa \geq \aleph_1)^M$*,* G $\mathbb{C}_\kappa$ *generic, then* $M[G] \models \kappa \leq 2^{\aleph_0} \leq (\kappa^{\aleph_0})^M$*. In particular, if* $\kappa^{\aleph_0} = \kappa$ *in* M *then* $M[G] \models 2^{\aleph_0} = \kappa$*.*

We will generally refer to $\mathbb{C}_\kappa$ as *Cohen forcing* named by Paul Cohen who first introduced forcing. We say that $\mathbb{C}_\kappa$ adds κ-many *Cohen reals*. Functions $\omega \to \omega$, or $\omega \to 2$ or subsets of ω are often called "reals" due to their one to one correspondence to the real numbers. We will also use the notation $\mathbb{C}_I$ for the forcing poset $\mathrm{Fn}(I \times \omega, \omega)$ for any set I in the ground model and $\mathbb{C}$ for $\mathrm{Fn}(\omega, \omega)$.

Lemma 1.29. *Let* $J_0 \subseteq I$ *be sets in* M *and let* $J_1 := I \setminus J_0$*. If* G *is* $\mathbb{C}_I$ *generic over* M*, then* $G \restriction J_0$ [1] *is* $\mathbb{C}_{J_0}$ *generic over* M*. Furthermore,* $G \restriction J_1$ *is* $\mathbb{C}_{J_1}$ *generic over* $M[G \restriction J_0]$*.*

[1] $G \restriction J_0$ means $\{p \restriction J_0 : p \in G\}$

Note that the notion $\mathbb{C}_I$ is absolute for transitive models of *ZFC*.

Proof. It easy to see that $G\restriction J_0$ is a filter on $\mathbb{C}_{J_0}$. Let $D \in M$ be dense in $\mathbb{C}_{J_0}$. Then $D' := \{p \cup q : p \in D, q \in \mathbb{C}_{J_1}\}$ is dense in $\mathbb{C}_I$ ($p = p\restriction J_0 \cup p\restriction J_1$, let $q \in D$ extend $p\restriction J_0$ then $p \leq q \cup p\restriction J_1$). Let $p \in G \cap D'$, then $p\restriction J_0 \in (G\restriction J_0) \cap D$.

Now let $D \in M[G\restriction J_0]$ be dense in $\mathbb{C}_{J_1}$. Let $\mathring{D}$ be a $\mathbb{C}_{J_0}$ name for D and $q \in G\restriction J_0$ that forces that $\mathring{D}$ is dense. We define $E := \{p \in \mathbb{C}_I : p \perp q \vee p\restriction J_0 \Vdash_{\mathbb{C}_{J_0}} p\restriction J_1 \in \mathring{D}\}$. $E \in M$ by the Definability Lemma.

Also E is dense in $\mathbb{C}_I$. Because let $r \in \mathbb{C}_I$ with $r \leq q$. As $r\restriction J_0 \Vdash \mathring{D}$ is dense, in particular $r\restriction J_0 \Vdash \exists t \leq r\restriction J_1, t \in \mathring{D}$ and so there is $p \leq r\restriction J_0, p \in \mathbb{C}_{J_0}$ and $t \leq r\restriction J_1, t \in \mathbb{C}_{J_1}$ with $p \Vdash t \in \mathring{D}$. $p \cup t \in E$ and extends r.

So $E \cap G \neq \emptyset$. Let $p \in E \cap G$. Then p and q must be compatible as $q \in G\restriction J_0 \subseteq G$, so $p\restriction J_0 \Vdash p\restriction J_1 \in \mathring{D}$, $p\restriction J_0 \in G\restriction J_0$ and $p\restriction J_1 \in G\restriction J_1$, which means that $p\restriction J_1 \in D \cap (G\restriction J_1)$. So $G\restriction J_1$ is generic. $\square$

The next Lemma is a consequence of Lemma 1.29, which tells us that in some sense Cohen forcing is commutative and it can be split up in different parts. This will be very useful later on.

Lemma 1.30 (Product Lemma)**.** *Let $J \subseteq I$ be sets in M, G $\mathbb{C}_I$ generic over M. Then $M[G] = M[G\restriction J][G\restriction(I \setminus J)]$.*

Proof. We can argue only in terms of Lemma 1.17, forgetting about the forcing poset: $M[G]$ is the smallest ctm of *ZFC* containg G, $M[G\restriction J][G\restriction(I \setminus J)]$ also contains G ($G = G\restriction J \cup G\restriction(I \setminus J)$) is a ctm of *ZFC* by Lemma 1.29, so $M[G] \subseteq M[G\restriction J][G\restriction(I \setminus J)]$. $M[G]$ on the other hand contains $G\restriction J$ and $G\restriction(I \setminus J)$. Thus $M[G\restriction J] \subseteq M[G]$ and in a second step $M[G\restriction J][G\restriction(I \setminus J)] \subseteq M[G]$. $\square$

Lemma 1.31. *Let G be $\mathbb{C}_I$ generic, $\mathring{A}$ be a name for a subset of some set $X \in M$, then there is a $J \subseteq I$ in M of size at most $|X| \cdot \aleph_0$ and a name $\vartheta \in M^{\mathbb{C}_J}$ such that* $\mathrm{val}_{\mathbb{C}_I}(\mathring{A}, G) = \mathrm{val}_{\mathbb{C}_J}(\vartheta, G\restriction J)$.

Proof. Let ϑ be a nice name for $\mathring{A}$ as in Lemma 1.26. It has the form $\vartheta = \bigcup_{\sigma \in \check{X}} \{\sigma\} \times A_\sigma$, where A_σ is an antichain. Let $J = \bigcup_{\sigma \in \check{X}} \bigcup_{p \in A_\sigma} \mathrm{dom}(p)$. By the ccc J has size at most $|X| \cdot \aleph_0$ and as ϑ is a nice name for $\mathring{A}$ and $\mathrm{val}_{\mathbb{C}_I}(\mathring{A}, G) \subseteq X$ we have $\mathrm{val}_{\mathbb{C}_I}(\mathring{A}, G) = \mathrm{val}_{\mathbb{C}_I}(\vartheta, G) = \mathrm{val}_{\mathbb{C}_J}(\vartheta, G\restriction J)$. $\square$

1.3 Dense embeddings

What we want to investigate next, is how different forcing posets are related; can two different posets produce the same generic extension? When are posets "equivalent"?

Definition 1.32. $\mathbb{P}$ is called atomless iff $\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(q, r \leq p \wedge q \perp r)$.

It is easy to see that $\mathbb{C}_\kappa$ is atomless. Actually all interesting posets will usually be atomless because of the following observation:

Lemma 1.33. *Let G be generic. If $G \in M$, then $M = M[G]$. If $\mathbb{P}$ is atomless then $G \notin M$ and $M \subsetneq M[G]$. If $\mathbb{P}$ is not atomless (that is, has an atom), then there is a generic $G \in M$.*

Proof. If $G \in M$, then val$(., G)$ would be definable in M, so $M[G] \subseteq M$. If $\mathbb{P}$ is atomless and $G \in M$ then $\mathbb{P} \setminus G \in M$ which is dense (use that $\mathbb{P}$ is atomless) and can not be met by G.
Let $a \in \mathbb{P}$, so that $\forall p, q \leq a$, $p \not\perp q$. Then $a{\uparrow}{\downarrow} = \{p \in \mathbb{P} : \exists q \leq a(q \leq p)\}$ is generic over M. □

For example forcing with a linear order is not at all useful because we would not get a proper extension (indeed the generic would be the whole poset). It is also easy to see that finite posets will never return a proper extension.

Definition 1.34. Let $\mathbb{P}, \mathbb{Q}$ be forcing posets. A function $i : \mathbb{P} \to \mathbb{Q}$ is called a dense embedding iff

- $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$
- $\forall q, p \in \mathbb{P}(q \leq p \to i(q) \leq i(p))$
- $\forall q, p \in \mathbb{P}(q \perp p \to i(q) \perp i(p))$
- $i(\mathbb{P})$ is dense in $\mathbb{Q}$

Example 1.35.

- The identity on any forcing poset is a dense embedding
- Consider $\mathbb{C} = \mathrm{Fn}(\omega, \omega)$ and $\mathbb{T} := \omega^{<\omega}$. Then the natural inclusion of $\mathbb{T}$ into $\mathbb{C}$ is a dense embedding.
- If $\mathbb{Q}$ and $\mathbb{P}$ are isomorphic, then an isomorphism provides a dense embedding.

If two posets are isomorphic (in M) then it is clear and not surprising that one can get the same extensions with one as with the other. They are basically the same. But the next Lemma tells us that dense embeddings are already sufficient for them to be "equivalent".

Lemma 1.36. *Let i be a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, G $\mathbb{Q}$ generic. Then $i^{-1}(G)$ is $\mathbb{P}$ generic and $M[G] = M[i^{-1}(G)]$. If H is $\mathbb{P}$ generic, then $i(H){\uparrow}$ is $\mathbb{Q}$ generic and $M[H] = M[i(H){\uparrow}]$.*

Proof. Let $p, q \in i^{-1}(G)$, then $q \perp p \to i(q) \perp i(p) \in G$ so p and q are compatible. That p and q have a common extension in $i^{-1}(G)$ will follow from $i^{-1}(G)$ being generic. Also if $r \geq q \in i^{-1}(G)$ then $i(r) \geq i(q)$ and so $i(r) \in G$, thus $r \in i^{-1}(G)$. Let D be dense in $\mathbb{P}$. Then $i(D)$ is dense in $\mathbb{Q}$. Indeed, let $q \in \mathbb{Q}$, then as $i(\mathbb{P})$ is dense, there is some $r \in \mathbb{P}$ with $i(r) \leq q$. For r there is a $p \in D$ with $p \leq r$ and thus $D \ni i(p) \leq i(r) \leq q$. We now have that $G \cap i(D) \neq \emptyset$, and therefore $i^{-1}(G) \cap D \neq \emptyset$ (take $p \in \mathbb{P}$ with $i(p) \in G \cap i(D)$). To see that for any $p, q \in i^{-1}(G)$ there is $r \in i^{-1}(G)$ extending them, note that $\{s : s \leq p, q \vee s \perp p \vee s \perp q\}$ is dense.

To see that that $i(H){\uparrow}$ is $\mathbb{Q}$ a filter is very similar. For genericity, consider $D' \subseteq \mathbb{P}$, $D' := \{p \in \mathbb{P} : \exists q \in D(i(p) \leq q)\}$ for D dense in $\mathbb{Q}$.

Now note that, as $i \in M$, $i^{-1}(G) \in M[G]$ so $M[i^{-1}(G)] \subseteq M[G]$. For $M[G] \subseteq M[i^{-1}(G)]$ we observe that $G = i(i^{-1}(G)){\uparrow}$. The argument for $M[H] = M[i(H){\uparrow}]$ is also similar. □

Being densely embeddable to another poset is not a symmetric relation, so it can happen that there is an embedding only in one direction, but this already is sufficient for them to yield the same extensions.

Lemma 1.37. *Let $\mathbb{Q}$ be a countable (in M) atomless forcing poset and let $\mathbb{T} := \omega^{<\omega}$. Then there is a dense embedding $i : \mathbb{T} \to \mathbb{Q}$ in M.*

Proof. First we observe that for any $p \in \mathbb{Q}$ there is an infinite maximal antichain below p (maximal in $p\downarrow$). As $\mathbb{Q}$ is atomless, we can recursively chose incompatible a_{k+1}, b_{k+1} that extend b_k, starting with $b_0 := p$. Then $\langle a_k \rangle$ is an infinite antichain as desired. It remains to make it maximal (in $p\downarrow$). ($\spadesuit$)

Also, having a q compatible with p we can find such an antichain, so that some element of it extends q. Having $r \leq p, q$ apply ($\spadesuit$) for r and make the result maximal in $p\downarrow$. ($\clubsuit$)

Fix an enumeration of $\mathbb{Q}$: $\langle q_n \rangle$. We recursively define an embedding as desired. First we let $i_0 := \{(\emptyset, \mathbb{1}_{\mathbb{Q}})\}$. At every step $\mathrm{dom}(i_n) \setminus \mathrm{dom}(i_{n-1}) = \omega^n$ and $i_n(\omega^n)$ will be a maximal antichain (maximal in all $\mathbb{Q}$). i_n being defined, we have that as $i_n(\omega^n)$ is a maximal antichain, for some $p \in \omega^n$, $i_n(p)$ is compatible with q_n. Chose an antichain $\langle a_k \rangle$ maximal below $i_n(p)$ as described at ($\clubsuit$) and extend i_n with $\{(p \cup \{(n+1, k)\}, a_k) : k \in \omega\}$. For the other elements of $i_n(\omega^n)$ extend i_n likewise using antichains as described at ($\spadesuit$) to get i_{n+1}. The property of $\mathrm{dom}(i_{n+1}) \setminus \mathrm{dom}(i_n) = \omega^{n+1}$ and $i_n(\omega^{n+1})$ being a maximal antichain is then preserved.

