Bachelor thesis

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The Axiom of Foundation

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Contents of the Bachelor Thesis

1. Introduction

This is the Axiom of Foundation:

$\exists y(y \in x) \to \exists y(y \in x \land \neg\exists z(z \in x \land x \in y))$

As this is an axiom, it is not deductively proven from other statements. While the question regarding the validity of this assumption is a philosophical question outside the field of mathematics, mathematically we can just assume it and discuss its implications. To state meaningful sentences, formal mathematics always needs to be based on axioms, so assuming Foundation is nothing extraordinary. In fact, it has become conventional to assume it. While the assumption of Foundation has minimal consequences in other fields of mathematics, it makes a considerable difference when practising set theory. This thesis provides an overview of the consequences of assuming the Axiom of Foundation.

The Axiom of Foundation is an axiom of the theory ZFC, which was developed in the 20th century to provide a formal basis for practicing mathematics.

Section 1 gives a historical overview of the axiomatization of mathematics, followed by some remarks about first-order logic, the logic of usual axiomatic set theory.

In Section 2 we state the axioms of ZFC and the various subtheories of ZFC. The second part of this section is concerned with the Incompleteness Theorems of Kurt Gödel. While we do not prove them in this thesis, we discuss their implications to provide an insight into the limitations of first-order theories.

Section 3 is focused on the implications of assuming Foundation. We begin with some formal definitions that allow us to prove a theorem justifying the use of transfinite recursion. As this allows us to recursively define functions, we introduce two recursively defined functions, namely the Mostowski collapsing function and the rank function. A particularly interesting form of the rank function is defined on the class of well-founded sets, which are the only sets existing when assuming Foundation.

In the remainder of Section 3 we show that the Axiom of Foundation is consistent with the other axioms of ZFC. This means that assuming Foundation does not lead to inconsistency when we assume that the other axioms of ZFC are consistent. Besides a metatheoretical proof regarding all of ZFC, we also do a similar proof within ZFC implying the consistency of Foundation for certain set models of subtheories of ZFC.

At the beginning of Section 4 we discuss the concept of absoluteness and the consequences of Foundation regarding absoluteness. Afterwards we state and prove the Reflection Theorem, which leads us to the final part of this thesis: While we cannot construct a set model of all of ZFC, we show that we can always construct a set model for each finite subtheory of ZFC. Moreover, we show that we can always find a countable transitive model that satisfies this subtheory.

As this is primarily a thesis on axiomatic set theory, very little knowledge of other fields of mathematics is required. It is assumed that the reader is familiar with first-order logic, but detailed knowlege of first-order logic is not required. Sections 1 to 3.3 even do not assume any significant knowledge of set theory. As otherwise it would notably increase the size of this thesis, Section 3.4 and the whole of Section 4 assume basic knowledge of ordinals and model theory. Additionally, a very limited knowledge of cardinals is required for Section 4.2.

1.1. **Axiomatization of Mathematics.** While mathematics has a long history spanning to ancient times, axiomatic set theory is very young in comparison, with its beginnings in the late 19th century, when mathematics gained its characteristic rigorism. The focus on deductive procedures began in analysis, with the concept of infinitesimals being replaced by the deductive concept of a limit [2, 2].

In 1872 both Georg Cantor and Richard Dedekind formulated the real numbers, which began to be thought of as a collection of objects instead of just a continuum. This shift towards arithmetic was another important step towards deductive formality and a focus on arithmetic rather than geometry. As this led Cantor to investigate the size of the continuum, resulting in the proof that the real numbers are uncountable, this formulation of real numbers marked the beginning of those mathematical problems from which set theory arose. The main problem leading to axiomatic set theory was that of the Continuum Hypothesis (CH), which was formulated by Cantor as the hypothesis that "[e]very infinite set of reals either is countable or has the power of the continuum". Finding a solution to the Continuum Problem (the question whether CH is true or false) necessitated the formulation of a suitable theory of mathematics [2, 2-6].

Axiomatization began in 1904 when Ernst Zermelo formulated the Axiom of Choice (AC), being equivalent to the assumption that every set can be well-ordered. While Cantor implicitly assumed that every well-defined set can be well-ordered, Zermelo replaced this assumption with an explicit axiom, giving a wider notion of "set" as set theory is also possible without assuming the Axiom of Choice [2, 12-13].

As the proof of Zermelo's Well-Ordering Theorem (AC holds $\rightarrow$ every set can be well-ordered) was controversial among mathematicians, Zermelo developed the first axiomatization of set theory in 1908 to clarify his assumptions for the proof of the Well-Ordering Theorem. Zermelo's theory consisted of seven axioms, similar but not equal to the theory Z (see 2.1). One difference is that Zermelo's theory allowed for urelements, which are objects without members still distinct from each other [2, 14-15]. This axiomatization provided a solution to Russell's paradox regarding s universal set (a set is called universal if it contains all sets). As Zermelo's axioms imply that for every x there exists a $y \in x$ such that $y \notin x$, it follows that there is no universal set [2, 19].

To integrate transfinite sets, which were of great interest to Cantor, into Zermelo's axiomatic set theory, the axiomatic system had to be expanded. As von Neumann's theory of ordinals depends on the validity of transfinite recursion, Zermelo's seven axioms are insufficient as even formulating the Theorem of Transfinite Recursion depends on the use of Replacement, which was not included in Zermelo's theory. This led to the independent proposal of Fraenkel (1921) and Skolem (1922) to adjoin the Replacement Scheme [2, 32-33].

The Axiom of Foundation was first stated by von Neumann in 1929, following a discussion about the benefits of restricting the universe to well-founded sets. Zermelo's axiomatization of 1930 included Replacement and Foundation, but contrary to the theory known today as ZFC it still allowed for urelements, rejected Infinity and considered Choice part of the underlying logic [2, 34-35].

1.2. **First-Order Logic.** The axiomatic system ZFC is a theory of *first-order logic* or *predicate logic*. As most of usual mathematics takes place within ZFC, first-order logic is by far the most common sort of logic used for formulating mathematical statements. It is assumed that the reader of this thesis has some knowledge of first-order logic and its usual notation, but is not necessary in the course of this thesis to know the exact details of first-order logical syntax and semantics (for a detailed introduction to first-order logic, refer to books such as [1] or [5]). However, we will explicate a few points that are often only implicitly assumed (according to [4]):

Each basic symbol of first-order logic is either a *logical symbol* or a *nonlogical symbol*. Logical symbols are fixed, including the equality symbol $=$, propositinal connectives such as $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$, quantifiers $\forall, \exists$, and variables. As many definitions of first-order logical syntax (such as the one used for this thesis) require parentheses, for example to distinguish between $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$, parentheses are also logical symbols in such cases.

For each application of logic, there is a finite set $\mathcal{L}$ called the *lexicon* or *signature* containing the nonlogical symbols. Each nonlogical symbol has an *arity*, which is a natural number, and a *type*, which is either "predicate symbol" or "function symbol". For example, given $\mathcal{L} = \{\in\}$, the signature of ZFC, $\in$ is a 2-ary predicate symbol. For discussing the theory of ordered rings with unity, we might use $\mathcal{L} = \{+, \Delta, <, 0, 1\}$. The 0-ary function symbols, such as $0, 1$ in this case, are usually called *constant symbols*. It is possible that $\mathcal{L}$ could be empty, in which case we will still have formulas such as $\exists x, y[x \neq y]$ and $\exists x[x = x]$.

As axioms of a theory are always sentences, having no free variables, the actual axiom is the *universal closure*, obtained by universally quantifying each free variable. For example, the Axiom of Pairing is written in Section 2.1 as $\exists z(x \in z \wedge y \in z)$, but it is understood that we really mean $\forall x, y \exists z(x \in z \wedge y \in z)$.

The exact definition of syntax is not fixed and varies between different definitions of formal logic. Using Polish (or prefix) notation, we would write $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ as $\vee \wedge pqr$ and $\wedge p \vee qr$, so one can define logical syntax without parentheses. It is also possible to have a more limited number of connectives, as for example all formulas could be reduced to semantically equivalent formulas using only the connectives $\neg, \wedge$.

While we cannot prove within ZFC that ZFC is consistent (see Section 2.2), we can nevertheless state such an assertion:

Definition 1.1. If Γ is a set of sentences of $\mathcal{L}$, then $\mathrm{Incon}(\Gamma)$ is the assertion that there is a formal proof of a contradiction from Γ. $\mathrm{Con}(\Gamma)$ is $\neg\mathrm{Incon}(\Gamma)$.

To make things more readable for humans, although formally $\mathcal{L} = \{\in\}$ for ZFC, we introduce new symbols. However, the new symbols introduced are always abbreviations of statements using $\mathcal{L} = \{\in\}$. For example, in Section 2.1, we introduce the symbol $\subseteq$, with $x \subseteq y$ meaning $\forall z(z \in x \rightarrow z \in y)$. All mathematical terminology beyond $\in$ and $=$ is an extension by defitions over ZFC.

2. The Axioms of Zermelo-Fraenkel

2.1. **The Axioms.** Most of usual mathematics, such as basic algebra, analysis or topology, takes place within the axiomatic system ZFC (the Zermelo-Fraenkel axioms with the Axiom of Choice) [4]. As ZFC is an axiomatic system of set theory, the language $\mathcal{L} = (\in, \emptyset, \emptyset)$ of set theory is used. While ZFC is commonly said to contain nine axioms, two of the "axioms" are actually schemes of axioms, consisting of a separate axiom for each logical formula. ZFC is not finitely axiomatizable, see [4] for a proof.
The axioms of ZFC are stated according to [3]. As axioms of a theory are always sentences (i.e. formulas with no free variables), the actual axioms are the universal closure of each of the following formulas, obtained by universally quantifying each free variable, see [4].

