COHERENT SYSTEMS OF FINITE SUPPORT ITERATIONS

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Abstract. We introduce a forcing technique to construct three-dimensional arrays of generic extensions through FS (finite support) iterations of ccc posets, which we refer to as 3D-coherent systems. We use them to produce models of new constellations in Cichoń's diagram, in particular, a model where the diagram can be separated into 7 different values. Furthermore, we show that this constellation of 7 values is consistent with the existence of a Δ^1_3 well-order of the reals.

1. Introduction

In this paper, we provide a generalization of the method of matrix iteration, to which we refer as *3D-coherent systems* of iterations and which can be considered a natural extension of the matrix method to include a third dimension. That is, if a matrix iteration can be considered as a system of partial orders $\langle \mathbb{P}_{\alpha,\beta} : \alpha \leq \gamma, \beta \leq \delta \rangle$ such that whenever $\alpha \leq \alpha'$ and $\beta \leq \beta'$ then $\mathbb{P}_{\alpha,\beta}$ is a complete suborder of $\mathbb{P}_{\alpha',\beta'}$, then our 3D-coherent systems are systems of posets $\langle \mathbb{P}_{\alpha,\beta,\xi} : \alpha \leq \gamma, \beta \leq \delta, \xi \leq \pi \rangle$ such that whenever $\alpha \leq \alpha'$, $\beta \leq \beta'$, $\xi \leq \xi'$ then $\mathbb{P}_{\alpha,\beta,\xi}$ is a complete suborder of $\mathbb{P}_{\alpha',\beta',\xi'}$. As an application of this method, we construct models where Cichoń's diagram is separated into different values, one of them with 7 different values. Moreover, these models determine the value of $\mathfrak{a}$, which is actually the same as the value of $\mathfrak{b}$, and we further show that such models can be produced so that they satisfy, additionally, the existence of a Δ^1_3-well order of the reals.

The method of matrix iterations, or *2D-coherent systems* of iterations in our terminology, has already a long history. It was introduced by Blass and Shelah in [BS89], to show that consistently $\mathfrak{u} < \mathfrak{d}$, where $\mathfrak{u}$ is the ultrafilter number and $\mathfrak{d}$ is the dominating number. The method was further developed in [BF11], where the terminology *matrix iteration* appeared for the first time, to show that if $\kappa < \lambda$ are arbitrary regular uncountable cardinals then there is a generic extension in which $\mathfrak{a} = \mathfrak{b} = \kappa < \mathfrak{s} = \lambda$. Here $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{s}$ denote the almost disjointness, bounding and splitting numbers respectively. In [BF11], the authors also introduce a new method for the preservation of a mad (maximal almost disjoint) family along a matrix iteration, specifically a mad family added by $\mathbb{H}_\kappa$ (Hechler's poset for adding a mad family, see Definition 4.1), a method which is of particular importance for our current work. Later, classical preservation properties for matrix iterations were improved by Mejía [Mej13a] to provide several examples of models where the cardinals in Cichoń's diagram assume many different values, in particular, a model with 6 different values. Since then, the question of *how many distinct values* there can be simultaneously in Cichoń's diagram has been of interest for many authors, see for example [FGKS] (a

2010 *Mathematics Subject Classification.* 03E17, 03E15, 03E35, 03E40, 03E45.

Key words and phrases. Cichoń's diagram, finite support iterations, 3D-coherent systems, 2D-coherent systems, Δ^1_3 well-orders.

The first author would like to thank the Austrian Science Fund (FWF) for the generous support through Lise-Meitner Project M1365-N13. The second and fourth authors were supported by the Austrian Science Fund (FWF) Project P25748. The third author was supported by the FWF Projects P23875-N13 and I1272-N25.

model of 5 values concentrated on the right) and [GMS16] (another model of 6 different values), and lies behind the development of many interesting forcing techniques. Very recently, the method of matrix iterations was used by Dow and Shelah [DS] to solve a long-standing open question in the area of cardinal characteristics of the continuum, namely, to show that it is consistent that the splitting number is singular.