It is then easy to check that $\bigcup_{n\in\omega} i_n$ is a dense embedding. $\square$

This Lemma is very useful. What it tells us is that all atomless countable forcing posets are equivalent.

Remark 1.38. There is no dense embedding $2^{<\omega} \to \mathbb{T}$. A way to see this is to note that in $2^{<\omega}$ everybody (except $\emptyset$) extends $\{(0,0)\}$ or $\{(0,1)\}$. But for two elements in $\mathbb{T}$ that are not equal to $\mathbb{1} = \emptyset$ we find a third one that is incompatible with both. Note that $\{(0,0)\}$ or $\{(0,1)\}$ cannot map on $\mathbb{1}$. Still there is a dense embedding $\mathbb{T} \to 2^{<\omega}$

1.4 The logic behind Forcing

Here we want to clarify how the results from the last sections can be used to obtain relative consistency results as presented in the introduction.

What we have done so far is we have shown that from a transitive model M that satisfies all *ZFC* axioms we can get a new transitive model N of *ZFC* that will eventually satisfy some other specific set of sentences Δ. What we actually want to do is we want to prove that Δ does not produce any contradiction relative to *ZFC*. So what one might do is just to take N as a counterexample. There are major problems with this approach. First of all we cannot assume from nothing there is even a model M of *ZFC* to begin with, so we cannot construct our counterexample. Of course we assume *ZFC* to be consistent and thus by the Completeness Theorem a model of *ZFC* should exist. But even then it is not guaranteed that such a model is transitive, which is very important for the construction. The next inelegant thing is that if we want to argue entirely with models and the Completeness Theorem we will produce a relative consistency proof inside *ZFC*, which is much more than we actually need. As already mentioned in the introduction, it is possible to carry out a proof in any system like Peano Arithmetic that can formalize predicate logic, define *ZFC*, and develop enough proof theory. We will refer to this system as "our metatheory" and we will carefully explain how the proof is done.

In Lemmata 1.14 and 1.15 (Truth and Definability Lemma) all assumption about M was that it is a ctm of *ZFC*. In the language of set theory this can be expressed by a single sentence. We can define the notion of "axiom of *ZFC*", what a model is and say that M is a ctm that satisfies *ZFC* and infer the Truth and the Definability Lemma. As already mentioned in the beginning this is actually not what we want. It is important to notice that when we proved the two Lemmata we really only needed M

to satisfy finitely many axioms that can explicitly be written down in our metatheory (as Peano Arithmetic for example). So we can show that for some $\varphi_i \in ZFC$, $ZFC \vdash \forall M \forall \mathbb{P} \left[\bigwedge_{i\leq n} \varphi_i^M \wedge M \text{ is a ctm } \rightarrow def_{M,\mathbb{P}} \wedge truth_{M,\mathbb{P}}\right]$, where φ_i^M is the formula $M \models \varphi_i$ (this can also be the relativization of φ_i to M as defined in [7][I.16]) and $def_{M,\mathbb{P}}$, $truth_{M,\mathbb{P}}$ are the Definability and the Truth Lemma.

Next, in Theorem 1.16 we showed that if $M \models ZFC$, also $M[G] \models ZFC$. Again we notice that every time we checked an axiom of *ZFC* for $M[G]$ we only used finitely many axioms in M, so Theorem 1.16 can be translated into the following, which is provable in our metatheory:

Theorem 1.39. *For any constant, term or defined notion c let* $\mathrm{ZFC}^c := \{\varphi^c : \varphi \in \mathrm{ZFC}\} \cup \{c \text{ is a ctm}\}$. *Let M be a constant symbol. Then we have that for any* $\varphi \in \mathrm{ZFC}$, $\mathrm{ZFC} + \mathrm{ZFC}^M \vdash \forall \mathbb{P} \forall G\, [G \text{ is } \mathbb{P} \text{ generic over } M \rightarrow M[G] \models \varphi]$.

Now from ZFC^M, we can prove that there exists a forcing poset $\mathbb{C}_\kappa$ in M (or any forcing poset that we are considering), by countability of M it follows that there is a generic G over M, we then get a ctm $M[G]$ and everything we did so far (this includes all absoluteness results that come from M and $M[G]$ being transitive) leads to (for example) $M[G] \models \neg CH$.

For now we avoided the problem that *ZFC* does not provide a model M, by just adding a constant for M and axioms that state that M is a ctm of *ZFC* (the *ZFC* from the metatheory and not what *ZFC* thinks is *ZFC*). This then lets us get a model of $\neg CH$ for example.

The last ingredient we need for our final proof is the following theorem, which is again provable in our metatheory.

Theorem 1.40 (Reflection Principle)**.** *For any finite* $\Sigma \subseteq \mathrm{ZFC}$, *we have that* $\mathrm{ZFC} \vdash \exists M(M \text{ is a ctm of } \Sigma)$.

For a reference, see [7][II.5]. For our metatheory, a finite subset of *ZFC* can be a natural number n_Σ (if our metatheory is arithmetical) coding a string of axioms of *ZFC* for example. The sentence $\exists M(M \text{ is a ctm of } \Sigma)$ is then another natural number definable from n_Σ (or rather the term or the notion denoting n_Σ in our specific theory).

Remember what our final goal is: We have a set of sentences Δ and we want to be convinced in our favorite, finitist metatheory, that $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + \Delta)$. For this of course, Δ must be meaningful for us, that is, it must have a representation in our metatheory, by a term or a defined notion (most of the time Δ will consist of a single sentence). The argument then goes as follows: Assume that $ZFC + \Delta$ is actually inconsistent, then we know that there is a finite subset Σ of $ZFC + \Delta$ which already provides a contradiction (finite subset means again the code of a finite subset ...). In particular we can define in *ZFC* this specific finite subset of $ZFC + \Delta$ by a formula plus we can define the contradiction and get that $ZFC \vdash [\Sigma \text{ is inconsistent}]$. We know that *ZFC* proves the soundness of first order logic and conclude that $ZFC \vdash \neg\exists N\, [N \models \Sigma]$. But we also know that $ZFC + ZFC^M \vdash \exists N\, [N \models \Sigma]$ (this involves the whole forcing construction and the arguments of why Δ holds in $N = M[G]$). We can therefore deduce that $ZFC + ZFC^M$ is inconsistent and in particular there are finitely many $\varphi_i \in ZFC$ so that $ZFC + M \text{ is a ctm} + \bigwedge_{i\leq n} \varphi_i^M$ is inconsistent. But by Theorem 1.40 $ZFC \vdash \exists M \left[M \text{ is a ctm} + \bigwedge_{i\leq n} \varphi_i^M\right]$, so *ZFC* itself is inconsistent – a contradiction.

Again we indicate that an important thing to keep in mind when doing forcing to get relative consistency is that we do not assume M to satisfy *ZFC* as seen from inside $\mathbf{V}$, but rather that every single *ZFC* axiom that our metatheory knows about, holds in M. This means that in the $\mathbf{V}$ we consider we are not allowed to use that M or $M[G]$ satisfies all of *ZFC* at once. But every time we need a specific axiom of *ZFC* to hold in $M[G]$ we can freely make use of it. For example, if we could use that in $\mathbf{V}$, $M[G]$ satisfies *ZFC* then in particular, in $\mathbf{V}$, Con(*ZFC*) is true. This is an arithmetical statement and by absoluteness $M[G]$ will also satisfy it. But we cannot infer the relative consistency of Con(*ZFC*). In fact, if we would prove $\text{Con}(ZFC) \to \text{Con}(ZFC + \text{Con}(ZFC))$, then as $ZFC + \text{Con}(ZFC) \vdash \text{Con}(ZFC)$ and the proof of $\text{Con}(ZFC) \to \text{Con}(ZFC + \text{Con}(ZFC))$ can be translated to a proof from $ZFC + \text{Con}(ZFC)$, we would obtain that $ZFC + \text{Con}(ZFC) \vdash \text{Con}(ZFC + \text{Con}(ZFC))$, which by Gödel's Second Incompleteness Theorem (that our metatheory knows about) means that $ZFC + \text{Con}(ZFC)$ and furthermore *ZFC* are inconsistent.

A good way to think about forcing, that avoids such insecurities as described above is the following. Consider the language $\mathcal{L} := \{\in, G, V\}$ where G is a constant and V a unary relation symbol and let $\varphi(x)$ be a formula that says that x is some specific forcing poset (for example $\mathbb{C}_{\aleph_{27}}$). Then we can consider the theory $ZFC(V[G])$, which is *ZFC* plus the assertion that V denotes a transitive class plus *ZFC* relativized to V plus a statement saying that the universe corresponds $V[G]$, where G is x generic over V and $\varphi(x)^V$. Then for this theory, the Definability and the Truth Lemma become a provable scheme and every statement in the language $\{\in\}$ provable from this theory is consistent with *ZFC*.

A last consideration we want to elaborate on is that often we will require M to satisfy more than just *ZFC*, especially we may need that M satisfies some set of sentences Δ of that we already have shown its consistency with *ZFC*. In that case we can carry out our construction in the theory $ZFC + \Delta + ZFC^M + \Delta^M$ (where Δ^M of course means $\{\varphi^M : \varphi \in \Delta\}$). Using the following stronger Reflection Principle, we can then apply the exact same argument and combine $\text{Con}(ZFC) \to \text{Con}(ZFC + \Delta)$ and $\text{Con}(ZFC + \Delta) \to \text{Con}(ZFC + \Theta)$ to $\text{Con}(ZFC) \to \text{Con}(ZFC + \Theta)$, where Θ is what we are forcing.

Theorem 1.41. *For any sentences $\langle \varphi_i \rangle_{i \leq n}$ of the language $\{\in\}$, we have that* ZFC $\vdash \exists M[M \textit{ is a ctm} \wedge \bigwedge_{i \leq n} (\varphi_i^M \leftrightarrow \varphi_i)]$.

2 Effects of Cohen Forcing on $\mathfrak{b}, \mathfrak{s}, \mathfrak{d}$ and $\mathfrak{a}$

We now will see how Cohen forcing affects the classical cardinal invariants $\mathfrak{b}, \mathfrak{s}, \mathfrak{d}$, $\mathfrak{a}$ and their corresponding families. We begin with the bounding and the splitting number.

2.1 Bounding and splitting families

First we investigate properties the generic objects in the Cohen extension have with respect to eventual dominance and splitting.

No function from the ground model will dominate a generic real:

Lemma 2.1. *Let G be $\mathbb{C}_I$ generic over M for any $I \in M$. Let $h := \bigcup G$. Then a generic function $f_i(n) := h(i, n)$ for $i \in M$ is not dominated by any $g \in (\omega^\omega)^M$. We say that the f_i's are unbounded over the ground model.*

Proof. Let $g \in (\omega^\omega)^M$. For any $i \in I$, the sets $D_n := \{p \in \mathbb{C}_I : \exists m > n[p(i, m) > g(m)]\}$ for $n \in \omega$ are dense and so G intersects all of these yielding $\forall n \in \omega \exists m > n[f_i(m) > g(m)]$. Thus g does not dominate f_i. □

On the other hand, by a similar argument it is easy to see that a generic real will not dominate any function from the ground model.