Axiom 0. Set Existence.
$\exists x(x = x)$

Axiom 1. Extensionality.
$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$

Axiom 2. Foundation.
$\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \neg\exists z(z \in x \wedge x \in y))$

Axiom 3. Comprehension Scheme. For each formula φ, without y free,
$\exists y \forall x(x \in y \leftrightarrow x \in v \wedge \varphi(x))$

Axiom 4. Pairing.
$\exists z(x \in z \wedge y \in z)$

Axiom 5. Union.
$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathcal{F} \rightarrow x \in A)$

Axiom 6. Replacement Scheme. For each formula, φ, without y free,
$\exists x \in A \exists! y \varphi(x, y) \rightarrow \exists B \forall x \in A \exists y \in B \varphi(x, y)$

To improve readability of the remaining axioms, we define the symbols $\subseteq$ (subset), $\emptyset$ or 0 (empty set), S (ordinal successor function), $\cap$ (intersection) and SING(x) (x is a singleton) on the basis of Axioms 1,3,4,5:

$x \subseteq y \iff \forall z(z \in x \rightarrow z \in y)$
$x = \emptyset \iff \forall z(z \notin x)$
$y = S(x) \iff \forall z(z \in y \leftrightarrow z \in x \vee z = x)$
$w = x \cap y \iff \forall z(z \in w \leftrightarrow z \in x \wedge z \in y)$
$\mathrm{SING}(x) \iff \exists y \in x \forall z \in x(z = y)$

Axiom 7. Infinity.
$\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$

Axiom 8. Power Set.
$\exists y \forall z(z \subseteq x \rightarrow z \in y)$

Axiom 9. Choice.
$\emptyset \in F \wedge \forall x \in F(x \neq y \rightarrow x \cap y = \emptyset) \rightarrow \exists C \forall x \in F(\text{SING}(C \cap x))$

Axiom 0, ensuring that the universe is non-empty, is usually omitted in presentations of ZFC. However, a non-empty universe is implicitly assumed in these cases.

Some subtheories of ZFC are of particular interest [4]:

- $ZFC :=$ Axioms 1-9
- $ZF :=$ Axioms 1-8
- ZC and Z are ZFC and ZF, respectively, without Axiom 6 (Replacement)
- $X^- := X$ without Axiom 2 (Foundation)
- $X - P := X$ without Axiom 8 (Power Set)
- $X - Inf := X$ without Axiom 7 (Infinity)

One reason for studying subtheories of ZFC could be to understand how the axioms are used in ordinary mathematics. For example, basic properties of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are developed within $ZF^- - P$, while basic facts about $\mathbb{R}, \mathbb{C}$ require the Power Set Axiom and are developed within ZF^- [4]. Additionally, it might be of interest to consider some of the subtheories for historical or philosophical reasons, especially regarding Replacement and Choice.

Foundation says that $\in$ is well-founded, meaning that every non-empty set has an $\in$-minimal element. This implies that there are no sets a, b with $a \in b \in a$ [3]. The Axiom of Foundation is never used in standard mathematics. As we will see in The consistency of Foundation, the proof of its relative consistency, implying that ZFC is consistent whenever ZFC^- is consistent, shows that Foundation is irrelevant for most of mathematics. However, it simplifies the discussion of models of set theory [4].

There exists an empty set as $\{x \in v : x \neq x\}$ is empty for any v. This set is unique by Extensionality.

We define the *universe* V of all sets:

Definition 2.1. $V := \{x : x = x\}$.

This definition is metatheoretical and cannot be stated within ZFC. V is not a set (it is a *proper class*):

Proof. If there was a set V with $\forall x[x \in V]$, the set $R := \{x \in V : x \notin x\}$ would exist. This would lead to $R \in R \leftrightarrow R \notin R$, a contradiction (Russell's Paradox). $\square$

2.2. **Incompleteness: the Theorems of Kurt Gödel.** While the Incompleteness Theorems of Kurt Gödel are not the focus of this thesis, they are nevertheless stated as they show the limitations of mathematics. While we have seen in Section 1.1 that eliminating self-referential statements eliminates paradoxes, it also eliminates the possibility of a formal theory to prove its own consistency. It is impossible for a formal theory to contain all of mathematics.

For proofs of the Incompleteness Theorems, see [1].

Theorem 2.2. *Gödel's First Incompleteness Theorem*
$\mathrm{Th}\mathfrak{N}$ *is not recursively axiomatizable.*

Theorem 2.3. *Gödel's Second Incompleteness Theorem*
Assume that T is a sufficiently strong recursively axiomatizable theory. Then $T \vdash \mathrm{Cons}T$ if and only if T is inconsistent.

The First Incompleteness theorem says that there is no algorithm capable of deciding the truth value of all true mathematical statements about the natural numbers. No matter what axiomatic system we use, either the system will be inconsistent or there will be statements about $\mathbb{N}$ that are true but undecidable.

The Second Incompleteness Theorem says that there is no sufficiently strong recursively axiomatizable theory ("sufficiently strong" meaning capable of statements about arithmetic) capable of proving its own consistency. So-called consistency proofs in mathematics always assume working in a specific theory and then proving the consistency of another theory. However, we cannot be sure if the theory in which we are doing a consistency proof is consistent. We can only inductively (in the empirical sense) assume its consistency as long as it is not shown to be inconsistent.

A consequence of the Incompleteness Theorems is that we must always practice arithmetic within a defined structure. Regardless of which structure we choose, there will always be a structure containing more arithmetical statements. No consistent formal system can contain all of arithmetic. In contrast to some metatheoretical statements that are abbreviations for a collection of statements within an axiomatic system, a statement regarding "all of arithmetic" cannot be such an abbreviation.

The Incompleteness Theorems are of great importance for the philosophy of science. A formal concept of absolute truth cannot exist as if otherwise, absolute truth would rely on some theory that is (from a hypothetical absolute point of view) of questionable truth. No consistent science can deductively proof its reliability.

3. The Well-Founded Universe

The definitions, theorems and proofs are stated according to [4].

3.1. **Theorem of Transfinite Recursion on Well-Founded, Set-Like Classes.** Given the Axiom of Pairing, we can construct pairs from two given sets:

Definition 3.1.
- $\{x, y\} = \{w : w = x \vee w = y\}$.
- $\{x\} = \{x, x\}$.
- $\langle x, y\rangle = (x, y) = \{\{x\}, \{x, y\}\}$.

$\{x, y\}$ is called an *unordered pair* while $\langle x, y\rangle$ is called an *ordered pair*.

Definition 3.2. R is a (binary) relation iff $\forall u \in R \exists x, y[u = \langle x, y\rangle]$.
xRy iff $\langle x, y\rangle \in R$.
$\neg xRy$ iff $\langle x, y\rangle \notin R$.

Definition 3.3. Let R be a relation.
- R is *transitive* on A iff $\forall xyz \in A[xRy \wedge yRz \to xRz]$.
- R is *irreflexive* on A iff $\forall x \in A[\neg xRx]$.
- R satisfies *trichotomy* on A iff $\forall xy \in A[xRy \vee yRx \vee x = y]$.
- *R totally orders A strictly* iff R is transitive and irreflexive on A and satisfies trichotomy on A.

Definition 3.4. R is a *function* iff R is a relation and $\forall xyz[(x, y) \in R \wedge (x, z) \in R \to y = z]$. If $\exists y[xRy]$, $R(x)$ denotes the unique y satisfying xRy.

Definition 3.5. Let R be any set.

- $\mathrm{dom}(R) := \{x : \exists y[(x, y) \in R]$
- $\mathrm{ran}(R) := \{y : \exists x[(x, y) \in R]$

These definitions are justified by Union and Comprehension. While $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are defined for any R, they are mainly applied to functions.

Definition 3.6. $R \upharpoonright A = \{(x, y) \in R : x \in A\}$.

This, called *restriction*, is mostly used for functions. The subset $R \upharpoonright A$ exists because of Comprehension.

Definition 3.7. Let R be a relation. $y \in X$ is *R-minimal* in X iff $\neg\exists z(z \in X \wedge zRy)$. R is *well-founded* on A iff for all non-empty $X \subseteq A$, there is a $y \in X$ that is R-minimal in X. *R well-orders A* iff R totally orders A strictly and is well-founded on A.

Definition 3.8. z is a *transitive set* iff $\forall y \in z[y \subseteq z]$.

An example of transitive sets are the ordinal numbers $0, 1, 2, \ldots, \omega, \omega + 1, \ldots,$ one definition of ordinals being transitive sets well-founded by $\in$. However, the class ON of all ordinals is not a set. If ON was a set, it would have to be an ordinal itself, implying $ON \in ON$, a contradiction to irreflexivity of $\in$.

Definition 3.9. Let R be a relation on a class A. If $y \in A$, let $y \downarrow = pred_R(y) = pred_{A,R}(y) = \{x \in A : xRy\}$. Then R is *set-like* on A iff $y\downarrow$ is a set for all $y \in A$.

For example, $\in$ is set-like on any class, in particular on ON. $\subseteq$ is set-like on V iff the Power Set Axiom holds.