Further motivation for this project was to determine the value of $\mathfrak{a}$ in classical FS (finite support) iterations of ccc posets models where no dominating reals are added. To recall some examples, a classical result of Kunen [Kun80] states that, under CH, any Cohen poset preserves a mad family of the ground model, which was improved by Steprans [Ste93] as he showed that, after adding ω_1-many Cohen reals, there is a mad family which is preserved by any further Cohen poset; Zhang [Zha99] proved that, under CH, any finite support iteration of $\mathbb{E}$ (the standard poset adding an eventually different real, see Definition 1.1) preserves a mad family from the ground model. As the family preserved in Steprans' result is added by $\mathbb{C}_{\omega_1} = \mathbb{H}_{\omega_1}$, we considered the preservation theory of Brendle and the first author [BF11] to see in which cases a mad family added by $\mathbb{H}_\kappa$ (for an uncountable regular κ) can be preserved through FS iterations of ccc posets. If such an FS iteration can be redefined as a matrix iteration where $\mathbb{H}_\kappa$ is used to add a mad family as in [BF11] and the preservation theory applies, then the mad family added by $\mathbb{H}_\kappa$ is preserved through the iteration. Thanks to this and to the fact that random forcing and $\mathbb{E}$ fit into the preservation framework (Lemmas 4.10 and 4.8), we generalize both Steprans and Zhang results by providing a general result about FS iterations preserving the mad family added by $\mathbb{H}_\kappa$ (Theorem 4.17).

In view of the previous result, it is worth asking whether such a result can be extended to matrix iterations like those in [Mej13a]. By analogy, if it is possible to add an additional coordinate for $\mathbb{H}_\kappa$ to a matrix iteration and produce a 3D iteration (3D-coherent system in our notation) where the preservation theory from [BF11] applies, then the mad family added by $\mathbb{H}_\kappa$ is preserved. Even more, the third dimension allows us to separate $\mathfrak{b}$ from other cardinals in Cichoń's diagram (which was not possible in [Mej13a]) and get a further division in Cichoń's diagram. In particular, the 3D-version of the matrix iteration from [Mej13a] for the consistency of 6 different values yields a model of 7 different values in Cichoń's diagram.

In addition, we show that these new constellations of Cichoń's diagram are consistent with the existence of a Δ^1_3 well-order of the reals. The study of those combinatorial properties of the real line (which can be expressed in terms of its cardinal characteristics) as well as the study of the existence of nicely definable combinatorial objects (like maximal almost disjoint families) in the presence of a projective well-order on the reals has been investigated intensively in the recent years. In [FF10] it is shown for example that various constellations involving $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{s}$ are consistent with the existence of a Δ^1_3 well-order, while in [FFK14] it is shown that every admissible assignment of $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's diagram is consistent with the existence of such projective well-order. There is one main distinction between the various known methods for generically adjoining projective well-orders: methods relying on countable support S-proper iterations like in [FF10, FFK14], and methods using finite support iterations of ccc posets, e.g. [FFZ11, FFT12, FFZ13]. In order to show that our new consistent constellations of Cichoń's diagram admit the existence of a Δ^1_3 well-order of the reals, we further develop the method of almost disjoint coding which was introduced in [FFZ11] and in particular answer one of the open questions stated in [FFK14].

The paper is organized as follows. In Section 2 we present some well known preservation theorems. In Section 3 we introduce the notion of 3D-iterations and review the preservation properties for matrix iterations from [BF11, Mej13a] which can be applied quite directly to 3D-coherent systems (even to arbitrary coherent systems). In Section 4 we review the method of preservation of a mad family along a matrix iteration as introduced in [BF11] and obtain similar results regarding $\mathbb{E}$ and the random algebra. As a consequence, we prove in Theorem 4.17 our generalization of Steprans' result discussed above, which is one of the main results of this paper. Recall:

Definition 1.1 (Standard forcing that adds an eventually different real)**.** Define the forcing notion $\mathbb{E}$ with conditions of the form (s,φ) where $s\in\omega^{<\omega}$ and $\varphi:\omega\to[\omega]^{<\aleph_0}$ such that $\exists n<\omega\forall i<\omega(|\varphi(i)|\leq n)$. Denote the minimal such n by width(φ). The order in $\mathbb{E}$ is defined as $(t,\psi)\leq(s,\phi)$ iff $s\subseteq t$, $\forall i<\omega(\varphi(i)\subseteq\psi(i))$ and $\forall i\in|t|\smallsetminus|s|(t(i)\notin\varphi(i))$.