Lemma 2.2. *Let f_i be a generic real as in Lemma 2.1, then for any $g \in (\omega^\omega)^M$, $g \not<^* f_i$.*

Proof. For $g \in (\omega^\omega)^M$ the sets $D_n := \{p \in \mathbb{C}_I : \exists m > n[p(i, m) \leq g(m)]\}$ are dense so G intersects all of them and we have that $|\{m \in \omega : g(m) \geq f_i(m)\}| = \aleph_0$, so $g \not<^* f_i$. □

Note that if we used $\leq^*$ (that is, $f \leq^* g$ iff $|\{n \in \omega : g(n) < f(n)\}| < \aleph_0$) instead of $<^*$, the above Lemma is false (take for example $g \equiv 0$ which is always dominated) and must be modified to:

Lemma 2.3. *Let f_i be a generic real as in Lemma 2.1, then for any $g \in (\omega^\omega)^M$ so that $|\{n \in \omega : g(n) > 0\}| = \aleph_0$, $g \not\leq^* f_i$.*

More generally we have that:

Lemma 2.4. *Let f_i be a generic real as in Lemma 2.1, then for any $n \in \omega$, $|\{m \in \omega : f_i(m) = n\}| = \aleph_0$. Or even $|\{m \in X : f_i(m) = n\}| = \aleph_0$ for any $X \in ([\omega]^\omega)^M$.*

Now for splitting we have very similar results. Until now the generic filter G was used to code functions $\omega \to \omega$, but it can also be used to code subsets of ω by taking the preimage of 0 (or any other natural number) of a generic function. By Lemma 2.4 these sets are infinite. We will see how they behave with respect to splitting.

Lemma 2.5. *Let G be $\mathbb{C}_I$ generic over M for any $I \in M$, $h := \bigcup G$. Then the generic sets $S_i := \{n \in \omega : h(i, n) = 0\}$ for $i \in I$ split all infinite ground model subsets of ω and they are split by all infinite ground model subsets that have infinite complement.*

Proof. Let $X \in ([\omega]^\omega)^M$ and $i \in I$. By Lemma 2.4, $|\{m \in X : f_i(m) = 0\}| = \aleph_0$ and $|\{m \in X : f_i(m) \neq 0\}| = \aleph_0$, so $|S_i \cap X| = \aleph_0$ and $|X \setminus S_i| = \aleph_0$. On the other hand, if $\omega \setminus X \in [\omega]^\omega$ also $|\{m \in \omega \setminus X : f_i(m) = 0\}| = \aleph_0$, thus $|S_i \setminus X| = \aleph_0$. □

The next theorem tells us what the values of $\mathfrak{b}$ and $\mathfrak{s}$ will be in $M[G]$.

Theorem 2.6. *Let G be $\mathbb{C}_\kappa$ generic over M for any uncountable κ. Then $M[G] \models \mathfrak{b} = \mathfrak{s} = \aleph_1$.*

Proof. Let $h := \bigcup G$ and $f_i(n) := h(i,n)$ for $i \in \kappa$. Then the family $\{f_i : i < \omega_1\}$ is unbounded in $M[G]$. Because let $g \in M[G]$, then by Lemma 1.31 there is a countable $J \subseteq \kappa$, so that $g \in M[G\restriction J]$. By Lemma 1.30, $M[G] = M[G\restriction J][G\restriction(\kappa \setminus J)]$. But then by Lemma 2.1 if $i \in \omega_1 \setminus J$, f_i is unbounded over $M[G\restriction J]$ and g does not dominate f_i. Let $S_i = \{n \in \omega : f_i(n) = 0\}$ for $i \in \kappa$, then $\{S_i : i < \omega_1\}$ is a splitting family in $M[G]$, by the same argument, applying Lemma 2.5. □

If we choose $\kappa > \aleph_1$ we get the relative consistency of $\mathfrak{b} = \mathfrak{s} < \mathfrak{c}$.

By Lemma 2.1 we know that a dominating family in M will not be dominating in $M[G]$ anymore, however unbounded families will stay unbounded.

Lemma 2.7. *For any countable forcing poset $\mathbb{P}$, G $\mathbb{P}$ generic over M, if $\mathcal{B}$ is unbounded in M then $\mathcal{B}$ is unbounded in $M[G]$.*

Proof. Let $\mathring{f}$ be a name for a function in $M[G]$, so by the Truth Lemma (Lemma 1.14) $\exists p \in G[p \Vdash \mathring{f}$ is a function $\omega \to \omega]$. Fix such a p and enumerate $p\!\downarrow := \{q \in \mathbb{P} : q \le p\}$ by $\langle p_i \rangle_{i\in\omega}$. In M we define functions g_i as follows: $g_i(n) := \min\{m \in \omega : \exists q \le p_i[q \Vdash \mathring{f}(n) = m]\}$. Note that this is well-defined as $p_i \Vdash \mathring{f}$ is a function $\omega \to \omega$ and so for some $q \le p_i$ and some m, $q \Vdash \mathring{f}(n) = m$. The family $\{g_i : i \in \omega\}$ is countable and thus bounded in M by some g. As $\mathcal{B}$ is unbounded in M, there is some $b \in \mathcal{B}$ so that $b \not<^* g$. We will show that also $b \not<^* f$. For this we observe that for all $n \in \omega$ the set $\{q : q \Vdash \exists k \ge n[b(k) \ge g(k) > \mathring{f}(k)]\}$ is dense below p. Take any $p_i \le p$. Then there is a $m \ge n$ so that $\forall k \ge m[g(k) > g_i(k)]$. Furthermore there is a $k \ge m$ with $b(k) \ge g(k)$ and some $q \le p_i$, with $q \Vdash g_i(k) = \mathring{f}(k)$ and so $q \Vdash b(k) \ge g(k) > \mathring{f}(k)$. So in particular $p \Vdash \exists k \ge n[b(k) > \mathring{f}(k)]$ for all $n \in \omega$ and so $p \Vdash \forall n \in \omega \exists k \ge n[b(k) > \mathring{f}(k)]$, which means $M[G] \models b \not<^* f$. This can be done for any $f \in M[G]$ and so $\mathcal{B}$ is unbounded in $M[G]$. □

Lemma 2.8. *Let G be $\mathbb{C}_I$ generic for any set $I \in M$, $\mathcal{B}$ unbounded in M, then $\mathcal{B}$ is also unbounded in $M[G]$.*

Proof. Let $\mathring{f}$ be a name for a function $f \in \omega^\omega$ in $M[G]$. Then $f \subseteq \omega \times \omega \in M$, so by Lemma 1.31 there is a set $J \subseteq I$ with $|J| \le \aleph_0$ and a name ϑ so that $f = \mathrm{val}_{\mathbb{C}_I}(\mathring{f}, G) = \mathrm{val}_{\mathbb{C}_J}(\vartheta, G\restriction J)$. $\mathbb{C}_J$ is then countable and $G\restriction J$ is $\mathbb{C}_J$ generic (Lemma 1.29), so by Lemma 2.7 $\mathcal{B}$ is unbounded in $M[G\restriction J]$ and so $f \in M[G\restriction J]$ cannot dominate $\mathcal{B}$ ("$<^*$" is absolute for transitive models). □

2.2 Dominating and almost disjointness

Theorem 2.9. *Let G be $\mathbb{C}_\kappa$ generic over M for any uncountable κ. Then $M[G] \models \mathfrak{d} \ge \kappa$. In particular, if $\kappa^{\aleph_0} = \kappa$ in M, then $M[G] \models \mathfrak{d} = \mathfrak{c}$*

Proof. Assume there is a dominating family $\mathcal{D}$ of size λ less then κ, then the family can be enumerated by a function $h : \lambda \times \omega \to \omega$. h is a subset of $\lambda \times \omega \times \omega$ and thus by Lemma 1.31 $h \in M[G\restriction J]$ for some $J \subseteq \kappa$ of size λ. So $\mathcal{D} \subseteq M[G\restriction J]$ but then $\mathcal{D}$ cannot be dominating in $M[G] = M[G\restriction J][G\restriction(\kappa \setminus J)]$, as the generic functions added by $G\restriction(\kappa \setminus J)$ are unbounded over $M[G\restriction J]$. The rest follows directly from Theorem 1.28. □

Proposition 2.10. *Let M satisfy* CH *and let $\mathbb{P}$ be a countable forcing poset in M. Then there is a mad family $\mathcal{A} \subseteq [\omega]^\omega \cap M$ that is still mad in $M[G]$ for any generic G.*

Proof. We will construct such a mad family in M recursively. First we observe that by *CH* and the countability of $\mathbb{P}$ there are $\aleph_1$ many nice names for a subset of ω $((|\mathbb{P}|^{\aleph_0})^{\aleph_0} = \aleph_1)$. We can thus enumerate the ordered pairs (τ_ξ, p_ξ) of nice names for a subset of ω and conditions in $\mathbb{P}$ with $\xi \in [\omega, \omega_1)$. Let $\langle A_n \rangle_{n\in\omega}$ be any almost disjoint family of size ω (for example a partition). A_δ for $\delta < \xi$ being defined we continue as follows:

- If $p_\xi \Vdash \forall\delta < \xi(|\tau_\xi \cap A_\delta| < \omega) \wedge |\tau_\xi| = \aleph_0$, we define a set A_ξ that "imitates" τ_ξ as follows. First enumerate $p_\xi{\downarrow} \times \omega$ by $(q_\alpha, n_\alpha), \alpha \in \omega$, and also reorder ξ by an ω-sequence $\langle \delta_\alpha \rangle_{\alpha\in\omega}$. For every $\alpha \in \omega$ we then chose an $a_\alpha \geq n_\alpha$ with the property: There is an $s \leq q_\alpha$ with $s \Vdash a_\alpha \in \tau_\xi \setminus (\bigcup_{\beta<\alpha} A_{\delta_\beta})$. This is possible because p_ξ and therefore q_α force that τ_ξ is almost disjoint from $\bigcup_{\beta<\alpha} A_{\delta_\beta}$. Let $A_\xi := \{a_\alpha : \alpha \in \omega\}$. Note that A_ξ is almost disjoint from all A_δ, $\delta < \xi$.
- Else let $A_\xi = \emptyset$.

$\mathcal{A} := \{A_\xi : \xi \in \omega_1\} \setminus \{\emptyset\}$ is then a mad family in $M[G]$ for any generic G. Because assume there is a $X \in [\omega]^\omega \cap M[G]$ that is almost disjoint from $\mathcal{A}$, then there is a nice name τ, a condition $p \in G$ and some ξ with $(\tau, p) = (\tau_\xi, p_\xi)$, so that $p_\xi \Vdash \forall\delta < \xi(|\tau_\xi \cap A_\delta| < \omega)$. The construction of A_ξ then yields that $|A_\xi \cap X| = \omega$. This is because for any $n \in \omega$ the set $\{s : s \Vdash \exists m \geq n(m \in A_\xi \cap \tau_\xi)\}$ is dense below p. Indeed, let $q \leq p$ then $(q, n) \in p_\xi{\downarrow} \times \omega$ and there is some s and some $a \geq n$ such that $s \Vdash a \in \tau_\xi \cap A_\xi$. Thus $\forall n \in \omega[p_\xi \Vdash \exists m \geq n(m \in A_\xi \cap \tau_\xi)]$ and thus $p_\xi \Vdash |A_\xi \cap \tau_\xi| = \omega$ and in particular $M[G] \models |A_\xi \cap X| = \omega$ – we have arrived at a contradiction.