Now we have all the concepts needed to prove the validity of transfinite induction for a transitive well-founded relation R. However, we introduce the *transitive closure* of a relation to prove that R need not be transitive:

Definition 3.10. For a relation R and a class A:

- s is a *path* (or R-path) of n *steps* in A iff $n \in \omega$, $n \geq 1$, s is a function, $\mathrm{dom}(s) = n+1$, $\mathrm{ran}(s) \subseteq A$, and $\forall j < n[s(j)Rs(j+1)]$. This s is called a path from $s(0)$ to $s(n)$.
- The *transitive closure* of R on A is the relation $R^* = R^*_A$ on A defined by xR^*y iff there exists a path in A from x to y.

Therefore $R \restriction A \subseteq R^* \restriction A$, and $R \restriction A = R^* \restriction A$ iff R is transitive on A.

Theorem 3.11. *Transfinite Induction on Well-founded Relations*
Assume that R is well-founded and set-like on A, and that X is a non-empty sub-class of A. Then X has an R-minimal element.

Proof. Fix any $a \in X$. Let b be an R-minimal element of the set $\{a\} \cup (\mathrm{pred}_{R^*}(a) \cap X)$. Then b is an R-minimal element of X, since $yRb \to y \in \mathrm{pred}_{R^*}(a)$. $\square$

As this theorem holds for all classes including proper classes, it is really a scheme in the metatheory. The scheme consists of a separate theorem for each combination of three formulas respectively defining R, A and X.

The theorem is typically used to justify inductive proofs. Assuming R being well-founded and set-like on A, if we want to prove that $\forall a \in A\varphi(a)$ for a given formula φ, we form $X := \{a \in A : \neg\varphi(a)\}$. To show that X is empty, we assume $X \neq \emptyset$ and derive a contradiction from the R-minimal element of X.

Theorem 3.12. *Transfinite Recursion on Well-founded Relations*
Assume that R is well-founded and set-like on A and $\forall x, s\exists! y\varphi(x, s, y)$. Define $G(x, s)$ to be the unique y such that $\varphi(x, s, y)$. Then we can write a formula ψ for which the following are provable:

(1) $\forall x\exists! y\psi(x, y)$*, so ψ defines a function F, where $F(x)$ is the y such that $\psi(x, y)$.*
(2) $\forall a \in A[F(a) = G(a, F \restriction (a\downarrow))]$

Proof. For sets d,h, let $App(d, h)$ say that h is a function, $\mathrm{dom}(h) = d \subseteq A$, $\forall y \in d[y\downarrow \subseteq d]$, and $\forall y \in d[h(y) = G(y, h \restriction (y\downarrow))]$; so we are saying that h is an *approximation* to F defined on d. Note that $\forall y \in d[y\downarrow \subseteq d]$ implies also $\forall y \in d[\mathrm{pred}_{A,R^*}(y) \subseteq d]$. An important set with this property is:
$d_x := \{x\} \cup \mathrm{pred}_{A,R^*}(y)$
for any $x \in A$, this is a set because R^* is set-like.
Assuming the theorem is true, $App(d, h)$ implies that $h = F \restriction d$. However, since we have to prove the theorem first, we will use $App(d, h)$ to define a ψ satisfying (1) and (2):

$\psi(x, y) \iff [x \notin A \wedge y = \emptyset] \vee [x \in A \wedge \exists d, h[\mathrm{App}(d, h) \wedge x \in d \wedge h(x) = y]]$.
We now need to prove the existence and uniqueness of these approximations. To prove uniqueness, we have to show that all the approximations agree wherever they are defined:
$\mathrm{App}(d, h) \wedge \mathrm{App}(d', h') \to \mathrm{App}(d \cap d', h \cap h')$. (U)
To verify this, note first that $\forall y \in (d \cap d')[y \downarrow \subseteq (d \cap d')]$. Then, note that $h(y) = h'(y)$ for all $y \in d \cap d'$, since an R-minimal element of $\{y \in d \cap d' : h(y) \neq h'(y)\}$ would be contradictory, using $h(y) = G(y, h \upharpoonright (y \downarrow))$. So, the intersection $h \cap h'$ is really a function with domain $d \cap d'$ that takes $y \in d \cap d'$ to $h(y) = h'(y)$. Then $\mathrm{App}(d \cap d', h \cap h')$ is clear.
By (U), we know that for all x, there is at most one y such that $\psi(x, y)$. To prove that such a y always exists, use:
$\forall x \in A \exists d, h[\mathrm{App}(d, h) \wedge x \in d]$. (E)
To prove (E), we apply transfinite induction on R. First observe that $\mathrm{App}(d, h) \wedge x \in d \to \mathrm{App}(d_x, h_x)$, where $h_x = h \upharpoonright d_x$. Assuming that (E) is false, let $X = \{x \in A : \neg \exists d, h[\mathrm{App}(d, h) \wedge x \in d]\} \neq \emptyset$. Observe that for $x \notin X$, we have an h_x such that $\mathrm{App}(d_x, h_x)$, and this h_x is unique by (U). Let $a \in X$ be R-minimal in X. Let $d = \mathrm{pred}_{A,R^*}(a) = \bigcup\{d_x : xRa\}$. By minimality, $xRa \to x \notin X$, so by the Replacement Axiom, we may define the set $\tilde{h} = \bigcup\{h_x : xRa\}$, which is a function by (U), and it is then easily verified that $\mathrm{App}(\tilde{d}, \tilde{h})$. Now $a \notin \tilde{d}$, but $a \downarrow \subseteq \tilde{d}$. Informally $F \upharpoonright \tilde{d}$ "should be" $G(a, \tilde{h} \upharpoonright (a \downarrow))$. Formally, let $d = \tilde{d} \cup \{a\}$ and let $h = \tilde{h} \cup \{(a, G(a, \tilde{h} \upharpoonright (a \downarrow)))\}$. Then $\mathrm{App}(d, h)$ and $a \in d$, contradicting $a \in X$.
Combining (U) and (E), we know that $\forall x \in A \exists! y \psi(x, y)$, so ψ defines a function F, as in (1). Then (2) follows from the definition of $\mathrm{App}(d, h)$. □

The theorem of transfinite recursion gives us the justification to recursively define functions on an (even proper) class A as long as the respective relation R is well-founded and set-like on A. Theorem 3.12 is false whenever R is not well-founded (see[4]) and meaningless whenever R is not set-like as the "theorem" is in fact a scheme of theorems. As proper classes do not exist in set theory, it is not even possible to state the respective theorems.
We will take advantage of this proof to introduce two recursively defined functions, namely the Mostowski collapsing function and the rank function.

3.2. **The Mostowski Collapsing Function.** While the Mostowski collapsing function is not necessary to define the class of well-founded sets, it will be needed later in Section 4. Additionally, this section contains a lemma stating that $\in$ is extensional on any transitive class. This result is used for a proof in Section 3.4.
In the following sections, we will only need the Mostowski function for proofs regarding the relation $\in$, but we define the Mostowski collapsing function for any well-founded and set-like relation:

Definition 3.13. Assume that R is well-founded and set-like on A. Define, recursively, for $y \in A$, $\mathrm{mos}(y) = \mathrm{mos}_{A,R}(y) = \{\mathrm{mos}(x) : x \in y \downarrow\}$. This is called the *Mostowski collapsing function*. Here, $y \downarrow = \mathrm{pred}_{A,R}(y)$.

This is justified by the Theorem of Transfinite Recursion:

Proof. Use Theorem 3.12. Let $G(x, s) = \mathrm{ran}(s)$; note that this does not depend on x, and is defined for all sets x, s. Then $F(y) = G(y, F \restriction (y \downarrow))$ translates to $F(y) = \{F(x) : x \in A \wedge zRy\} = \{F(x) : x \in y \downarrow\}$. □

Lemma 3.14. *Assume that R is well-founded and set-like on A. Then* mos“A *is transitive.*

Proof. By definition, every $\mathrm{mos}(y) = \{\mathrm{mos}(x) : x \in y \downarrow\}$ is a subset of mos“A. □

The Mostowski function on A need not be injective, but if (A, R) is a model for the Axiom of Extensionality, then mos is injective and provides an isomorphism from (A, R) onto the membership relation on some transitive set:

Definition 3.15. The relation R is *extensional* on A iff (A, R) satisfies the Axiom of Extensionality; equivalently $\forall x, y \in A[x \downarrow = y \downarrow \to x = y]$.

Since $x \downarrow = x$ when R is $\in$ and A is transitive,

Lemma 3.16. *The $\in$ relation is extensional on A whenever A is transitive.*

Lemma 3.17. *Assume that R is well-founded and set-like on A. Then the function* $\mathrm{mos}_{A,R}$ *is 1-1 iff R is extensional on A, in which case* mos *provides and isomorphism from (A, R) onto* (mos“$A, \in$).