Clearly $\mathbb{E}$ is σ-centered and adds a real which is eventually different from the reals in the ground model. We will use also the following notation. If Ω is a non-empty countable set, $\mathbb{B}_\Omega$ is the cBa (complete Boolean algebra) $2^{\Omega\times\omega}/\mathcal{N}(2^{\Omega\times\omega})$. Here, $\mathcal{N}(2^{\Omega\times\omega})$ denotes the σ-ideal of null subsets of $2^{\Omega\times\omega}$ with respect to the standard product measure. Note that $\mathbb{B}_\Omega\simeq\mathbb{B}:=\mathbb{B}_\omega$. In general, for any set Γ, $\mathbb{B}_\Gamma:=\text{limdir}\{\mathbb{B}_\Omega:\Omega\subseteq\Gamma\text{ countable}\}$. Denote by $\mathfrak{R}$ the class of all random algebras, that is, $\mathfrak{R}:=\{\mathbb{B}_\Gamma:\Gamma\neq\emptyset\}$. Recall Cohen forcing $\mathbb{C}_\Gamma:=\text{Fn}(\Gamma\times\omega,2)$ which is the poset of finite partial functions from $\Gamma\times\omega$ to 2 ordered by reverse inclusion. Put $\mathbb{C}=\mathbb{C}_\omega$. Another well-known poset which we will make use of is the localization poset (see for example [BJ95]). For convenience, we repeat its definition:

Definition 1.2. $\mathbb{LOC}$ is the poset of all $\varphi\in([\omega]^{<\aleph_0})^\omega$ such that

(i) for all $n\in\omega$, $|\varphi(n)|\leq n$, and
(ii) there is a $k\in\omega$ such that for all but finitely many n, $|\varphi(n)|\leq k$.

The extension relation is defined as follows: $\varphi'\leq\varphi$ if and only if $\varphi(n)\subseteq\varphi'(n)$ for all $n<\omega$.

Section 5 contains our main results about Cichoń's diagram. We evaluate the almost disjointness number in various constellations in which the value of $\mathfrak{a}$ was previously not known, and obtain a model in which there are 7 distinct values in Cichoń's diagram. Let $\theta_0\leq\theta_1\leq\kappa\leq\mu\leq\nu$ be regular uncountable cardinals, and let $\lambda\geq\nu$.

Theorem. *Assume $\lambda^{<\theta_1}=\lambda$. Then, there is a ccc poset forcing* $\text{add}(\mathcal{N})=\theta_0$, $\text{cov}(\mathcal{N})=\theta_1$, $\mathfrak{b}=\mathfrak{a}=\kappa$, $\text{non}(\mathcal{M})=\text{cov}(\mathcal{M})=\mu$, $\mathfrak{d}=\nu$ *and* $\text{non}(\mathcal{N})=\mathfrak{c}=\lambda$.

Elaborating on the method of almost disjoint coding as developed in [FFZ11], we show in Section 6 that the constellations of Section 5 are consistent with the existence of a projective wellorder of the reals whenever the associated cardinal values do not exceed $\aleph_\omega$ (even though we conjecture that the result remains true with arbitrarily large cardinal values). In particular, we outline the proof of the following:

Theorem. *In L, let $\theta_0<\theta_1<\kappa<\mu<\nu<\lambda$ be uncountable regular cardinals and, in addition, $\lambda<\aleph_\omega$. Then, there is a cardinal preserving forcing extension of L in which there is a Δ^1_3 well-order of the reals and, in addition,* $\text{add}(\mathcal{N})=\theta_0$, $\text{cov}(\mathcal{N})=\theta_1$, $\mathfrak{b}=\mathfrak{a}=\kappa$, $\text{non}(\mathcal{M})=\text{cov}(\mathcal{M})=\mu$, $\mathfrak{d}=\nu$ *and* $\text{non}(\mathcal{N})=\mathfrak{c}=\lambda$.

Section 7 contains some further discussion and open questions.

2. Preservation properties for FS iterations

We review the theory of preservation properties for FS iterations developed by Judah and Shelah [JS90] and Brendle [Bre91]. A similar presentation also appears in [GMS16, Sect. 3].

Definition 2.1. $\mathbf{R} := \langle X, Y, \sqsubset\rangle$ is a *Polish relational system* if the following is satisfied:

(i) X is an uncountable Polish space,
(ii) Y is a non-empty analytic subspace of some Polish space and
(iii) $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ for some increasing sequence $\langle \sqsubset_n\rangle_{n<\omega}$ of closed subsets of $X \times Y$ such that $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is nwd (nowhere dense) for all $y \in Y$.