To see that $\mathcal{A}$ is mad in M, note that "almost disjoint" is absolute. □

Theorem 2.11. *Let M satisfy* CH *and let $\kappa > (\aleph_1)^M$ be a cardinal in M. Let G be $\mathbb{C}_\kappa$ generic over M, then $M[G] \models \mathfrak{a} = \aleph_1 < \mathfrak{c}$*

Proof. Let $\mathbb{T} := (\omega^{<\omega})^M$ as in Lemma 1.37. Then by Proposition 2.10 there is a mad family $\mathcal{A}$ of size $(\aleph_1)^M$ that stays mad in $M[H]$ for any $\mathbb{T}$ generic H. Such a family will also stay mad in $M[G]$. By absolutness it will be almost disjoint in $M[G]$. Now assume there is a $X \in M[G]$ that witnesses that $\mathcal{A}$ is not mad in $M[G]$. Then by Lemma 1.31 there is a countable $J \subseteq \mathbb{C}_\kappa$ so that $X \in M[G{\restriction}J]$. As $\mathbb{C}_J$ is countable, by Lemma 1.37, there is a dense embedding $i\colon \mathbb{T} \to \mathbb{C}_J$. But then by Lemma 1.36 $M[G{\restriction}J] = M[i^{-1}(G{\restriction}J)]$ and $i^{-1}(G{\restriction}J)$ is $\mathbb{T}$ generic and so $\mathcal{A}$ must be mad in $M[i^{-1}(G{\restriction}J)]$. But $X \in M[i^{-1}(G{\restriction}J)]$ – a contradiction.

In $M[G]$ there is now a mad family of size $\aleph_1$ and so $M[G] \models \mathfrak{a} = \aleph_1 < \kappa \leq \mathfrak{c}$ (by the ccc $\kappa > \aleph_1$). □

In Theorem 2.11 we used that M satisfies the *CH*. For being able to produce a relative consistency from the last result, we need a proof of *CH* being consistent. This is usually covered in any course on axiomatic set theory as in [7, II.6], using Gödel's constructible universe from [4]. Nevertheless we will provide a forcing argument for Con(*ZFC*) $\to$ Con(*ZFC*+ *CH*) in Example 4.9.

The results from this section then yield the following relative consistency:

Theorem 2.12. Con(ZFC) $\to$ Con(ZFC $+ \mathfrak{b} = \mathfrak{s} = \mathfrak{a} < \mathfrak{d} = \mathfrak{c}$).

3 Further posets

3.1 Products and Iterations

In this section we want to present very brievly the general method of product forcing and an outline to the basic idea of iterated forcing.

Remember how we proved many results from the last section. We very often used that Cohen forcing could somehow be split up into two parts. Small objects of our extension already existed in a smaller model over which we have a generic. The reason for this is that Cohen forcing looks like what is called a product.

Definition 3.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets. Then their product is defined as $\mathbb{P} \times \mathbb{Q}$ together with $\mathbb{1} := (\mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\mathbb{Q}})$ and $(p, q) \leq (p', q')$ iff $p \leq p'$ and $q \leq q'$.

Example 3.2. $\mathbb{C}_\kappa \cong \mathbb{C}_I \times \mathbb{C}_J$ whenever J and I partition κ.

Theorem 3.3. *Let $\mathbb{P}, \mathbb{Q} \in M$. If G is $\mathbb{P} \times \mathbb{Q}$ generic over M, then $K := \mathrm{dom}(G)$ is $\mathbb{P}$ generic over M and $H := \mathrm{ran}(G)$ is $\mathbb{Q}$ generic over $M[K]$. Furthermore $M[G] = M[K][H] = M[K \times H]$.*

Theorem 3.4. *Let $\mathbb{P}, \mathbb{Q} \in M$. If K is $\mathbb{P}$ generic over M and H is $\mathbb{Q}$ generic over $M[K]$ then $K \times H$ is $\mathbb{P} \times \mathbb{Q}$ generic over M and $M[K][H] = M[K \times H]$.*

These theorems are proved very similarly to Lemma 1.30.

What the last two theorems tell us, is that forcing twice is actually equivalent to forcing only once, but an important assumption there, is that the second poset already lives in M. This was always the case when we used the Cohen poset because $\mathbb{C}_I$ is absolute. We can do even more then only forcing twice. If $\mathbb{P}_\xi$ is a sequence of forcing posets, then its product $\prod \mathbb{P}_\xi$ is defined in the analogous way (so $\langle p_\xi \rangle \leq \langle q_\xi \rangle$ iff $p_\xi \leq q_\xi$ for all ξ). We can then split it up into $\prod_{\xi \leq \alpha} \mathbb{P}_\xi \times \prod_{\xi > \alpha} \mathbb{P}_\xi$ and the two theorems hold. Another choice would be the finite support product which consists only of those sequences $\langle p_\xi \rangle$ which are $\neq \mathbb{1}$ only finitely often.

To give an intuition about products, consider the following example that we already know, but seen from a different perspective: Assume we want to make $\mathfrak{d}$ large. We know that $\mathbb{C}$ adds an unbounded real and thus destroys all dominating families from the ground model. But in $M[G]$ there are new small dominating families. So we do the same again to get $M[G][G']$. But still there are new dominating families appearing. A natural idea would be to define a chain of models $M_0 = M \subseteq M_1 = M[G] \subseteq \cdots \subseteq M_{\omega_2}$ so that a small dominating family should already have appeared in one of those initial models and would have been destroyed. The problem is that there is no good way of defining the limit of such models. The union of models of *ZFC* does not have to be a model of *ZFC* again. So what we do, is we take the finite support product of ω_2 copies of $\mathbb{C}$. This poset is isomorphic to $\mathbb{C}_{\omega_2}$ and in the end we only force once over that poset, without taking care about limit steps. This is extremely nice because it lets us carry out the typical arguments we would do when we would have a chain of models, but without ever really iterating the forcing procedure. All we need in our arguments, is that locally we can split up our model into two forcings.

In the above description the posets $\mathbb{P}_\xi$ all live in M. Now assume that we have a poset $\mathbb{P}$ in M and then in $M[G]$ we want to force with a $\mathbb{Q}$ which did not already exist in M. Then $\mathbb{P} \times \mathbb{Q}$ makes no sense in M. But there is still a way to make sense of it and that is where iterated forcing comes into play. We really only want to give an outline to how iterated forcing approximately works and so we won't be careful about questions of coding, for example.

Definition 3.5. Let $\mathbb{P}$ be a forcing poset and $\mathring{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing poset (that is $\mathbb{1} \Vdash \mathring{\mathbb{Q}}$ is a forcing poset). Then their product is defined as $\mathbb{P} * \mathring{\mathbb{Q}} := \{(p, \mathring{q}) : p \in \mathbb{P}, \mathring{q} \in \operatorname{dom}(\mathring{\mathbb{Q}}), p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}\}$ and $(p, \mathring{q}) \leq (p', \mathring{q}')$ iff $p \leq p'$ and $p \Vdash q \leq q'$.

One can easily prove analogous statements as for products (for example you have to change H to $\{q : \exists p \in \mathbb{P}[(p, \mathring{q}) \in K]\}$). Now to define an iteration we cannot just take a product as before but we have to make a recursion to get a sequence $\langle \mathbb{P}_\xi \rangle_{\xi \leq \alpha}$ of posets. For finite suport iteration (fsi) this usually goes as follows:

For $\mathbb{P}_0$ just take a forcing poset you want to begin with. If $\mathbb{P}_\xi$ is defined we take $\mathbb{P}_{\xi+1} := \mathbb{P}_\xi * \mathring{\mathbb{Q}}_\xi$ where $\mathring{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$ name for a poset. At limits we take $\mathbb{P}_\mu = \bigcup_{\xi<\mu} \mathbb{P}_\xi$. Conditions in $\mathbb{P}_\mu$ can be considered as μ-sequences of conditions where only finitely many are $\neq \mathbb{1}$. $\leq$ is then defined in the same way as usual. Non finite support iterations can be defined by changing the construction appropriately at the limit steps.

We give an overview of how we can get $\mathfrak{b} = \omega_2 = \mathfrak{c}$ (and therefore $\mathfrak{a} = \mathfrak{d} = \omega_2$). There is a way to define a poset that destroys unbounded families by adding a dominating real (called the Hechler poset $\mathbb{H}$). But the notion of Hechler poset is not absolute so that we cannot take a product as for $\mathfrak{d}$. What we do, is we define the finite support iteration, $\langle \mathbb{P}_\xi \rangle_{\xi \leq \omega_2}$, where at successor steps we take the product with a name for the Hechler poset. When we force with $\mathbb{P}_{\omega_2}$ we can argue in the usual way. First of all our poset is ccc (this is because we took the fsi of ccc posets). $\mathfrak{c} \leq \omega_2$ by the counting of nice names. Assume there is an unbounded family of size ω_1, then it must appear in some initial forcing with $\mathbb{P}_\xi$, but its unboundedness is destroyed by $\mathbb{P}_{\xi+1}$. It was shown by Baumgartner and Dordal (see [1]) that $\mathfrak{s} = \omega_1$ in this construction.

3.2 The spectrum of mad families

In this section we want to consider not only what the least size of a mad family is, but what the possible sizes of mad families are in general. For splitting, unbounded or dominating families this question is clearly uninteresting. We will show that it is consistent to have mad families of any size. For this we will introduce a ccc forcing poset that adds such a mad family. The original idea comes from Hechler's [5].

Definition 3.6. For any set I we define the poset $\mathbb{H}_I$ to be the set $\operatorname{Fn}(I, [\omega]^{<\omega})$ together with the order $q \leq p$ iff

- $\operatorname{dom}(q) \supseteq \operatorname{dom}(p)$
- for any $i \in \operatorname{dom}(p)$, $q(i) \supseteq p(i)$
- for any $i, j \in \operatorname{dom}(p)$, $q(i) \cap q(j) = p(i) \cap p(j)$

Lemma 3.7. *$\mathbb{H}_I$ is ccc.*

Proof. Note that if $p \restriction \operatorname{dom}(p) \cap \operatorname{dom}(q) = q \restriction \operatorname{dom}(p) \cap \operatorname{dom}(q)$, then p and q are compatible. Assume $\mathcal{B}$ is an uncountable set of conditions in $\mathbb{H}_I$. By the Δ-root Lemma (see [7, Lemma III.2.6]) there is an uncountable $\mathcal{A} \subseteq \mathcal{B}$ and a $R \subseteq I$ so that $\operatorname{dom}(p) \cap \operatorname{dom}(q) = R$ for any $p \neq q \in \mathcal{A}$. But the number of functions $R \to [\omega]^{<\omega}$ is countable (R is finite), so at least two p, q must coincide on R and thus must be compatible. $\square$

We will consider $\mathbb{H}_\kappa$ where κ is any uncountable cardinal. For any generic G and $\alpha \in \kappa$ we let $A_\alpha := \bigcup_{p \in G} p(\alpha)$ (where $p(\alpha) := \emptyset$, when $\alpha \notin \operatorname{dom}(p)$). By the typical density argument it is clear that every A_α is infinite. Also $\mathcal{A} := \{A_\alpha : \alpha \in \kappa\}$ is almost disjoint. Because if $\alpha, \beta \in \kappa$ there is some $p \in G$ with $\alpha, \beta \in \operatorname{dom}(p)$ and then $A_\alpha \cap A_\beta = p(\alpha) \cap p(\beta)$. Furthermore $\mathcal{A}$ is maximal in $M[G]$:

Proposition 3.8. *The family $\mathcal{A} := \{A_\alpha : \alpha \in \kappa\}$ is maximal in $M[G]$.*

Proof. Let X be an infinite subset of ω in $M[G]$. We are going to show that X cannot be almost disjoint from $\mathcal{A}$. Let $\mathring{X}$ be a nice name for X. First notice that $\mathring{X}$ is countable by the ccc. Thus there is some $\alpha \in \kappa$, so that α is not in the domain of any condition occurring in $\mathring{X}$. Now assume $p \in G$ is a condition that forces $\mathring{X} \cap A_\alpha \subseteq n$ for $n \in \omega$ and $A_\beta \cap X \subseteq n$ for $\beta \in \operatorname{dom}(p)$ (this is possible because $\operatorname{dom}(p)$ is finite). Let $(q, \check{k}) \in \mathring{X}$ so that $k \geq n$ and $q \in G$. Then we define the condition r as follows: $\operatorname{dom}(r) = \operatorname{dom}(p) \cup \operatorname{dom}(q) \cup \{\alpha\}$, for $\beta \in \operatorname{dom}(p) \cup \operatorname{dom}(q)$ we define $r(\beta) := q(\beta) \cup p(\beta)$ and $r(\alpha) := p(\alpha) \cup \{k\}$. We have that $r \leq q$. This is because q and p are compatible and because $\alpha \notin \operatorname{dom}(r)$. Also $r \leq p$, again because p and q are compatible and because for any $\beta \in \operatorname{dom}(p) \cup \operatorname{dom}(q)$, $r(\alpha) \cap r(\beta) = p(\alpha) \cap p(\beta)$. This is, because $k \notin p(\beta)$, as $p \in G$ and $n \leq k$ was chosen so that $A_\beta \cap X \subseteq n$. But now $r \Vdash k \in A_\alpha$ and also $r \Vdash k \in \mathring{X}$. We have arrived at a contraction, as we assumed that $p \Vdash \mathring{X} \cap A_\alpha \subseteq n$. □

All in all we get that:

Theorem 3.9. *If G is $\mathbb{H}_\kappa$ generic over M, where κ is an uncountable cardinal, then in $M[G]$ there is mad family of size κ.*

At first sight, it is not clear how to refute a theorem as: If $\mathfrak{a} < \kappa < \mathfrak{c}$, then there is no mad family of size κ. We can show that this is indeed not the case and that the spectrum of mad families can be very wide. For this we use the poset $\mathbb{H}_\kappa$ for various κ and take their finite support product. In the end we get mad families for all these κ.