Proof. If R is not extensional on A, then there are $a \neq b$ in A with $a \downarrow = b \downarrow$, which implies that $\mathrm{mos}(a) = \mathrm{mos}(b)$; thus, $\mathrm{mos}_{A,R}$ is not 1-1.
Conversely, assume that R is extensional on A. We shall prove that mos is 1-1. We wish to show $a \neq b \to \mathrm{mos}(a) \neq \mathrm{mos}(b)$ by "induction on (a, b)", but rather than intorucing a concept of "double induction", we quantify out the b, and let $X = \{a \in A : \exists y \in A[a \neq y \wedge \mathrm{mos}(a) = \mathrm{mos}(y)]$. If $X = \emptyset$, then mos is 1-1, so assume that $X \neq \emptyset$. Applying induction (Theorem 3.11), let $a \in X$ be R-minimal in X. Then fix any $b \in X$ such that $a \neq b$ and $\mathrm{mos}(a) = \mathrm{mos}(b)$. Since R is extensional, $a \downarrow \neq b \downarrow$. There are now two cases:
Case I. There is a c with cRa and $\neg cRb$. Since $\mathrm{mos}(c) \in \mathrm{mos}(a) = \mathrm{mos}(b) = \{\mathrm{mos}(z) : z \in b \downarrow\}$, there is a d with dRb and $\mathrm{mos}(d) = \mathrm{mos}(c)$, and note that $c \neq d$ because $\neg cRb$. But then $c \in X$ and cRa, contradiction minimality of a.
Case II. There is a d with dRb and $\neg dRa$. Exactly as above, we find a c with cRa and $c \neq d$ and $\mathrm{mos}(d) = \mathrm{mos}(c)$. Again, $c \in X$ and cRa, constradicting minimality of a.
Now that we know that mos is 1-1, the fact that mos is an isomorphism (that is, $\forall x, y \in A[xRy \leftrightarrow \mathrm{mos}(x) \in \mathrm{mos}(y)]$), follows immediately from the definition of $\mathrm{mos}(y)$ as $\{\mathrm{mos}(x) : x \in y\}$. □

When $R = \in$ and A is transitive, then mos is the identity function on A. More generally:

Lemma 3.18. *Assume that $\in$ is well-founded and extensional on A. Let $T \subseteq A$ be transitive. Then* $\mathrm{mos}_{A,\in}(y) = y$ *for all $y \in T$.*

Proof. If a is $\in$-minimal in $\{y \in T : \mathrm{mos}(a) \neq a\}$, then $\mathrm{mos}(a) = \{\mathrm{mos}(y) : y \in A \wedge y \in a\} = \{y : y \in a\} = a$, a contradiction. The second = used minimality of a and the fact that $y \in a \to y \in T \subseteq A$. □

3.3. **Definition of the Well-Founded Sets and Well-Founded Universe.** The following lemma is not needed until the end of this chapter, but it is stated here as it serves as an introduction to the rank function:

Lemma 3.19. *Let R be a relation on a class A, and assume that we have defined a function $\Phi : A \to ON$ such that $xRy \to \Phi(x) < \Phi(y)$ for all $x, y \in A$. Then R is well-founded.*

Proof. If $X \subseteq A$, then any $a \in X$ with $\Phi(a) = \min\{\Phi(x) : x \in X\}$ is R-minimal in X. □

The Φ is not uniquely defined. Transfinite recursion lets us define the rank function, which is the optimum Φ for well-founded relations:

Definition 3.20. Assume that R is well-founded and set-like on A. Define, recursively, for $y \in A$, $\mathrm{rank}(y) = \mathrm{rank}_{A,R}(y) = \bigcup\{S(\mathrm{rank}(x)) : x \in y\downarrow\}$. Let $\mathrm{rank}(y) = \emptyset$ for $y \notin A$.

This is justified by Theorem 3.12:

Proof. Let $G(x, s) = \bigcup\{S(t) : t \in \mathrm{ran}(s)\}$; note that this does not depend on x, and is defined for all sets x, s. Then $F(a) = G(a, F \restriction (a\downarrow))$ translates to $F(a) = \bigcup\{S(F(c)) : c \in A \wedge cRa\}$. □

The rank is always an ordinal. Therefore, the following lemma uses sup for $\bigcup$ and $\alpha + 1$ for $S(\alpha)$:

Lemma 3.21. *If R is well-founded ond set-like on A, then* $\mathrm{rank}(y)$ *is an ordinal for all* $y \in A$, *so* $\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 : x \in A \wedge xRy\}$. *Furthermore,* $xRy \to \mathrm{rank}(x) < \mathrm{rank}(y)$ *for all* $x, y \in A$.

Proof. Prove $\mathrm{rank}(y) \in ON$ by transfinite induction; an R-minimal y such that $\mathrm{rank}(y) \notin ON$ would be contradictory because the union of a set of ordinals is an ordinal. Then the "Furthermore" is clear, since we are taking a sup, so $xRy \to \mathrm{rank}(x) + 1 \leq \mathrm{rank}(y)$. □

The rank function does not skip any ordinals:

Lemma 3.22. *Assume that R is well-founded and set-like on A. Fix $b \in A$, and fix* $\alpha < \mathrm{rank}(b)$. *Then* $\alpha = \mathrm{rank}(a)$ *for some $a \in A$ such that $aR^*_A b$.*

Proof. Fix α, and let b be a minimal counter-example. More precisely, let
$X = \{b \in A : \mathrm{rank}(b) > \alpha \wedge \neg\exists a \in A[\mathrm{rank}(a) = \alpha \wedge aR^*_A b]\}$.
If the lemma fails, then $X \neq \emptyset$, so fix some $b \in X$ that is R-minimal in X and let $\beta = \mathrm{rank}(b)$. Now $\alpha < \beta = \mathrm{rank}(b) = \sup\{\mathrm{rank}(t) + 1 : t \in A \wedge tRb\}$, so fix $t \in A$ such that tRb and $rank(t) + 1 > \alpha$, so $\mathrm{rank}(t) \geq \alpha$.
If $\mathrm{rank}(t) = \alpha$: Then $b \notin X$ (since tR^*b), a contradiction.
If $\mathrm{rank}(t) > \alpha$: Since $t \notin X$ (by tRb and minimality of b), fix $a \in A$ such that $\mathrm{rank}(a) = \alpha$ and aR^*t. But then aR^*b, contradicting $b \in X$. □

We now specifically consider the relation $\in$. Because of the Axiom of Foundation, $\in$ is well-founded on the universe V:

Lemma 3.23. *The Axion of Foundation is equivalent to the statement that the $\in$ relation is well-founded on V.*

Proof. "$\in$ is well founded on V" means that for every non-empty subset $x \subseteq V$, there is a $y \in x$ that is $\in$-minimal in x. But $x \subseteq V$ is trivially true and minimality of y in x means $\neg\exists z(z \in x \wedge z \in y)$, so we are asserting
$\forall x[\exists y(y \in x) \to \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))]$,
which is the statement of Foundation. $\square$

Therefore, we can consider $\text{rank}_{V,\in}(x)$ for any x. For any transitive set A, the values of $\text{rank}_{V,\in}(x)$ and $\text{rank}_{A,\in}(x)$ agree. In the case of $A = ON$:

Lemma 3.24. *For $\alpha \in ON$: $\text{rank}_{ON,\in}(\alpha) = \alpha$; also, assuming the Axiom of Foundation, $\text{rank}_{V,\in}(\alpha) = \alpha$.*

Proof. First, observe that $\in$ is a well-order of ON, so we do not need Foundation to define $\text{rank}_{ON,\in}$. To prove $\text{rank}_{ON,\in}(\alpha) = \alpha$, assume this fails and let α be $\in$-minimal in $\{\xi \in ON : \text{rank}_{ON,\in}(\xi) \neq \xi\}$. Applying minimality, $\text{rank}_{ON,\in}(\alpha) = \sup\{\xi + 1 : \xi < \alpha\} = \alpha$, a contradiction.
Exactly the same proof works for $\text{rank}_{V,\in}$, assuming Foundation so that $\text{rank}_{V,\in}$ is defined. $\square$

A generalization of $\text{rank}_{ON,\in}(\alpha) = \text{rank}_{V,\in}(\alpha)$ is the following lemma:

Lemma 3.25. *Suppose that $A \subseteq B$, and R is well-founded and set-like on B. Let R^* denote R^{*B}. If $b \in A$ then $\text{rank}_{A,R}(b) \leq \text{rank}_{B,R}(b)$. If $b \in A$ and $\text{pred}_{B,R^*}(b) \subseteq A$ then $\text{rank}_{A,R}(b) = \text{rank}_{B,R}(b)$.*

Proof. First prove $\text{rank}_{A,R}(b) \leq \text{rank}_{B,R}(b)$ by induction: If b is R-minimal in $\{x \in A : \text{rank}_{A,R}(x) > \text{rank}_{B,R}(x)\}$, then we get a contradiction from
$\text{rank}_{A,R}(b) = \sup\{\text{rank}_{A,R}(x) + 1 : x \in A \wedge xRb\} \leq \{\text{rank}_{B,R}(x) + 1 : x \in B \wedge xRb\} = \text{rank}_{B,R}(b)$.
The "$\leq$" uses minimality of b.
Next, prove $\text{rank}_{A,R}(b) = \text{rank}_{B,R}(b)$ when $\text{pred}_{B,R^*}(b) \subseteq A$ by induction: If b is R-minimal in $\{x \in A : \text{rank}_{A,R}(x) \neq \text{rank}_{B,R}(x) \wedge \text{pred}_{B,R^*}(x) \subseteq A\}$ we get a contradiction from
$\text{rank}_{A,R}(b) = \sup\{\text{rank}_{A,R}(x) + 1 : x \in A \wedge xRb\} = \{\text{rank}_{B,R}(x) + 1 : x \in B \wedge xRb\} = \text{rank}_{B,R}(b)$.
The second "$=$" uses minimality of b, plus the fact that from $\text{pred}_{B,R^*}(b) \subseteq A$ we conclude that xRb implies both $x \in A$ and $\text{pred}_{B,R^*}(x) \subseteq A$. $\square$

For a fixed B and $b \in B$, the smallest A for which the lemma holds is $\{b\} \cup \text{pred}_{B,R^*}(b)$. Once again, we consider the specific case $B = V$ and $R = \in$. In this case, $\text{pred}_{V,\in^*}(b)$ is called the *transitive closure* of b:

Definition 3.26. For any set a, the *transitive closure* of a, $\text{trcl}(a)$, is $\{x : x \in^* a\}$.