For $x \in X$ and $y \in Y$, $x \sqsubset y$ is often read y $\sqsubset$*-dominates* x. A family $F \subseteq X$ is $\mathbf{R}$*-unbounded* if there is <u>no</u> real in Y that $\sqsubset$-dominates every member of F. Dually, $D \subseteq Y$ is a $\mathbf{R}$*-dominating* family if every member of X is $\sqsubset$-dominated by some member of D. $\mathfrak{b}(\mathbf{R})$ denotes the least size of a $\mathbf{R}$-unbounded family and $\mathfrak{d}(\mathbf{R})$ is the least size of a $\mathbf{R}$-dominating family.

Say that $x \in X$ is $\mathbf{R}$*-unbounded over a model* M if $x \not\sqsubset y$ for all $y \in Y \cap M$. Given a cardinal λ say that $F \subseteq X$ is λ-$\mathbf{R}$*-unbounded* if, for any $Z \subseteq Y$ of size $< \lambda$, there is an $x \in F$ which is $\mathbf{R}$-unbounded over Z.

By (iii), $\langle X, \mathcal{M}(X), \in\rangle$ is Tukey-Galois below $\mathbf{R}$ where $\mathcal{M}(X)$ denotes the σ-ideal of meager subsets of X. Therefore, $\mathfrak{b}(\mathbf{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathbf{R})$. Fix, for this section, a Polish relational system $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ and an uncountable regular cardinal θ.

Remark 2.2. Without loss of generality, $Y = \omega^\omega$ can be assumed. The reason is that, by the existence of a continuous surjection $f : \omega^\omega \to Y$, the Polish relational system $\mathbf{R}' := \langle X, \omega^\omega, \sqsubset'\rangle$, where $x \sqsubset'_n z$ iff $x \sqsubset_n f(z)$, behaves much like $\mathbf{R}$ in practice. Namely, $\mathbf{R}$ is Tukey-Galois equivalent to $\mathbf{R}'$ and moreover, the notions λ-$\mathbf{R}$-unbounded and λ-$\mathbf{R}'$-unbounded are equivalent. Also, for posets, the notions of θ-$\mathbf{R}$-good and θ-$\mathbf{R}'$-good (see the definition below) are equivalent.

Definition 2.3 (Judah and Shelah [JS90])**.** A poset $\mathbb{P}$ is θ-$\mathbf{R}$-*good* if, for any $\mathbb{P}$-name $\dot{h}$ for a real in Y, there is a non-empty $H \subseteq Y$ of size $< \theta$ such that $\Vdash x \not\sqsubset \dot{h}$ for any $x \in X$ that is $\mathbf{R}$-unbounded over H.

Say that $\mathbb{P}$ is $\mathbf{R}$-*good* when it is $\aleph_1$-$\mathbf{R}$-good.

Definition 2.3 describes a property, respected by FS iterations, to preserve specific types of $\mathbf{R}$-unbounded families. Concretely,

(a) any θ-$\mathbf{R}$-good poset preserves every θ-$\mathbf{R}$-unbounded family from the ground model and
(b) FS iterations of θ-cc θ-$\mathbf{R}$-good posets produce θ-$\mathbf{R}$-good posets.

Posets that are θ-$\mathbf{R}$-good work to preserve $\mathfrak{b}(\mathbf{R})$ small and $\mathfrak{d}(\mathbf{R})$ large since, whenever F is a θ-$\mathbf{R}$-unbounded family, $\mathfrak{b}(\mathbf{R}) \leq |F|$ and $\theta \leq \mathfrak{d}(\mathbf{R})$.

Clearly, θ-$\mathbf{R}$-good implies θ'-$\mathbf{R}$-good whenever $\theta \leq \theta'$ and any poset completely embedded into a θ-$\mathbf{R}$-good poset is also θ-$\mathbf{R}$-good.

Consider the following particular cases of interest for our main results.

Lemma 2.4 ([Mej13a, Lemma 4])**.** *Any poset of size $< \theta$ is θ-$\mathbf{R}$-good. In particular, Cohen forcing is $\mathbf{R}$-good.*

Example 2.5. (1) *Preserving non-meager sets:* Consider the Polish relational system $\mathbf{Ed} := \langle \omega^\omega, \omega^\omega, \neq^*\rangle$ where $x \neq^* y$ iff x and y are eventually different, that is, $x(i) \neq y(i)$

for all but finitely many $i<\omega$. By [BJ95, Thm. 2.4.1 and 2.4.7], $\mathfrak{b}(\mathbf{Ed})=\mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Ed})=\mathrm{cov}(\mathcal{M})$.