Definition 3.10. Fix a set C of uncountable cardinals. Then we let $\mathbb{P} := \prod_{\kappa \in C}^{<\omega} \mathbb{H}_\kappa$ be the finite support product of those $\mathbb{H}_\kappa$ for $\kappa \in C$.

Lemma 3.11. *$\mathbb{P}$ is ccc.*

Proof. We can view the elements of $\mathbb{P}$ as finite partial functions from $\{(\kappa, \alpha) : \kappa \in C, \alpha \in \kappa\}$ to $[\omega]^{<\omega}$. Then we can apply the same argument as for $\mathbb{H}_\kappa$. □

Theorem 3.12. *If G is $\mathbb{P}$ generic over M, then there is, for any $\kappa \in C$, a mad family of size κ in $M[G]$.*

Proof. Let $\kappa \in C$. Then $\mathbb{P} \cong \prod_{\lambda \in C \setminus \{\kappa\}}^{<\omega} \mathbb{H}_\lambda \times \mathbb{H}_\kappa$, so we can view $M[G]$ as $N[K]$ where $M \subseteq N$ and K is $\mathbb{H}_\kappa$ generic over N. Note that $\mathbb{H}_\kappa$ is an absolute notion, so the $\mathbb{H}_\kappa$ of M (the one we actually force with) is the $\mathbb{H}_\kappa$ of N. But then by Theorem 3.9 there is a mad family of size κ in $N[K] = M[G]$. □

If we start with an appropriate ground model (say with GCH) and if $\sup C$ has uncountable cofinality then in $M[G]$, $\mathfrak{c} = \sup C$ by the usual counting of nice names. In particular we can get the following:

Theorem 3.13. *If $\lambda \in M$ has uncountable cofinality and $\lambda^{\aleph_0} = \lambda$ in M, then there is a forcing extension in which $\mathfrak{c} = \lambda$ and for any cardinal $\kappa \in [\aleph_1, \lambda]$ there is a mad family of size κ.*

4 Generalized bounding, splitting and almost disjointness

4.1 Generalized Cohen Forcing

The use of Cohen Forcing was to make the continuum very large so that *CH* fails. Additionally Cohen Forcing had various effects on the structure of the real line. In this section we aim to generalize Cohen Forcing in order to make the Continuum Hypothesis false at an arbitrary cardinal. This means that we want to generalize the ideas from Section 1.2 to get $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ for arbitrary $\aleph_\alpha$. Again the forcing poset we will use, will have similar effects on the cardinal characteristics at $\aleph_\alpha$. As before, we will focus on bounding, splitting and almost disjointness.

The forcing poset we used (or rather an isomorphic poset to the one we used) to raise the continuum at $\aleph_0$ was $\mathrm{Fn}(\kappa \times \omega, 2)$, which can be interpreted as finite approximations to a κ-sequence of subsets of ω. A natural way to generalize this idea for some arbitrary $\aleph_\alpha$, would be to take $\mathrm{Fn}(\kappa \times \aleph_\alpha, 2)$ ($\kappa > \aleph_{\alpha+1}$) to make $2^{\aleph_\alpha}$ bigger than $\aleph_{\alpha+1}$. It is not very difficult to see that this really works. In fact you can notice that $\mathrm{Fn}(\kappa \times \aleph_\alpha, 2)$ is actually isomorphic to $\mathrm{Fn}(\kappa \times \omega, 2)$ and then it is trivial that $\kappa \leq 2^{\aleph_0} \leq 2^{\aleph_\alpha}$ in the generic extension. But there is something very inelegant in doing this; namely that when we want to raise the continuum at $\aleph_\alpha$, we simultaneously blow it up at all infinite cardinals below to the same high level and "damage" the universe at regions where we don't want to. What would be more satisfying, is to be able to operate only at single levels that we are interested in. This would give us the possibility to change the structures of the continua (the respective invariants, etc...) in a very independent way and it would provide a wider range of possible consistency results. For example it would then be possible to let *CH* fail first only at some specific cardinal. From the forcing idea given above, it is not clear why $2^\lambda > \lambda^+$ should not imply $2^\kappa > \kappa^+$ for $\lambda \geq \kappa$.

The forcing posets we will use are the following:

Definition 4.1. Let λ be an infinite cardinal and I, J sets, then we define $\mathrm{Fn}_\lambda(I, J) := \{p \in [I \times J]^{<\lambda} : p \text{ is a partial function } I \to J\}$, the poset with the order $\supseteq$, and $\mathbb{1} := \emptyset$ as usual. Furthermore, if κ is an infinite cardinal and $I \in M$ some index set, we let $\mathbb{C}_{\kappa,I} := \mathrm{Fn}_\kappa(I \times \kappa, \kappa)$.

Definition 4.2. Let λ be an infinite cardinal and $\mathbb{P}$ a forcing poset. We say that $\mathbb{P}$ has the λ chain condition (λ-cc) iff every antichain in $\mathbb{P}$ has size less than λ.

The ccc is then the $\aleph_1$-cc. There are some remarks that can be made about this generalized notion of ccc. First of all the λ-cc clearly implies the θ-cc for $\theta \geq \lambda$. We will generally be interested in the smallest such λ (this is called the Suslin number of $\mathbb{P}$, denoted as $S(\mathbb{P})$). $S(\mathbb{P})$ will never be $\aleph_0$ and also $S(\mathbb{P})$ will be regular unless $S(\mathbb{P})$ is finite (but in that case $M[G] = M$).

Lemma 4.3. $\mathrm{Fn}_\lambda(I, J)$ *has the* $(|J|^{<\lambda})^+$*-cc.*

Remember that the ccc implies that cardinals are preserved. This result can easily be generalized to the following for the λ-cc. The full proof can be found in [7, IV.7].

Theorem 4.4. *If* $\mathbb{P}$ *has the* λ*-cc, then* $\mathbb{P}$ *preserves cardinals greater or equal to* λ*.*

To preserve cardinals $\geq \lambda$, just means that if $\alpha \geq \lambda$ is a cardinal in M, then it is still one in $M[G]$. Having the λ-cc is good in the sense that being a cardinal becomes absolute for ordinals $\geq \lambda$. But what happens bellow λ? If our goal is to make the CH false first at say $\aleph_3$ but everything bellow collapses to ω, then we end up with $(\aleph_3)^M = (\aleph_1)^{M[G]}$. To handle what happens bellow λ we use the following notion:

Definition 4.5. $\mathbb{P}$ is called λ-closed iff every decreasing sequence $\langle p_\xi \rangle_{\xi<\delta}$ for any $\delta < \lambda$ has a lower bound.

The use of this notion is made clear by the following Lemma:

Lemma 4.6. *Let $\mathbb{P}$ be λ-closed (in M of course), $A, B \in M$ with $|A| < \lambda$. Then $(B^A)^M = (B^A)^{M[G]}$.*

Proof. $(B^A)^M \subseteq (B^A)^{M[G]}$ is clear by absolutness. For $(B^A)^{M[G]} \subseteq (B^A)^M$ fix $f \in (B^A)^{M[G]}$ and enumerate A by $\langle a_\xi \rangle_{\xi<\kappa}$ ($\kappa = |A| < \lambda$). By the Truth Lemma there is some $p \in G$ so that $p \Vdash \mathring{f} : A \to B$. Let $q \leq p$ and define sequences $\langle q_\xi \rangle_{\xi<\kappa}$, $\langle b_\xi \rangle_{\xi<\kappa}$ recursively as follows: chose $q_0 \leq q$ such that $q_0 \Vdash \mathring{f}(a_0) = \check{b}$ for some $b \in B$ and let $b_0 = b$. q_ξ being defined for $\xi < \delta$, chose $r \leq q_\xi$ for all $\xi < \delta$ (use λ-closed) and then let $q_\delta \leq r$ be such that $q_\delta \Vdash \mathring{f}(a_\delta) = \check{b}$ for some $b \in B$ and let $b_\delta = b$. $\langle q_\xi \rangle_{\xi<\kappa}$ then still has an upper bound r and we get the function $h \in (B^A)^M$, with $h(a_\xi) = b_\xi$. In particular our construction yields $r \Vdash \mathring{f} = \check{h}$. r does not have to be an element of G but notice that, as $q \leq p$ was arbitrary, the set $D := \{r \in \mathbb{P} : \exists h \in (B^A)^M (r \Vdash \mathring{f} = \check{h})\}$ is dense below p. As p was in G, we have that $G \cap D \neq \emptyset$ and thus there is some $r \in G$ and some $h \in (B^A)^M$ with $r \Vdash \mathring{f} = \check{h}$. In particular, $f = h \in (B^A)^M$. $\square$

In particular ${}^\kappa\kappa$ is unchanged for $\kappa < \lambda$, so no κ-reals are added.

Lemma 4.7. $\mathrm{Fn}_\lambda(I, J)$ *is* $\mathrm{cf}(\lambda)$*-closed. In particular, if λ is regular,* $\mathrm{Fn}_\lambda(I, J)$ *is λ-closed.*

Proof. Just take $\bigcup_{\xi<\delta} p_\xi$ for a sequence $\langle p_\xi \rangle_{\xi<\delta}$ where $\delta < \mathrm{cf}(\lambda)$. $\square$

Theorem 4.8. *Let λ be regular, $|J| \leq \lambda$ and $2^{<\lambda} = \lambda$ in M. Then* $\mathrm{Fn}_\lambda(I, J)$ *preserves (all) cardinals. In particular, for every $\alpha \in o(M)$, $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$.*

Proof. By Lemma 4.3, $\mathrm{Fn}_\lambda(I, J)$ has the $(|J|^{<\lambda})^+ \leq ((2^{<\lambda})^{<\lambda})^+ = (2^{<\lambda})^+ = \lambda^+$-cc (use regularity) and thus the Lemma follows for cardinals $\geq \lambda^+$ by Theorem 4.4. For a cardinal $\kappa \leq \lambda$, notice that κ not being a cardinal would be witnessed by a bijection $f : \delta \to \kappa$ for some $\delta < \kappa$, which by Lemma 4.6 would be in M. $\square$

In the first section we used Cohen forcing to make the Continuum large. We now give an example of how to force the Continuum Hypothesis, which completes the proof of Theorem 2.12.

Example 4.9. Let G be $\mathrm{Fn}_{\aleph_1}(\aleph_1, \mathfrak{c})$ generic over M, then $M[G] \models CH$.