We now define a set to be well-founded iff it is possible to compute its rank. This definition does not assume the Axiom of Foundation. However, the definition needs the following lemma to justify it:

Lemma 3.27. *For any set b: $\in$ is well-founded on $\text{trcl}(b)$ iff $\in$ is well-founded on $\{b\} \cup \text{trcl}(b)$.*

Proof. This is obvious if $b \in \mathrm{trcl}(b)$, so assume that $b \notin \mathrm{trcl}(b)$; equivalently $\neg(b \in^* b)$. The $\leftarrow$ direction is still clear from the fact that $\mathrm{trcl}(b) \subseteq \{b\} \cup \mathrm{trcl}(b)$.
For the $\rightarrow$ direction: Assume that $\in$ is well-founded on $\mathrm{trcl}(b)$, and let X be a non-empty subset of $\{b\} \cup \mathrm{trcl}(b)$; we need to produce an $\in$-minimal element a of X. This is obvious if $b \notin X$ or if $X = \{b\}$. But if $\{b\} \subsetneq X$, then any $\in$-minimal element $a \in X \setminus \{b\}$ is $\in$-minimal in X, since $b \in a$ would imply that $b \in^* b$. □

This fact lets us state the following definition:

Definition 3.28. The set b is a *well-founded* set iff $\in$ is well-founded on $\mathrm{trcl}(b)$, in which case $\mathrm{rank}(b)$ denotes $\mathrm{rank}_{\{b\}\cup\mathrm{trcl}(b),\in}(b)$. WF denotes the class of all well-founded sets.

As we will see in the following lemmas, WF is a proper class, containing all of ON. The class WF is closely related to the Axiom of Foundation. Assuming Foundation means assuming that only well-founded sets exist, so WF becomes equivalent to the universe V:

Lemma 3.29. *If T is a transitive class and $\in$ is well-founded on T, then $T \subseteq WF$ and* $\mathrm{rank}(b) = \mathrm{rank}_{T,\in}(b)$ *for all $b \in T$.*

Proof. Use Lemma 3.27, since $b \in T \rightarrow \mathrm{pred}_{T,\in^*}(b) = \mathrm{trcl}(b) \subseteq T$. □

Setting $T = ON$:

Lemma 3.30. *$ON \subseteq WF$, so WF is a proper class, and* $\mathrm{rank}(\alpha) = \alpha$ *for $\alpha \in ON$.*

Lemma 3.31. *The Axiom of Foundation is equivalent to $V = WF$.*

Proof. For $\rightarrow$: $\in$ is well-founded on every set.
For $\leftarrow$: If x is a non-empty set with no $\in$-minimal elements, then $\in$ is not well-founded on $\mathrm{trcl}(x)$, so $x \notin WF$. □

So, assuming Foundation means assuming that only well-founded sets exist. This implies that for every set b, we can calculate $\mathrm{rank}(b)$, so the universe can be hierarchically structured by the rank function, with $\emptyset = 0$ being the only set with rank 0. We can define a hierarchy on V by the following recursive definition [2, 33-34]:

Definition 3.32. $V_0 = \emptyset$; $V_{\alpha+1} = \mathcal{P}(V_\alpha)$; $V_\delta = \bigcup_{\alpha<\delta} V_\alpha$ for limit ordinals δ.

Every V_α contains all sets with rank $\leq \alpha$. Additionally, $V = \bigcup_\alpha V_\alpha$.
While the universe is smaller when assuming Foundation, most objects that are of mathematical interest are objects within a well-founded universe, so this is not a problem in most cases. Furthermore, assuming the well-foundedness of all sets allows for the simplification of many constistency proofs.
The following lemmas imply that WF is model of set theory. We will use these lemmas to prove the consistency of Foundation in Section 3.4:

Lemma 3.33. (1) $x \in b \in WF \rightarrow x \in WF \wedge \mathrm{rank}(x) < \mathrm{rank}(b)$.
(2) *$\in$ is well-founded on WF.*
(3) *For all sets b, $b \in WF$ iff $b \subseteq WF$, so that WF is a transitive class.*
(4) *For $b \in WF$,* $\mathrm{rank}(b) = \sup\{\mathrm{rank}(x) + 1 : x \in b\}$.
(5) $\mathrm{rank}(b) = \mathrm{rank}_{WF,\in}(b)$ *for all $b \in WF$.*

Proof. For (1): $x \in WF$ because $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(b)$, and then, using Lemma 3.30 with $T = \{b\} \cup \mathrm{trcl}(b)$: $\mathrm{rank}(x) = \mathrm{rank}_{T,\in}(x) < \mathrm{rank}_{T,\in}(b) = \mathrm{rank}(b)$; the "$<$" is by Lemma 3.22. Now, (2) follows by Lemma 3.19. Note that $x \in b \to \mathrm{trcl}(x) \subseteq \mathrm{trcl}(b)$ is clear from the definition of trcl using $\in^*$.
For (3): The $\to$ direction is clear from (1). For $\leftarrow$: If $b \subseteq WF$ then $\mathrm{trcl}(b) \subseteq WF$ (since WF is transitive), so $\in$ iss well-founded on $\mathrm{trcl}(b)$ by (2).
(5) is immediate by Lemma 3.30 with $T = WF$. Then, (4) follows using the definition of rank_{WF}. $\square$

The next two lemmas are easily derived from Lemma 3.33:

Lemma 3.34. *If $z \subseteq y \in WF$ then $z \in WF$ and* $\mathrm{rank}(z) \leq \mathrm{rank}(y)$.

Lemma 3.35. *Suppose that $x, y \in WF$. Then:*

(1) $\{x, y\} \in WF$ *and* $\mathrm{rank}(\{x, y\}) = \max(\mathrm{rank}(x), \mathrm{rank}(y)) + 1$.
(2) $\langle x, y\rangle \in WF$ *and* $\mathrm{rank}(\langle x, y\rangle) = \max(\mathrm{rank}(x), \mathrm{rank}(y)) + 2$.
(3) *If $\mathcal{P}(x)$ exists, then $\mathcal{P}(x) \in WF$ and* $\mathrm{rank}\mathcal{P}(x) = \mathrm{rank}(x) + 1$.
(4) $\bigcup x \in WF$ *and* $\mathrm{rank}(\bigcup x) \leq \mathrm{rank}(x)$.
(5) $x \cup y \in WF$ *and* $\mathrm{rank}(x \cup y) = \max(\mathrm{rank}(x), \mathrm{rank}(y))$.
(6) $\mathrm{trcl}(x) \in WF$ *and* $\mathrm{rank}(\mathrm{trcl}(x)) = \mathrm{rank}(x)$.

3.4. **The Consistency of Foundation.** This section assumes basic knowledge of ordinals and model theory.

Now that we have established the definition of a well-founded universe and proven that $V = WF$ is equivalent to the Axiom of Foundation, we shall prove the consistency of the Axiom of Foundation. In other words, we shall see that ZFC (respectively ZF) is consistent if and only if ZFC^- (respectively ZF^-) is consistent (for a definition of consistency, see 1.1). We shall now define the notion of a relative consistency proof. Notice that this is a metatheoretical definition:

Definition 3.36. Define $\Gamma \leq \Lambda$ iff we have a finitistic proof that $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$; such a proof is called a *relative consistency proof.* Define $\Gamma \sim \Lambda$ iff $\Gamma \leq \Lambda$ and $\Lambda \leq \Gamma$; we say that Γ and Λ are *proof-theroretically equivalent.*

So we shall prove that $ZFC \leq ZFC^-$ and $ZF \leq ZF^-$, hence $ZFC \sim ZFC^-$ and $ZF \sim ZF^-$. For any $\varphi \in ZFC$, we get a sentence φ^{WF}, obtained by restricting all bound variables to elements of WF. We will show that φ^{WF} is a theorem of ZFC^- whenever φ is an axiom of ZFC.
While this is a metatheoretical proof, there is also a similar proof within ZFC^-, using the set $R(\gamma)$, which may be seen as an approximation to WF:

Definition 3.37. For any ordinal γ, $R(\gamma) = \{x \in WF : \mathrm{rank}(x) < \gamma\}$.

Definition 3.38. $HF = R(\omega)$ is called the set of *hereditarily finite sets.*

Theorem 3.39. (ZFC^-) *$R(\gamma) \models Z$ whenever $\gamma > \omega$ and γ is a limit ordinal. The Axiom of Choice implies that $R(\gamma) \models ZC$.*

We shall see that in the case of HF, Replacement is true but Infinity is false.
As the proof of the above theorem is very similar to the metatheoretical proof of the consistency of Foundation, we shall state lemmas regarding sufficient conditions for each of the axioms of ZFC. Afterwards we shall respectively adapt the lemmas to the metatheoretical proof and to the proof within $R(\gamma)$.
We begin with conditions for the first six axioms (see Section 2.1):

Lemma 3.40. $(ZF^- - P)$: *For any class M:*

(1) *If M is transitive, then the Extensionality Axiom holds in M.*
(2) *If $M \subseteq WF$, then the Foundation Axiom holds in M.*
(3) *If $\forall z \in M \forall y \subseteq z[y \in M]$, then the Comprehension Axiom holds in M.*
(4) *If $\forall x, y \in M[\{x, y\} \in M]$, then the Pairing Axiom holds in M.*
(5) *If $\forall \mathcal{F} \in M[\bigcup \mathcal{F} \in M]$, then the Union Axiom holds in M.*
(6) *Assume that M is transitive and for all functions f: If $\mathrm{dom}(f) \in M$ and $\mathrm{ran}(f) \subseteq M$, then $\mathrm{ran}(f) \in M$. Then the Replacement Axiom holds in M.*

