PROSEMINAR AXIOMATIC SET THEORY I (S2018): 17.03.2018

Exercise 1:

(1) Let A be an infinite set of ordinals with the property that for every $\gamma \in A$ there is $\delta \in A$ such that $\gamma < \delta$. Show that $\bigcup A$ is a limit ordinal.
(2) Show that the collection of all limit ordinals (resp. successor cardinals) is a proper class.

Exercise 2: Let γ be a limit ordinal. Show that the following are equivalent:

(1) $\forall \alpha, \beta < \gamma(\alpha + \beta < \gamma)$
(2) $\forall \alpha < \gamma(\alpha + \gamma = \gamma)$
(3) $\forall X \subseteq \gamma(\text{type}(X) = \gamma \vee \text{type}(\gamma \backslash X) = \gamma)$
(4) $\exists \delta(\gamma = \omega^\delta)$

Such γ are called *indecomposable*. The least γ such that $\gamma = \omega^\gamma$ is called $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\}$.

Exercise 3: Prove the *uniqueness* of the presentation in the Cantor Normal Form Theorem.

Exercise 4: If $R_1 \subseteq R_2$ are both well-founded and set-like on A, then $\text{rank}_{A,R_1}(y) \leq \text{rank}_{A,R_2}(y)$ for all $y \in A$. Also, $\text{rank}_{A,R_1}(y) = \text{rank}_{A,R_2}(y)$ if $R_1 \subseteq R_2 \subseteq R_1^{\text{TC}}$.

Exercise 5: For any relation R on a class A and $a \in A$: R is well-founded on $\text{pred}_{A,R^{\text{TC}}}(a)$ iff R is well-founded on $\{a\} \cup \text{pred}_{A,R^{\text{TC}}}(a)$.

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