Proof. $\mathrm{Fn}_{\aleph_1}(\aleph_1, \mathfrak{c})$ is $\aleph_1$-closed and thus $({}^{\aleph_0}2)^M = ({}^{\aleph_0}2)^{M[G]}$. Also $(\aleph_1)^M = (\aleph_1)^{M[G]}$, because no bijection from a smaller ordinal to $(\aleph_1)^M$ could have been added and so it stays the next cardinal after ω (also no ordinal between ω and $\aleph_1$ can become a cardinal because of downwards absolutness). $\bigcup G$ is a function from $(\aleph_1)^M$ onto $(\mathfrak{c})^M$. Furthermore we get a function from $(\aleph_1)^{M[G]} = (\aleph_1)^M$ onto $({}^{\aleph_0}2)^{M[G]}$ and thus in $M[G]$, $\mathfrak{c} = \aleph_1$. $\square$

We now give an example of forcing the negation of the *CH* first at a $\aleph_7$. For this we begin with a model of *GCH*. This is legitimated as *GCH* was shown to be consistent with *ZFC* by Gödel (see [4]). For *GCH* there is no simple forcing extension as presented in this thesis (it can be done with a proper class poset that simultaneously collapses continua). The advantage of starting with *GCH* (or even with the stronger $V = L$) is that the combinatorics and especially cardinal arithmetic are easier to handle with these assumptions.

Example 4.10. Let M satisfy *GCH*, G be $\mathrm{Fn}_{\aleph_7}(\aleph_9 \times \aleph_7, 2)$ generic over M, then $M[G] \models 2^{\aleph_7} = \aleph_9 \wedge \forall n \neq 7(2^{\aleph_n} = \aleph_{n+1})$.

Proof. As usual G codes an injection of $\aleph_9$ into $({}^{\aleph_7}2)^{M[G]}$. $\mathrm{Fn}_{\aleph_7}(\aleph_9 \times \aleph_7, 2)$ is $\aleph_7$-closed and thus $\forall n < 7(2^{\aleph_n} = \aleph_{n+1})$ follows. The poset has the $\aleph_7$-cc and furthermore there are at most $({\aleph_9}^{\aleph_6})^{\aleph_6 \cdot \kappa} \leq \kappa^+$ nice names for subsets of κ ($\kappa \geq \aleph_8$). All in all we can conclude the above statement. □

Generalized Cohen forcing has similar "product-like" properties as the usual Cohen forcing, that will be useful for determining the structure of the reals in $M[G]$. The proofs for the for the following three Lemmata are the same as for Lemma 1.29, 1.30 and 1.31.

Lemma 4.11. *Let $I \in M$, J_0, J_1 a partition of I in M. If G is $\mathbb{C}_{\kappa,I}$ generic over M, then $G{\restriction}J_0$ is $\mathbb{C}_{\kappa,J_0}$ generic over M and $G{\restriction}J_1$ is $(\mathbb{C}_{\kappa,J_1})^M$ generic over $M[G]$.*

It is important to notice that $\mathbb{C}_{\kappa,J_1}$ is not necessarily absolute between M and $M[G]$ unless κ is regular. When κ is collapsed, then $\mathbb{C}_{\kappa,J_1}$ doesn't even mean anything in $M[G]$. That is why we wrote $(\mathbb{C}_{\kappa,J_1})^M$.

Lemma 4.12. *Let $I \in M$, J_0, J_1 a partition of I in M, G $\mathbb{C}_{\kappa,I}$ generic over M. Then $M[G] = M[G{\restriction}J_0][G{\restriction}J_1]$.*

Lemma 4.13. *Let G be $\mathbb{C}_{\kappa,I}$ generic over M, $\mathring{A}$ a name for a subset in $M[G]$ of some set $X \in M$, then there is a $J \subseteq I$ in M of size at most $|X| \cdot (\kappa^{<\kappa})$ and a $\mathbb{C}_{\kappa,J}$ name ϑ such that* $\mathrm{val}(\mathring{A}, G) = \mathrm{val}(\vartheta, G{\restriction}J)$.

4.2 Applications of generalized Cohen Forcing

We want to use the generalized Cohen forcing to obtain similar results for $\mathfrak{b}(\kappa)$, $\mathfrak{s}(\kappa)$, $\mathfrak{d}(\kappa)$ and $\mathfrak{a}(\kappa)$ as in Section 2, where κ is an uncountable infinite cardinal. Most of the time we will require κ also to be regular. One reason for this is that, when κ is singular, $\mathbb{C}_{\kappa,\lambda}$ is only $\mathrm{cf}(\kappa)$-closed and so $\mathbb{C}_{\kappa,\lambda}$ does not have to preserve cardinals. Even the contrary is the case, namely that κ will always collapse to its cofinality. So if we want to get some relative consistency about the cardinal characteristics at a singular cardinal using the Cohen forcing, in the generic extension we will end up again with a regular cardinal (or rather an ordinal which has regular cardinality). Also, when κ is collapsed the notion of $<^*$ in κ^κ is not necessarily absolute anymore.

Proposition 4.14. *Let $\kappa \in M$ be a singular cardinal in M. Let G be $\mathbb{C}_{\kappa,\lambda}$ generic for any λ. Then in $M[G]$, $|\kappa| = \mathrm{cf}(\kappa)$.*

Proof. First of all we note that $\mathrm{cf}(\kappa)$ is really a cardinal in $M[G]$ by $\mathrm{cf}(\kappa)$-closedness. Let $h := \bigcup G$, then h is a total function $\lambda \times \kappa \to \kappa$. Let $f \colon \kappa \to \kappa$ be defined by $f(\alpha) = h(0, \alpha)$. Let $\langle \alpha_i \rangle_{i<\mathrm{cf}(\kappa)}$ be a strictly increasing cofinal sequence of regular cardinals in κ (in M).

Then we can define a surjection $g\colon \mathrm{cf}(\kappa) \to \kappa$, where $g(i) := \alpha$ if $f([\beta, \alpha_i)) = \{\alpha\}$ for some β (α is then unique) and 0 else. To see that g is a surjection, just consider the following dense sets: $D_\alpha := \{p \in \mathbb{C}_{\kappa,\lambda} : p(0, [\beta, \alpha_i)) = \{\alpha\}$ for some β and $i\}$. □

Let's get back to the cardinal characteristics. From now on κ will always be a regular cardinal if not explicitly stated otherwise. We remind the reader also that, then κ is still a regular cardinal in $M[G]$ when forcing with $\mathbb{C}_{\kappa,\lambda}$ and it stays the same $\aleph_\alpha$. We will very often use the following in our arguments:

Lemma 4.15. *If $N \supseteq M$ is a transitive extension of M which preserves the cardinals $\leq \kappa$, then the notions of bounding, splitting and almost disjointness at κ are absolute for N and M. More precisely, if $f, g \in (\kappa^\kappa)^M$ and $A, B \in (\mathcal{P}(\kappa))^M$, then $(f <^* g)^M$ iff $(f <^* g)^N$, $(A$ splits $B)^M$ iff $(A$ splits $B)^N$ and $(|A \cap B| < \kappa)^M$ iff $(|A \cap B| < \kappa)^N$.*

The analogues of Lemmata 2.1-2.5 can now be proven in the very same way:

Lemma 4.16. *Let G be $\mathbb{C}_{\kappa,I}$ generic for some $I \in M$. Let $h := \bigcup G$. Then any function f_i, where $f_i(\alpha) := h(i, \alpha)$, is unbounded over M, that is, no $f \in (\kappa^\kappa)^M$ dominates f_i.*

Lemma 4.17. *Let f_i be as in Lemma 4.16, then for any $g \in (\kappa^\kappa)^M$, $g \not<^* f_i$.*

Lemma 4.18. *Let f_i be as in Lemma 4.16, then for any $g \in (\kappa^\kappa)^M$ so that $|\{\alpha \in \kappa : g(\alpha) > 0\}| = \kappa$, $g \not\leq^* f_i$.*

Lemma 4.19. *Let f_i be as in Lemma 4.16, then for any $\alpha \in \kappa$, $|\{\beta \in \kappa : f_i(\beta) = \alpha\}| = \kappa$. Or even $|\{\beta \in X : f_i(\beta) = \alpha\}| = \kappa$ for any $X \in ([\kappa]^\kappa)^M$.*

Lemma 4.20. *The generic sets $S_i := \{\alpha \in \kappa : f_i(\alpha) = 0\}$ for $i \in I$ split all sets in $([\kappa]^\kappa)^M$ and are split by all sets $X \in ([\kappa]^\kappa)^M$, where $|\kappa \setminus X| = \kappa$.*

We want to add at this point that the statements made in the last five Lemmata all have to be read relativized to $M[G]$. Saying a set has cardinality κ will always be false in $\mathbf{V}$, unless $\kappa = \omega$, because for $\mathbf{V}$, κ is merely a countable ordinal. We didn't have this problem in Lemmata 2.1-2.5. It should always be clear whether a statement is made in M, $M[G]$ or $\mathbf{V}$.

Theorem 4.21. *Let M satisfy $\kappa^{<\kappa} = \kappa$ (for example via the* GCH*). Let $\lambda \geq \kappa^+$, G $\mathbb{C}_{\kappa,\lambda}$ generic. Then $M[G] \models \mathfrak{b}(\kappa) = \mathfrak{s}(\kappa) = \kappa^+$.*

Proof. $\mathbb{C}_{\kappa,\lambda}$ is κ-closed and has the $(\kappa^{<\kappa})^+ = \kappa^+$-cc and thus it preserves cardinals.

Let $h := \bigcup G$ and $f_i(\alpha) := h(i, \alpha)$ for $i \in \lambda$ as always. Then $\mathcal{B} := \{f_i : i < \kappa^+\}$ is an unbounded family of size κ^+ in $M[G]$. Because let $g \in M[G]$, then $g \subseteq \kappa \times \kappa$, so there is a $J \subseteq \lambda$ of size $|\kappa \times \kappa| \cdot \kappa = \kappa$ in M so that $g \in M[G{\restriction}J]$ by Lemma 4.13. As $|J| = \kappa$, there is some $i < \kappa^+$ so that $i \notin J$; that is $i \in \lambda \setminus J$. Then $f_i \in M[G{\restriction}J][G{\restriction}(\lambda \setminus J)] = M[G]$ and in particular f_i is unbounded over $M[G{\restriction}J]$, so $f_i \not<^* g$ and $\mathcal{B}$ must be unbounded in $M[G]$. For splitting, define $S_i := \{\alpha \in \kappa : f_i(\alpha) = 0\}$, then $\mathcal{S} := \{S_i : i < \kappa^+\}$ is a splitting family by the same argument. □

Lemma 4.22. *For any κ-closed poset $\mathbb{P}$ of size κ, G $\mathbb{P}$ generic over M, if $\mathcal{B} \subseteq \kappa^\kappa$ is unbounded in M, then $\mathcal{B}$ is unbounded in $M[G]$.*

Proof. Let $f \in (\kappa^\kappa)^{M[G]}$, $\mathring{f}$ a $\mathbb{P}$ name for f. Let $p \in G$ with $p \Vdash \mathring{f} \in \kappa^\kappa$ and enumerate $p{\downarrow}$ by $\langle p_i \rangle_{i \in \kappa}$ (conditions may be repeated). We define functions $g_i \in (\kappa^\kappa)^M$ as follows: $g_i(\kappa) := \min\{\beta \in \kappa : \exists q \leq p_i[q \Vdash g_i(\alpha) = \beta]\}$. The family $\{g_i : i < \kappa\}$ has size at most κ and thus is bounded in M by some g. $\mathcal{B}$ is unbounded and thus for some $b \in \mathcal{B}$, $b \not\leq^* g$. For every α the sets $\{q : q \Vdash \exists \beta \geq \alpha[b(\beta) \geq g(\beta) > \mathring{f}(\beta)]\}$ is dense below p. Then for any $\alpha \in \kappa$, $p \Vdash \exists \beta \geq \alpha[b(\beta > \mathring{f}(\beta)]$ and so $M[G] \models b \not\leq^* f$. □

Remark 4.23. If a $\mathbb{P}$ as in the last Lemma that is additionally atomless (i.e. useful) exists, than $\kappa^{<\kappa} = \kappa$ must hold. This is, because one can by a recursion construct a copy of $\kappa^{<\kappa}$ in $\mathbb{P}$.