Proof. For (4): The Pairing Axiom is $\forall x, y \exists z[x \in z \wedge y \in z]$, so relativized to M we get the sentence $\forall x, y \in M \exists z \in M[x \in z \wedge y \in z]$, so let $z = \{x, y\} \in M$.
For (1): The Axiom of Extensionality relativized to M is just the statement that the $\in$ relation is extensional on M, which is true whenever M is transitive, see Lemma 3.16.
For (2): The Foundation Axiom relativized to M is
$\forall x \in M[\exists y \in M(y \in x) \to \exists y \in M(y \in x \wedge \neg \exists z \in M(z \in x \wedge z \in y))]$. To prove this, use the fact that $\in$ is well-founded on WF (Lemma 3.33) and let y be $\in$-minimal in $x \cap M$.
For (5): The Union Axiom relativized to M is
$\forall \mathcal{F} \in M \exists A \in M \forall Y \in M \forall x \in M(x \in Y \wedge Y \in \mathcal{F} \to x \in A)$, so let $A = \bigcup \mathcal{F} \in M$.
For (3): Fix a formula φ without y free. φ may have x, z free, along with possibly other free variables $v_0, \ldots, v_{n-1}$, so write it as $\varphi(x, z, v_0, \ldots, v_{n-1})$. Then we must verify
$\forall z, v_0, \ldots, v_{n-1} \in M \exists y \in M \forall x \in M[x \in y \leftrightarrow x \in z \wedge \varphi^M(x, z, \overrightarrow{v})]$.
This holds, since we may let $y = \{x \in z : \varphi^M(x, z, \overrightarrow{v})\}$; then $y \in M$ because $y \subseteq z$. Note that φ may hava quantified variables, and φ^M relativizes all of these to M. The formulas φ and φ^M need not be equivalent, but that is irrelevant here. We are using the Comprehension Axiom (which is part of $ZF^- - P$), applied with the formula φ^M (not φ), to assert that the set y exists, and then we are using the hypothesis of (3) to assert that $y \in M$.
(6) is similar to (3), so we shall be briefer. Assume that $A \in M$ and that $\forall x \in M[x \in A \to \exists! y \in M \varphi^M(x, y)]$. We need to produce a $B \in M$ such that $(\forall x \in A \exists y \in B \varphi(x, y))^M$. So, let f be the function with $\mathrm{dom}(f) = A$ such that $f(x)$ is the (unique) $y \in M$ such that $\varphi^M(x, y)$; the set f exists by the Replacement Axiom (which is part of $ZF^- - P$) applied with the formula $\varphi^M(x, y) \wedge y \in M$ (not $\varphi(x, y)$). Then, let $B = \mathrm{ran}(f)$. □

Corollary 3.41. $(ZF^- - P)$ Axioms 1,2,3,4,5,6 hold in WF.

Proof. The conditions of Lemma 3.40 are easily verified using the results of Section 3.3. Those results, along with Lemma 3.40, were derived from $ZF^- - P$. □

We get a similar result for $R(\gamma)$:

Corollary 3.42. $(ZF^-)R(\gamma) \models$ *Axioms1, 2, 3, 4, 5* whenever γ is a limit ordinal. $HF = R(\omega) \models$ *Axiom6.*

Note that $R(\alpha+1) \models \exists x \forall y x \notin y$, so the Pairing Axiom fails for successor ordinals.

We now consider the remaining axioms. In contrast to the first six axioms, which were stated in Section 2.1 using only the language $\mathcal{L} = \{\in\}$, Axioms 7,8,9 were stated using the defined notions $\subseteq$, $\emptyset$, S, $\cap$ and SING in addition to $\mathcal{L} = \{\in\}$. While it is practically possible to write Axioms 7,8,9 only using $\mathcal{L} = \{\in\}$, it would be impractical for more complicated notions. Regarding $\subseteq$, we are using the fact that $\subseteq$ is a Δ_0 formula, and thus absolute for transitive models (for more on absoluteness, see Section 4.1). As most of the "interesting" models are transitive, this allows us to handle Axiom 8:

Lemma 3.43. (ZF^-) *Let M be a transitive class. Then:*
8. $\forall x \in M((\mathcal{P}(x) \cap M) \in M) \to$ the Power Set Axiom holds in M.
Also, the $\leftarrow$ direction holds if M satisfies the Comprehension Axiom.

Proof. By the absoluteness of $\subseteq$, the Power Set Axiom holds in M iff
$\forall x \in M \exists y \in M \forall z \in M(z \subseteq x \to z \in y)$.
For the $\leftarrow$, let $y = \mathcal{P}(x) \cap M$. Then $(\text{PowerSet})^M$ gives us a $y \in M$ such that $\mathcal{P}(x) \cap M \subseteq y$. But then Comprehension in M plus that absoluteness of $\subseteq$ implies that $\mathcal{P}(x) \cap M = \{z \in y : (z \subseteq x)^M\} \in M$. □

Corollary 3.44. (ZF^-) The Power Set Axiom holds in WF, and in $R(\gamma)$ for any limit γ.

Proof. If $x \in WF$, then applying Lemmas 3.34 and 3.35, $\mathcal{P}(x) \subseteq WF$ and $\mathcal{P}(x) \cap WF = \mathcal{P}(x) \in WF$. Likewise for the $R(\gamma)$. □

Lemma 3.45. $(ZF^- - P)$ *Let M be a transitive class, and assume that the Axioms of Extensionality, Comprehension, Pairing, and Union hold in M.*
7. The Axiom of Infinity holds in M if $\omega \in M$.
9. Axiom 9 holds in M iff every disjoint family of non-empty sets in M has a choice set in M.

Proof. For Axiom 9: The form of AC stated in Section 2 is logically equivalent to $\forall F \exists C[\text{df}(F) \to \text{cs}(C, F)$, where $\text{df}(F)$ is
$\emptyset \notin F \wedge \forall x \in F \forall y \in F(x \neq y \to x \cap y = \emptyset$,
asserting that F is a disjoint family of non-empty sets and $\text{cs}(C, F)$ is
$\forall x \in F(\text{SING}(C \cap x))$,
asserting that C is a choice set for F. Both df and cs are Δ_0 in the notions $\emptyset$, $\cap$ and SING, which are absolute for M (see Section 4.1), so $(AC)^M$ is equivalent to $\forall F \in M \exists C \in M[\text{df(F)} \to \text{cs(C, F)}]$.
For Axiom 7: The Axiom of Infinity holds in M iff
$\exists x \in M(\varphi(x)^M)$, where $\varphi(x)$ is $\emptyset \in x \wedge \forall y \in x(S(y) \in x)$.
But φ is Δ_0 in the notions $\emptyset$ and S, which are absolute for M, so we can replace the $(\varphi(x))^M$ by $\varphi(x)$, which holds of ω. □

Corollary 3.46. $(ZF^- - P)$ Axiom 7 holds in WF. Furthermore, AC implies that Axiom 9 holds in WF.

Proof. $\omega \in WF$ by Lemma 3.30. If $F \in WF$ is a disjoint family of non-empty sets, then F has a choice set C (assuming AC). $C \cap \bigcup F$ is also a choice set for F, and $C \cap \bigcup F \in WF$ by Lemmas 3.34 and 3.35. □

Theorem 3.47. *Let* Γ *be one of the theories* $ZF - P, ZFC - P, ZF, ZFC$. *Let* Γ^- *be* Γ *with the Axiom of Foundation deleted. Then* $\Gamma \leq \Gamma^-$*; that is, there is a finitistic proof of* $\mathrm{Con}(\Gamma^-) \rightarrow \mathrm{Con}(\Gamma)$.

Proof. Applying Corollaries 3.41, 3.44 and 3.46, we may work in Γ^- and prove each axiom of Γ relativized to WF. □

Proof of Theorem 3.39. Exactly as for WF. $\omega \in R(\gamma)$ because $\gamma > \omega$. For Axiom 9, observe that if C is a choice set for F, then $\mathrm{rank}(C \cap \bigcup F) \leq \mathrm{rank}(F)$. □

In the case of $HF = R(\omega)$, Replacement is true but Infinity is false:

Theorem 3.48. (ZF^-) $HF \models ZFC - Inf$, *and the Axiom of Infinity is false in* HF.

Proof. Infinity fails because $\neg\exists x \in HF \varphi(x)$, where φ is as in the proof of Lemma 3.45. For Choice, note that we do not have to work in ZFC^-. Without assuming AC, HF can be well-ordered, so every disjoint family in HF has a choice set in HF. □

4. Countable Transitive Models

Again, basic knowledge of ordinals and model theory is assumed. Section 4.2 also requires minimal knowledge of cardinals. The definitions, theorems and proofs are stated according to [4].

4.1. **Absoluteness and Reflection.** Recall the following definitions and the following lemma from model theory:

Definition 4.1. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures for $\mathcal{L}$. Then $\mathfrak{A} \subseteq \mathfrak{B}$ means that $A \subseteq B$ and the functions and predicates of $\mathfrak{A}$ are the restrictions of the corresponding functions and predicates of $\mathfrak{B}$.
$\mathfrak{A}$ is called a *substructure* (or *submodel*) of $\mathfrak{B}$, and $\mathfrak{B}$ is called an *extension* of $\mathfrak{A}$.

Definition 4.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. If φ is a formula of $\mathcal{L}$, then $\mathfrak{A} \preceq_\varphi \mathfrak{B}$ means that $\mathfrak{A} \models \varphi[\sigma]$ iff $\mathfrak{B} \models \varphi[\sigma]$ for all assignments σ for φ in A. $\mathfrak{A} \preceq \mathfrak{B}$ (*elementary substructure* or *elementary submodel*) means that $\mathfrak{A} \preceq_\varphi \mathfrak{B}$ for all formulas φ of $\mathcal{L}$.

Lemma 4.3. *If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preceq_\varphi \mathfrak{B}$ whenever φ is quantifier-free, and* $\mathrm{val}_{\mathfrak{A}}(\tau)[\sigma] = \mathrm{val}_{\mathfrak{B}}(\tau)[\sigma]$ *whenever τ is a term of $\mathcal{L}$ and σ is an assignment for τ in A.*

We now introduce the the notion of a Δ_0 formula:

Definition 4.4. Assume that $\mathcal{L}$ contains the symbol $\in$ plus possibly other predicate and function symbols. Then the Δ_0 formulas of $\mathcal{L}$ are those formulas constructed by the rules:

a. All atomic formulas are Δ_0 formulas.
b. If φ is a Δ_0 formula, y is a variable, and τ is a term that does not contain y, then $\forall y \in \tau\varphi$ and $\exists y \in \tau\varphi$ are Δ_0 formulas.
c. If φ is a Δ_0 formula then so is $\neg\varphi$.
d. If φ and ψ are Δ_0 formulas then so are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.

Lemma 4.5. *Let $\mathcal{L}$ be as in Definition 4.4, and assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Also, assume that A is a transitive set and $\in_A = \{(a,b) \in A \times A : a \in b\}$ and $\in_B = \{(a,b) \in B \times B : a \in b\}$. Then $\mathfrak{A} \preceq_\varphi \mathfrak{B}$ for all Δ_0 formulas φ of $\mathcal{L}$.*

Proof. Induct on φ. The basis, when φ is atomic, is just Lemma 4.3. The induction steps for propositional connectives are trivial. For the induction step for $\exists$, assume that $\varphi(\vec{x}, z)$ is $\exists y[y \in \tau(\vec{x}, z) \wedge \psi(\vec{x}, y, z)]$, where ψ is Δ_0, and assume (inductively) that $\mathfrak{A} \preceq_\psi \mathfrak{B}$. For any tuple $\vec{a}$ from A and $c \in A$, let $\tau_{\vec{a},c}$ abbreviate $\mathrm{val}_{\mathfrak{A}}\tau[\vec{a}, c]$, which is the same as $\mathrm{val}_{\mathfrak{B}}\tau[\vec{a}, c]$, because $\mathfrak{A} \subseteq \mathfrak{B}$. Then, for such $\vec{a}, c$, the definition of $\models$ yields:
$A \models \varphi[\vec{a}, c] \leftrightarrow \exists b \in A\{b \in \tau_{\vec{a},c} \wedge A \models \psi[\vec{a}, b, c]\} \leftrightarrow \exists b \in B\{b \in \tau_{\vec{a},c} \wedge B \models \psi[\vec{a}, b, c]\} \leftrightarrow B \models \varphi[\vec{a}, c]$.
The second $\leftrightarrow$ uses $\mathfrak{A} \preceq_\psi \mathfrak{B}$ along with the fact that A is transitive, so that $\tau_{\vec{a},c} \subseteq A \subseteq B$, so the "$\exists b \in A$" and "$\exists b \in B$" could both be replaced with $\exists b$. The induction step for $\forall$ is similar. □

When $\mathcal{L} = \in$, the Δ_0 formulas are formulas in which all quantifiers are bounded. Many properties of sets are expressed using Δ_0 formulas. For example, the formulas used in

Section 2.1 to define the notions $\subseteq, \emptyset, S, \cap, \mathrm{SING}$ are all Δ_0 formulas of $\mathcal{L} = \in$. While these notions are not defined in 2.1 using Δ_0 formulas, the formulas used are logically equivalent to Δ_0 formulas, such as $\forall z(z \in x \to z \in y)$, the formula used for $\subseteq$, being logically equivalent to $\forall z \in x(z \in y)$.
A more general concept is the concept of *absoluteness*:

Definition 4.6. φ is absolute for A, B iff $A \preceq_\varphi B$.

As we have seen, Δ_0 formulas are absolute for transitive models. For example, we say "$\subseteq$ is absolute for transitive models", meaning $A \preceq_{x \subseteq y} B$ for any transitive classes A, B. This definition even makes sense with proper class models, in which case a single statement becomes a scheme in the metatheory. Therefore, we can define:

Definition 4.7. φ is absolute for A iff $A \preceq_\varphi V$.

While the following lemma might seem obvious, it is very important regarding consistency proofs and is often only used implicitly:

Lemma 4.8. *If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are two formulas in $\mathcal{L} = \{\in\}$ and $\forall \vec{x}[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$ holds in V and in M, then φ is absolute for M iff ψ is absolute for M.*

This lemma is used when ψ is an "official" definition of a concept and φ is a logically equivalent formula that is known to be absolute (for example, when φ is Δ_0 and M is transitive).

We shall consider absoluteness for transitive models of the theory BST, which is even weaker than ZF, so we can also consider models of the sort $R(\gamma)$ for limit γ:

Definition 4.9. BST (Basic Set Theory) denotes the axioms of Extensionality, Foundation, Comprehension, Pairing and Union, plus the disjunction: the Power Set Axiom holds or the Replacement Axiom holds. Then BST^- denotes these same axioms without Foundation.

Assuming Foundation, the notion of "ordinal" and related concepts are absolute for transitive models:

Lemma 4.10. *The following set-theoretic notions are defined by formulas that BST proves are equivalent to Δ_0 formulas. Hence, they are absolute for every transitive model of BST:*

(1) *x is a transitive set.*
(2) *x is an ordinal.*
(3) *x is a successor ordinal.*
(4) *$x = 0$.*
(5) *x is a limit ordinal.*
(6) *x is a natural number.*
(7) *$x \subseteq \omega$.*
(8) *$x = \omega$.*

Proof. For (1), x is transitive iff $\forall y \in x(\forall z \in y(y \in x))$; this is clearly equivalent to Definition 3.8.

For (2): Transitivity is Δ_0 by (1), and the statement that x is totally ordered by $\in$ is easily expressed by quantifying over x. For x to be an ordinal, it is required that x is also well-ordered by $\in$, but this follows by Foundation.
(3 - 8) are easy, using (2). For example, x is a successor ordinal iff x is an ordinal and $\exists y \in x \forall z \in x[z = y \vee z \in y]$; and x is a natural number iff x is an ordinal and x and all its elements are either successor ordinals or 0. □

Some other properties that are absolute for transitive models of BST are the following (for a proof, see [4]):

(1) The 0-ary function $\emptyset$.
(2) The 1-ary successor function S.
(3) The 2-ary intersection function $\cap$.
(4) The 2-ary union function $\cup$, and the 1-ary union and intersection functions $\bigcup$ and $\bigcap$, where we define $\bigcap \emptyset = \emptyset$.
(5) The ternary relation: $\{x, y\} = z$.
(6) The 2-ary unordered pairing function $\{x, y\}$, and the 1-ary singleton function $\{x\}$, and the 2-ary ordered pairing function $\langle x, y\rangle$.
(7) The properties: z is an ordered pair, and x is a relation.
(8) $\mathrm{dom}(x)$ and $\mathrm{ran}(x)$.
(9) The properties: f is a function, f is an injection, f is a surjection, and f is a bijection.
(10) The binary function $f(x)$, defined to be $\emptyset$ unless f is a function and $x \in \mathrm{dom}(f)$.
(11) The binary function $x \times y$.
(12) All relational properties of R and A defined in Definition 3.3.

We shall now show that recursively defined functions are absolute. This implies the absoluteness of the rank function, which will be important for Section 4.2:

Theorem 4.11. *Assume that R is a defined 2-ary relation, G is a defined 2-ary function and A is a class (a defined 1-ary relation). Assume also that R is well-founded and set-like on A and let F be the defined 1-ary function, as in Theorem 3.12, such that $\forall a \in A[F(a) = G(a, F \restriction (a\downarrow))]$; assume that $F(a) = \emptyset$ for $a \notin A$.*
Now, let M be a transitive model for $ZF - P$, and assume that R, A, G are all absolute for M, and that (R is set-like on $A)^M$, and that $a\downarrow \subseteq M$ for all $a \in M$. Then $F^M(a)$ is defined for $a \in M$, and F is absolute for M.

Proof. Note that (R is well-founded on $A)^M$, since a non-empty $X \in M$ with $X \subseteq A$ and no R-minimal element would contradict well-foundedness of R in V. Also, the function $a \mapsto a\downarrow$ is absolute for M.
Now, Theorem 3.12 was a theorem of $ZF - P$, so that F^M is defined, and note that by absoluteness of G, if $F \restriction (a\downarrow) = F^M \restriction (a\downarrow)$ then $F \restriction (a) = F^M \restriction (a)$. If F were not absolute, then an R-minimal element of $\{a \in M : F(a) \neq F^M(a)\}$ would be contradictory. □

As the proof of Theorem 3.12 relies on the Replacement Axiom, one cannot substitute the $ZF - P$ with BST or Z.
A direct implication of Theorem 4.11 is the following:

Lemma 4.12. *The functions α^β and* $\mathrm{rank}(x)$ *are absolute for transitive $M \models ZF - P$.*

While many results of model theory do not make sense for proper classes (for example, $A \preceq V$ is not even a valid metatheoretical statement), they might make sense for proper classes if we restrict them to finitely many formulas at once. If we have a given list of formulas $\varphi_0, \varphi_1, \ldots, \varphi_n - 1$, we can write the sentence $\exists A[\bigwedge_{i<n}(A \preceq_{\varphi_i} V)]$, and we shall see that we can also prove it by applying the Reflection Theorem (4.15).
Before we can prove the Reflection Theorem, we need a "class version" of the Tarski-Vaught criterion from model theory:

Definition 4.13. A list of formulas $\varphi_0, \varphi_1, \ldots, \varphi_n - 1$ is *subformula-closed* iff every subformula of each φ_i is also on the list, and no formula on the list uses the universal quantifier $\forall$.