Lemma 4.24. *Let $I \in M$, G $\mathbb{C}_{\kappa,I}$ generic over M. Let M satisfy $\kappa^{<\kappa} = \kappa$. If $\mathcal{B} \subseteq (\kappa^\kappa)^M$ is unbounded in M, then it stays unbounded in $M[G]$.*

Proof. If f is in $(\kappa^\kappa)^{M[G]}$, then $f \in M[G{\restriction}J]$ for some $J \subseteq J$ of size κ. $G{\restriction}J$ is generic over $\mathbb{C}_{\kappa,J}$ which has size $\kappa^{<\kappa} = \kappa$ and is κ-closed. Thus $\mathcal{B}$ is unbounded in $M[G{\restriction}J]$ and f is not a bound for $\mathcal{B}$. □

Theorem 4.25. *Let M satisfy $\kappa^{<\kappa} = \kappa$. Let $\lambda \geq \kappa^+$, G $\mathbb{C}_{\kappa,\lambda}$ generic. Then $M[G] \models \mathfrak{d}(\kappa) \geq \lambda$. If $\lambda^\kappa = \lambda$, then $M[G] \models \mathfrak{d}(\kappa) = 2^\kappa = \lambda$.*

Proof. $\mathbb{C}_{\kappa,\lambda}$ is κ-closed and has the $(\kappa^{<\kappa})^+ = \kappa^+$-cc and thus it preserves cardinals.

Assume there is a dominating family $\mathcal{D}$ of size $\lambda^* < \lambda$. Then it can be enumerated in $M[G]$ by a function $h\colon \lambda^* \times \kappa \to \kappa$. This is a subset of $\lambda^* \times \kappa \times \kappa$ and thus by Lemma 4.13, $h \in M[G{\restriction}J]$, where $J \subseteq \lambda$ has size at most λ^*. We then have $M[G{\restriction}J][G{\restriction}(\lambda \setminus J)] = M[G]$ and in particular there are unbounded κ-reals over $M[G{\restriction}J]$ in $M[G]$, but then $\mathcal{D}$ cannot be dominating.

If there are at most $\lambda^\kappa = \lambda$ nice names for κ-reals, we get $\lambda = 2^\kappa \leq \mathfrak{d}(\kappa) \leq 2^\kappa$. □

Remark 4.26. Theorem 4.25 actually also holds when $\kappa^{<\kappa} > \kappa$. In that case, the size of J, namely $\lambda^* \cdot \kappa^{<\kappa}$ will still equal to λ^* because λ^* must be greater than $\kappa^{<\kappa}$, as $\kappa^{<\kappa}$ collapses to κ in $M[G]$ (a generic real $f \in \kappa^\kappa$ can code a surjection by identifying functions of the form $f{\restriction}[\alpha, \alpha + \delta)$ with elements in $\kappa^{<\kappa}$) and $\mathcal{D}$ cannot have size κ.

It is clear that $\star(\kappa) < \star(\lambda)$ is consistent for $\kappa < \lambda$ regular when $\star$ is any of the four cardinal characteristics. Also, clearly $2^\kappa \leq 2^\lambda$. So it is natural to ask if we can have $\star(\lambda) < \star(\kappa)$. In the Cohen extensions, when starting with a model of *GCH*, $\mathfrak{b}(\kappa) = \mathfrak{s}(\kappa) < \mathfrak{b}(\lambda) = \mathfrak{s}(\lambda)$ for $\kappa < \lambda$ regular, they are just equal to κ^+ and λ^+ respectively. But for $\mathfrak{d}(\kappa)$ the situation is different:

Theorem 4.27. *Let $\kappa < \lambda$ be regular cardinals and let κ', λ' be such that* $\operatorname{cf}(\kappa') > \kappa$ *and* $\operatorname{cf}(\lambda') > \lambda$. *Then there is a forcing extension in which $\mathfrak{d}(\kappa) = \kappa'$, $\mathfrak{d}(\lambda) = \lambda'$.*

First a Lemma that will be useful:

Lemma 4.28 (Approximation Lemma)**.** *If $\mathbb{P}$ has the κ^+-cc, G is $\mathbb{P}$ generic over M, $f\colon A \to B$ is a function in $M[G]$, where A,B are sets in M. Then there is $F\colon A \to [B]^{\leq\kappa}$ in M such that $f(a) \in F(a)$ for every $a \in A$.*

Proof. For $p \in G$ that forces $f\colon A \to B$, we let $F(a) := \{b \in B : \exists q \leq p[q \Vdash f(a) = b]\}$. □

This approximation Lemma is usually used for the proof that κ^+-cc posets preserve cardinals above κ.

Proof of Theorem 4.27. Start with $M \models GCH$. First force with $\mathbb{C}_{\lambda,\lambda'}$ to get a model $N = M[G]$. Then we have that in M, $(\lambda')^\lambda = \lambda'$ and so by Theorem 4.25, $N \models \mathfrak{d}(\lambda) = \lambda'$. Now take $\mathbb{C}_{\kappa,\kappa'}$ in N and let K be $\mathbb{C}_{\kappa,\kappa'}$ generic over N. We check if the asumptions of Theorem 4.25 are still met. In N, $\kappa^{<\kappa}$ is still equal to κ as $\mathbb{C}_{\lambda,\lambda'}$ was $\lambda(>\kappa)$-closed. And $(\kappa')^\kappa$ is still equal to κ' by the same argument. Also κ is still regular. Thus $N[K] \models \mathfrak{d}(\kappa) = \kappa'$ and cardinals are still preserved. We have to argue that $\mathfrak{d}(\lambda)$ has not changed. First of all, dominating families in N, stay dominating. Because let $f \in (\lambda^\lambda)^{N[K]}$, then by Lemma 4.28 there is $F\colon \lambda \to [\lambda]^{\leq\kappa}$ in N so that $f(\alpha) \in F(\alpha)$. Define $h(\alpha) := \sup F(\alpha)$. This is well-defined as λ is regular. But now, any function that dominates h also dominates f. Next, for any dominating family in $N[K]$ there is a dominating family in N of the same size. A dominating family can be seen as a function $d\colon \mu \times \lambda \to \lambda$. By the same argument as before we can get a family of size less or equal in N, which is also dominating. Thus $(\mathfrak{d}(\lambda))^{N[K]} = (\mathfrak{d}(\lambda))^N = \lambda'$. □

We now get that for arbitrary regular λ, κ everything is possible: $\mathfrak{d}(\kappa) < \mathfrak{d}(\lambda)$, $\mathfrak{d}(\kappa) = \mathfrak{d}(\lambda)$ and $\mathfrak{d}(\lambda) < \mathfrak{d}(\kappa)$. For example we have the consistency of $\mathfrak{d}(\omega_1) = \aleph_3 < \mathfrak{d} = \aleph_7$. Moreover we have shown $\mathfrak{d}(\kappa)$ can be separated from 2^κ when κ is uncountable. Something similar works for $\mathfrak{b}$ if we use the fsi poset presented at the end of section 3.1.

We now move to mad families.

Proposition 4.29. *Let M satisfy* GCH *and let $\mathbb{P}$ be a κ-closed forcing poset of size κ in M. Then there is a mad family $\mathcal{A} \subseteq [\kappa]^\kappa \cap M$ of size κ^+ that is still mad in $M[G]$ for any generic G.*

Proof. By the *GCH* the number of nice names for subsets of κ is bounded by $(\kappa^\kappa)^\kappa = \kappa^+$ ($\mathbb{P}$ clearly has the κ^+-cc). Let us enumerate the ordered pairs (τ_ξ, p_ξ) of nice names for a subset of κ and conditions in $\mathbb{P}$ with $\xi \in [\kappa, \kappa^+)$. We start with an almost disjoint family $\langle A_i \rangle_{i\in\kappa}$ of size κ (say a partition of κ). If A_δ are defined for $\delta < \xi$, we continue as follows:

- If $p_\xi \Vdash \forall \delta < \xi(|A_\delta \cap \tau_\xi| < \kappa) \wedge |\tau_\xi| = \kappa$, we imitate τ_ξ by A_ξ as follows. We enumerate $p_\xi{\downarrow} \times \kappa$ by $(q_\alpha, i_\alpha), \alpha \in \kappa$. We also reorder ξ by $\langle \delta_\alpha \rangle_{\alpha\in\kappa}$. For $\alpha \in \kappa$ we chose an $a_\alpha \geq i_\alpha$, so that there is some $s \leq q_\alpha$ with $s \Vdash a_\alpha \in \tau_\xi \setminus (\bigcup_{\beta<\alpha} A_{\delta_\beta})$. This is possible because $q_\alpha \leq p_\xi$ force that $\tau_\xi \cap (\bigcup_{\beta<\alpha} A_{\delta_\beta})$ has size smaller than κ (by κ-closedness, κ stays a regular cardinal). Let $A_\xi := \{a_\alpha : \alpha \in \kappa\}$. A_ξ is then of size κ as it is unbounded in κ and κ is regular. A_ξ also is almost disjoint from all A_δ, $\delta < \xi$.
- Else let $A_\xi = \emptyset$

We let $\mathcal{A} := \{A_\xi : \xi \in \kappa^+\} \setminus \{\emptyset\}$. Assume $X \in [\kappa]^\kappa \cap M[G]$ is almost disjoint from $\mathcal{A}$. Then there is a nice name τ, a condition $p \in G$ and some $\xi \in [\kappa, \kappa^+)$ with $(\tau, p) = (\tau_\xi, p_\xi)$, such that $p_\xi \Vdash \forall \delta < \xi(|\tau_\xi \cap A_\delta| < \kappa) \wedge |\tau_\xi| = \kappa$. But then X cannot be almost disjoint from A_ξ, because for any $i \in \kappa$, $D := \{s \in \mathbb{P} : s \Vdash \exists j \geq i(j \in \tau_\xi \cap A_\xi)\}$ is dense below p. We get that $p \Vdash |\tau_\xi \cap A_\xi| = \kappa$ because of the regularity of κ. So $M[G] \models |X \cap A_\xi| = \kappa$. X cannot be almost disjoint from $\mathcal{A}$. $\mathcal{A}$ is almost disjoint in both M and $M[G]$ because of the absoluteness of "almost disjointness" and it is maximal in M by maximality in $M[G]$. □

Theorem 4.30. *Let M satisfy* GCH *and $\lambda \geq \kappa^+$. Let G be $\mathbb{C}_{\kappa,\lambda}$ generic over M, then $M[G] \models \mathfrak{a}(\kappa) = \kappa^+$.*

Proof. By the *GCH*, $\mathbb{C}_{\kappa,\kappa}$ is a κ-closed forcing poset of size κ. Thus we can apply the last proposition to get a mad $\mathcal{A}$ that stays mad when forcing over $\mathbb{C}_{\kappa,\kappa}$. Let $X \in [\kappa]^\kappa \cap M[G]$. Then there is some $J \subseteq \lambda$ of size κ in M with $X \in M[G \restriction J]$. But then $\mathbb{C}_{\kappa,\kappa}$ and $\mathbb{C}_{J,\kappa}$ are isomorphic and $X \in M[i^{-1}(G \restriction J)] = M[G \restriction J]$ where i is an isomorphism from $\mathbb{C}_{\kappa,\kappa}$ to $\mathbb{C}_{J,\kappa}$ in M. Thus X is almost disjoint from $\mathcal{A}$ and $\mathcal{A}$ must be maximal in $M[G]$. □

The results from this section yield for example the following consistency result:

Theorem 4.31. $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \mathfrak{b}(\aleph_2) = \mathfrak{s}(\aleph_2) = \mathfrak{a}(\aleph_2) = \aleph_3 < \mathfrak{d}(\aleph_2) = 2^{\aleph_2} = \aleph_{17})$.

We want to add that the results of section 3.2 about the spectrum of mad families can easily be generalized to any regular cardinal.

4.3 Some results for singular cardinals

In [8] we considered bounding, splitting and almost-disjointness also for singular cardinals. For λ a singular cardinal and $f, g \in \lambda^\lambda$, we say that f dominates g whenever $|\{\alpha \in \lambda : f(\alpha) \leq g(\alpha)\}| < \lambda$. S splits X means that $|S \cap X| = \lambda$ and $|X \setminus S| = \lambda$. A, B are almost disjoint iff $|A \cap B| < \lambda$. The definitions of $\mathfrak{d}(\lambda), \mathfrak{b}(\lambda), \mathfrak{s}(\lambda)$ are the same as for regular cardinals. $\mathfrak{a}(\lambda)$ is the least size of a mad family $\geq \mathrm{cf}(\lambda)$. In this section, we want to add some results concerning the relationship between these invariants for different λ.