Lemma 4.14. *Let $\varphi_0, \varphi_1, \ldots, \varphi_n - 1$ be a subformula-closed list of formulas of $\mathcal{L} = \{\in\}$. Let A, B be classes with $\emptyset \neq A \subseteq B$. Then the following are equivalent:*

(1) $\bigwedge_{i<n}(A \preceq_{\varphi_i} B)$.
(2) *For all existential formulas $\varphi_i(x_1, \ldots x_r)$, of the form $\exists y \varphi_j(\vec{x}, y)$, the following holds:*
$\forall a_1, \ldots, a_r \in A[\varphi_i^B(\vec{a}) \to \exists b \in A \varphi_j^B(\vec{a}, b)]$.

Proof. (1) $\to$ (2): $\varphi_i^B(\vec{a}) \to \varphi_i^A(\vec{a}) \to \exists b \in A \varphi_j^A(\vec{a}, b) \to \exists b \in A \varphi_j^B(\vec{a}, b)$.
Here, the middle $\to$ used the meaning of φ_i^A, and the other two used $A \preceq_{\varphi_i} B$ and $A \preceq_{\varphi_j} B$.
(2) $\to$ (1): Assume (2), and prove $A \preceq_{\varphi_i} B$ by induction on the length of φ_i, so assume that we have proved $A \preceq_{\varphi_j} B$ whenever φ_j is shorter than φ_i. The basis cases (φ_i is atomic) and the induction steps for propositional connectives are trivial, so now assume that $\varphi_i(\vec{x})$ is existential, of tha form $\exists y \varphi_j(\vec{x}, y)$. Fix $a_1, \ldots a_r \in A$. To show $\varphi_i^B(\vec{a}) \leftrightarrow \varphi_i^A(\vec{a})$, use
$\varphi_i^B(\vec{a}) \leftrightarrow \exists b \in A \varphi_j^B(\vec{a}, b) \leftrightarrow \exists b \in A \varphi_j^A(\vec{a}, b) \leftrightarrow \varphi_i^A(\vec{a})$.
Here, the middle $\leftrightarrow$ used $A \preceq_{\varphi_j} B$, the last $\leftrightarrow$ used the meaning of φ_i^A, and the first $\leftrightarrow$ used the meaning of φ_i^B for the $\leftarrow$ and (2) for the $\to$. □

This result lets us state the Reflection Theorem:

Theorem 4.15 (Reflection Theorem)**.** *Let $\varphi_0, \varphi_1, , \varphi_{n-1}$ be any list of formulas of $\mathcal{L} = \{\in\}$. Assume that B is a non-empty class and $A(\xi)$ is a set for each $\xi \in ON$, and assume that:*

(1) $\xi < \eta \to A(\xi) \subseteq A(\eta)$.
(2) $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ *for limit* η.
(3) $B = \bigcup_{\xi \in ON} A(\xi)$.

Then $\forall \xi \exists \eta > \xi[A(\eta) \neq \emptyset \wedge \bigwedge_{i<n}(A(\eta) \preceq_{\varphi_i} B) \wedge \eta$ is a limit ordinal].

Proof. We may assume that our list is subformula-closed; if not, replace each φ_i by a a logically equivalent formula not using $\forall$ (replace the $\forall$ by $\neg\exists\neg$), and then add to the list all subformulas of formulas appearing on the list.
For each existential $\varphi_i(\vec{x})$ (of form $\exists y \varphi_j(\vec{x}, y)$, where $\vec{x}$ denotes an r-tuple; $r = r_i$), define $F_i : B^r \to ON$ as follows: If $\varphi_i^B(\vec{a})$, then $F_i(\vec{a})$ is the least ζ such that $\exists b \in A(\zeta)\varphi_j^B(\vec{a}, b)$. If $\neg\varphi_i^B(\vec{a})$, then $F_i(\vec{a}) = 0$.

Next, define $G_i : ON \to ON$ by: $G_i(\xi) = \sup\{F_i(a_1, \ldots, a_r) : a_1, \ldots a_r \in A(\xi)\}$ whenever φ_i is existential, with $r = r_i$. When φ_i is not existential, let $G_i(\xi) = 0$. Finally, let $K(\xi)$ be the larger of $\xi + 1$ and $\max\{G_i(\xi) : i < n\}$.
Now, fix ξ; it is sufficient to produce an $\eta > \xi$ such that $A(\eta) \neq \emptyset$ and (2) of Lemma 4.14 holds for $A(\eta), B$. So, let ζ_0 be the least $\zeta > \xi$ such that $A(\zeta) \neq \emptyset$, and let $\zeta_{n+1} = K(\zeta_n)$. Then $\xi < \zeta_0 < \zeta_1 < \cdots$. Let $\eta = \sup\{\zeta_k : k \in \omega\}$. $\square$

The most well-known application is the case $B = V$ and $A(\xi) = R(\xi)$. In Section 4.2, this allows us to provide a transitive model for each finite set of axioms of ZF.

4.2. **Countable Transitive models.** While we cannot find a set model that satisfies all of ZF, we shall see that we can find a set model that satisfies any finite set of axioms of ZF by applying the Reflection Theorem.
We have seen in Section 3.4 that, working in ZF^-, models of the sort $R(\gamma)$ satisfy $ZF - Inf$ for $\gamma = \omega$ and Z for limit ordinals $\gamma > \omega$. While we cannot find an $R(\gamma)$ that satisfies both Infinity and the whole Replacement Scheme, we can always find a model for any finite subtheory. Moreover, as we shall prove in this section, we can always find a transitive model.

Recall the following theorem from model theory (for a proof, see [4]):

Theorem 4.16 (Downward Löwenheim-Skolem-Tarski Theorem)**.** *Work in ZFC^-. Let $\mathfrak{B}$ be any structure for $\mathcal{L}$. Fix κ such that $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |B|$, and fix $S \subseteq B$ with $|S| \leq \kappa$. Then there is an $\mathfrak{A} \preceq \mathfrak{B}$ such that $S \subseteq A$ and $|A| = \kappa$.*

The models $R(\gamma)$ for $\gamma > \omega$ are transitive and uncountable. The following lemma implies that for any finite set of axioms satisfied by an $R(\gamma)$ we can find a countable transitive model satisfying the same axioms:

Lemma 4.17. *Assume AC. Let B be any infinite set such that $(B, \in)$ satisfies the Axiom of Extensionality. Let κ be any infinite cardinal with $\kappa < |B|$. Fix $S \subseteq B$ such that S is transitive and $|S| \leq \kappa$. Then there is a transitive M such that $S \subseteq M$, $(M, \in) \equiv (B, \in)$, and $|M| = \kappa$. In particular, there is a countable transitive M such that $(M, \in) \equiv (B, \in)$.*

Proof. By the Downward Löwenheim-Skolem-Tarski Theorem (4.16), let $A \preceq B$ with $S \subseteq A$ and $|A| = \kappa$. Then A also satisfies the Axiom of Extensionality. Since $\in$ is well-founded on A (by the Foundation Axiom), the Mostowski function mos is defined on A (Definition 3.13) and is an isomorphism from $(A, \in)$ onto $(M, \in)$ for some transitive set M (see lemmas 3.14) and 3.17). Forthermore, since S is transitive, $\mathrm{mos}(y) = y$ for all $y \in S$ (see Lemma 3.18), so $S \subseteq M$.
$|M| = \kappa$ because mos is a bijection and $|A| = \kappa$, and $(M, \in) \equiv (A, \in) \equiv (B, \in)$ because $M \cong A$ and $A \preceq B$.
The "in particular" follows by letting $\kappa = \aleph_0$, and $S = \emptyset$. $\square$

Using the consistency of Foundation for $R(\gamma)$ models (Theorem 3.39), the Reflection Theorem (4.15) and the above lemma (4.17) we can finally construct a countable transitive model for each finite set of axioms of ZF:

Corollary 4.18. Let Λ be a finite set of axioms of ZF. Then

(1) $ZF \vdash \exists\eta[R(\eta) \models Z \cup \Lambda]$.
(2) $ZFC \vdash \exists\eta[R(\eta) \models ZC \cup \Lambda]$.
(3) $ZFC \vdash \exists M[M \models ZC \cup \Lambda \wedge |M| = \aleph_0 \wedge M$ is transitive$]$.

Proof. For (1)(2), let $B = V$ and $A(\xi) = R(\xi)$. Let $\{\varphi_0, \ldots, \varphi_{n-1}\} = \Lambda$, and apply the Reflection Theorem (4.15) to get a limit $\eta > \omega$ such that $\bigwedge_{i<n}(R(\eta) \preceq_{\varphi_i} V)$. Since φ_i is a sentence and an axiom of ZF, each $R(\eta) \preceq_{\varphi_i} V$ yields $\varphi_i^{R(\eta)} \leftrightarrow \varphi_i$ and then $R(\eta) \models \varphi_i$. And, $ZF \vdash (R(\eta \models Z)$ and $ZFC \vdash (R(\eta \models ZC)$ by Theorem 3.39. For (3), apply Lemma 4.17 to $(R(\eta), \in)$. $\square$

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