Lemma 4.32. *Let λ be singular, then $\mathfrak{a}(\lambda) \leq \mathfrak{a}(\mathrm{cf}(\lambda))$ and $\mathfrak{s}(\lambda) \leq \mathfrak{s}(\mathrm{cf}(\lambda))$.*

Proof. Let $\kappa := \mathrm{cf}(\lambda)$, $\langle \alpha_i \rangle_{i<\kappa}$ an increasing cofinal sequence of cardinals in λ and let $\langle X_i \rangle_{i<\kappa}$ be a partition of λ so that $|X_i| = \alpha_i$ for each $i \in \kappa$.

For any set $A \in [\kappa]^\kappa$ we define $\tilde{A} \in [\lambda]^\lambda$ as $\tilde{A} = \bigcup_{i \in A} X_i$.

Let $\mathcal{A} \subseteq [\kappa]^\kappa$ be a κ-mad family. Then we define $\tilde{\mathcal{A}} := \{\tilde{A} : A \in \mathcal{A}\}$. We then have that $\tilde{\mathcal{A}} \subseteq [\lambda]^\lambda$ and $\tilde{\mathcal{A}}$ is almost disjoint ($\tilde{A} \cap \tilde{B} = \bigcup_{i \in A \cap B} X_i$). Now assume $X \in [\lambda]^\lambda$ is almost disjoint from $\tilde{\mathcal{A}}$. For any $i \in \kappa$ define $y_i := \min\{j \in \kappa : |X \cap X_j| \geq \alpha_i\}$. This is well defined because if $|X \cap X_j| < \alpha_i$ for all $j \in \kappa$, then $|X| = |\bigcup_{j \in \kappa} X \cap X_j| \leq \alpha_i \cdot \kappa < \lambda$. Let $Y = \{y_i : i \in \kappa\}$, then $Y \in [\kappa]^\kappa$ and Y is almost disjoint from $\mathcal{A}$ because if $|Y \cap A| = \kappa$ for some $A \in \mathcal{A}$, then $|X \cap \tilde{A}| \geq |X \cap \left(\bigcup_{j \in Y \cap A} X_j\right)| \geq \bigcup_{i, y_i \in Y \cap A} \alpha_i = \lambda$. We have arrived at a contradiction, thus $\tilde{\mathcal{A}}$ is maximal. Also $\tilde{\mathcal{A}}$ has the same size as $\mathcal{A}$.

For $\mathcal{S} \subseteq [\kappa]^\kappa$ a splitting family, let $\tilde{\mathcal{S}} := \{\tilde{S} : S \in \mathcal{S}\}$. $\tilde{\mathcal{S}}$ has the same size as $\mathcal{S}$ and is a λ-splitting family. Because let $X \in [\lambda]^\lambda$ and define $Y \in [\kappa]^\kappa$ as before. Then some $S \in \mathcal{S}$ splits Y, as $\mathcal{S}$ is a splitting family. But then also $\tilde{S}$ splits X: $|X \cap \tilde{S}| \geq |X \cap \left(\bigcup_{j \in S \cap Y} X_j\right)| \geq \bigcup_{i, y_i \in S \cap Y} \alpha_i = \lambda$ and $|X \setminus \tilde{S}| \geq |X \cap \left(\bigcup_{j \in Y \setminus S} X_j\right)| \geq \bigcup_{i, y_i \in Y \setminus S} \alpha_i = \lambda$. □

Lemma 4.33. *Let λ be singular, then $\mathfrak{b}(\lambda) \leq \mathfrak{b}(\mathrm{cf}(\lambda))$ and $\mathfrak{d}(\mathrm{cf}(\lambda)) \leq \mathfrak{d}(\lambda)$.*

Proof. Let $\kappa := \mathrm{cf}(\lambda)$, $\langle \alpha_i \rangle_{i<\kappa}$ an increasing cofinal sequence of regular cardinals greater than κ in λ and let $\langle X_i \rangle_{i<\kappa}$ be a partition of λ so that $|X_i| = \alpha_i$ for each $i \in \kappa$.

For any $f \in \kappa^\kappa$ we define $\tilde{f} \in \lambda^\lambda$ by $\tilde{f}(\beta) := \alpha_{f(i)}$, whenever $\beta \in X_i$. Let $\mathcal{B} \subseteq \kappa^\kappa$ be an unbounded family, then define $\tilde{\mathcal{B}} := \{\tilde{f} : f \in \mathcal{B}\}$. $\tilde{\mathcal{B}}$ has the same size as $\mathcal{B}$ and is unbounded. Assume there is some $g \in \lambda^\lambda$, so that $\tilde{f} <^* g$ for all $f \in \mathcal{B}$. Define $h \in \kappa^\kappa$ by $h(i) := \min\{j \in \kappa : |g^{-1}[\alpha_j] \cap X_i| \geq \alpha_i\}$. This is well defined because assume $|g^{-1}[\alpha_j] \cap X_i| < \alpha_i$ for every j, then $\bigcup_{j \in \kappa} (g^{-1}[\alpha_j] \cap X_i) = X_i$ contradicts regularity of

α_i and $\kappa < \alpha_i$. Then h is a bound for $\mathcal{B}$, because assume for some $f \in \mathcal{B}$ and some $X \in [\kappa]^\kappa$, $f(i) \geq h(i)$ for all $i \in X$. But this means that for any $i \in X$ and for all $\beta \in X_i$, $\tilde{f}(\beta) = \alpha_{f(i)} \geq \alpha_{h(i)}$. But $h(i)$ was chosen so that g maps at least α_i elements of X_i below $\alpha_{h(i)}$. Thus for any $i \in X$ there are at least α_i many β's with $\tilde{f}(\beta) \geq g(\beta)$. As X is unbounded in κ, we have that $\tilde{f}(\beta) \geq g(\beta)$ for at least $\bigcup_{i \in X} \alpha_i = \lambda$ many β's. We have arrived at a contradiction.

Now let $\mathcal{D} \subseteq \lambda^\lambda$ be a λ-dominating family. For any $g \in \mathcal{D}$ we define $g' \in \kappa^\kappa$ as before, that is $g'(i) := \min\{j \in \kappa : |g^{-1}[\alpha_j] \cap X_i| \geq \alpha_i\}$. The family $\mathcal{D}' := \{g' : g \in \mathcal{D}\} \subseteq \kappa^\kappa$ has then size less or equal to $\mathcal{D}$ and it is dominating. For any $f \in \kappa^\kappa$, $\tilde{f}$ is dominated by some $g \in \mathcal{D}$. By the same arguments as above, g' dominates f. □

The last two Lemmata were proved in a very similar fashion. What we did in Lemma 4.33, was we defined a map $\varphi : \kappa^\kappa \to \lambda^\lambda, f \mapsto \tilde{f}$ and another one $\psi : \lambda^\lambda \to \kappa^\kappa, g \mapsto g'$, so that for any $f \in \kappa^\kappa$, $g \in \lambda^\lambda$, $\varphi(f) <^* g \to f <^* \psi(g)$. In the context of "generalized Galois-Tukey connections", the pair (φ, ψ) is called a morphism from one so called "relation" $(\lambda^\lambda, \lambda^\lambda, <^*)$ to another one $(\kappa^\kappa, \kappa^\kappa, <^*)$. The corresponding maps from Lemma 4.32 also produce a morphism between specific relations. Finding such morphisms is a very general method to prove inequalities between cardinal characteristics. For more on Galois-Tukey connections see [2, 4].

The inequalities fit perfectly in the situation we have after forcing with $\mathbb{C}_{\kappa,\lambda^+}$ over a model of *GCH*, where $\kappa = \mathrm{cf}(\lambda)$. $\mathfrak{d}(\kappa) = \lambda^+ = 2^\lambda$, and thus $\mathfrak{d}(\lambda) = \lambda^+$. $\mathfrak{a}(\lambda), \mathfrak{b}(\lambda)$ be equal to κ^+ (In [8] we showed that $\mathfrak{b}(\lambda), \mathfrak{a}(\lambda) \geq \mathrm{cf}(\lambda)^+$). $\mathfrak{s}(\lambda)$ is less or equal to κ^+.

A natural question is if we can separate, say $\mathfrak{b}(\lambda)$ from $\mathfrak{b}(\mathrm{cf}(\lambda))$, or if the cardinal characteristics at cardinals of the same cofinality are equal. For dominating, the question is answered by the following Lemma:

Lemma 4.34. *Let λ be an infinite cardinal. Then $\mathfrak{d}(\lambda) > \lambda$.*

Of course for regular λ, we know this, because $\mathfrak{d}(\lambda) \geq \mathfrak{b}(\lambda)$ and $\mathfrak{b}(\lambda) > \kappa$ by the typical diagonalization argument. But we can actually directly carry out a diagonalization for $\mathfrak{d}(\lambda)$, regardless of whether λ is regular or not.

Proof. Let $\mathcal{D} \subseteq \lambda^\lambda$ be of size $\kappa \leq \lambda$ and $\mathcal{D} = \{f_i : i \in \kappa\}$. Partition λ into κ many sets $\langle X_i \rangle_{i<\kappa}$ all of size λ. Define $f : \lambda \to \lambda$ by $f(\alpha) = f_i(\alpha)$ whenever $\alpha \in X_i$. Clearly f is not (strictly) dominated by any of the f_i's. □

Now when we have *GCH*, $\mathfrak{d}(\mathrm{cf}(\lambda)) = \mathrm{cf}(\lambda)^+ < \lambda < \mathfrak{d}(\lambda)$. For example $\mathfrak{d} = \aleph_1 < \mathfrak{d}(\aleph_\omega) = \aleph_{\omega+1}$.

Also we know that $\mathfrak{d}(\lambda)$ is not fixed at λ^+, but can be arbitrarily large, as $\mathfrak{d}(\mathrm{cf}(\lambda))$ can be arbitrarily large. But even without lifting $\mathfrak{d}(\mathrm{cf}(\lambda))$, we can get $\mathfrak{d}(\lambda)$ large. If we take any regular cardinal α between $\mathrm{cf}(\lambda)$ and λ and we add many α Cohen reals, we will add unbounded λ reals. By partitioning λ into parts of size α and using a cofinal sequence, λ many α reals can code a λ real. By a density argument they will be unbounded (for $f \in \lambda^\lambda$, chose dense sets that witness, that for any cardinal δ between α and λ, our real is greater than f on an unbounded set in δ).

Note that in the proofs of $\mathfrak{a}(\lambda) \leq \mathfrak{a}(\mathrm{cf}(\lambda))$, $\mathfrak{s}(\lambda) \leq \mathfrak{s}(\mathrm{cf}(\lambda))$, $\mathfrak{b}(\lambda) \leq \mathfrak{b}(\mathrm{cf}(\lambda))$ and $\mathfrak{d}(\mathrm{cf}(\lambda)) \leq \mathfrak{d}(\lambda)$ we never used the regularity of $\mathrm{cf}(\lambda)$ but only that there is a cofinal increasing sequence of regular cardinals in λ of type $\mathrm{cf}(\lambda)$. These inequalities can be generalized to the following:

Theorem 4.35. *Let $\lambda < \mu$ be cardinals of the same cofinality. Then $\mathfrak{a}(\mu) \leq \mathfrak{a}(\lambda)$, $\mathfrak{s}(\mu) \leq \mathfrak{s}(\lambda)$, $\mathfrak{b}(\mu) \leq \mathfrak{b}(\lambda)$ and $\mathfrak{d}(\lambda) \leq \mathfrak{d}(\mu)$*

What this Lemma tells us is that for $\mathfrak{a}(\lambda)$, $\mathfrak{s}(\lambda)$, $\mathfrak{b}(\lambda)$ there are only finitely many possible values when we fix the cofinality of λ (else we would get an infinite decreasing chain of cardinals).

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