(2) *Preserving unbounded families:* Let $\mathbf{D}:=\langle\omega^\omega,\omega^\omega,\leq^*\rangle$ be the Polish relational system where $x\leq^* y$ iff $x(i)\leq y(i)$ for all but finitely many $i<\omega$. Clearly, $\mathfrak{b}(\mathbf{D})=\mathfrak{b}$ and $\mathfrak{d}(\mathbf{D})=\mathfrak{d}$.

Miller [Mil81] proved that $\mathbb{E}$ is $\mathbf{D}$-good. Besides, ω^ω-bounding posets are $\mathbf{D}$-good, like the random algebras.

(3) *Preserving null-covering families:* Let $b:\omega\to\omega\smallsetminus\{0\}$ such that $\sum_{i<\omega}\frac{1}{b(i)}<+\infty$ and let $\mathbf{Ed}_b:=\langle\mathbb{R}_b,\mathbb{R}_b,\neq^*\rangle$ be the Polish relational system where $\mathbb{R}_b:=\prod_{i<\omega}b(i)$. Since $\mathbf{Ed}_b$ is Tukey-Galois below $\langle\mathcal{N}(\mathbb{R}_b),\mathbb{R}_b,\not\ni\rangle$ (for $x\in\mathbb{R}_b$ the set $\{y\in\mathbb{R}_b:\neg(x\neq^* y)\}$ has measure zero with respect to the standard Lebesgue measure on $\mathbb{R}_b$), $\mathrm{cov}(\mathcal{N})\leq\mathfrak{b}(\mathbf{Ed}_b)$ and $\mathfrak{d}(\mathbf{Ed}_b)\leq\mathrm{non}(\mathcal{N})$.

By a similar argument as in [Bre91, Lemma 1*], any ν-centered poset is θ-$\mathbf{Ed}_b$-good for any $\nu<\theta$ infinite. In particular, σ-centered posets are $\mathbf{Ed}_b$-good.

(4) *Preserving "union of null sets is not null":* For each $k<\omega$ let $\mathrm{id}^k:\omega\to\omega$ such that $\mathrm{id}^k(i)=i^k$ for all $i<\omega$ and put $\mathcal{H}:=\{\mathrm{id}^{k+1}:k<\omega\}$. Let $\mathbf{Lc}:=\langle\omega^\omega,\mathcal{S}(\omega,\mathcal{H}),\in^*\rangle$ be the Polish relational system where

$$\mathcal{S}(\omega,\mathcal{H}):=\{\varphi:\omega\to[\omega]^{<\aleph_0}:\exists h\in\mathcal{H}\forall i<\omega(|\varphi(i)|\leq h(i))\},$$

and $x\in^*\varphi$ iff $\exists n<\omega\forall i\geq n(x(i)\in\varphi(i))$, which is read *$x$ is localized by φ*. As a consequence of Bartoszyński characterization (see [BJ95, Thm. 2.3.9]), $\mathfrak{b}(\mathbf{Lc})=\mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc})=\mathrm{cof}(\mathcal{N})$.

Any ν-centered poset is θ-$\mathbf{Lc}$-good for any $\nu<\theta$ infinite (see [JS90]) so, in particular, σ-centered posets are $\mathbf{Lc}$-good. Moreover, subalgebras (not necessarily complete) of random forcing are $\mathbf{Lc}$-good as a consequence of a result of Kamburelis [Kam89].

The following are the main general results concerning the preservation theory presented so far.

Lemma 2.6. *Let $\langle\mathbb{P}_\alpha\rangle_{\alpha<\theta}$ be a $\lessdot$-increasing sequence of ccc posets and $\mathbb{P}_\theta=\mathrm{limdir}_{\alpha<\theta}\mathbb{P}_\alpha$. If $\mathbb{P}_{\alpha+1}$ adds a Cohen real $\dot{c}_\alpha$ over $V^{\mathbb{P}_\alpha}$ for any $\alpha<\theta$, then $\mathbb{P}_\theta$ forces that $\{\dot{c}_\alpha:\alpha<\theta\}$ is a θ-$\mathbf{R}$-unbounded family of size θ.*

Theorem 2.7. *Let $\delta\geq\theta$ be an ordinal and $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\delta}$ a FS iteration of non-trivial θ-$\mathbf{R}$-good ccc posets. Then, $\mathbb{P}_\delta$ forces $\mathfrak{b}(\mathbf{R})\leq\theta$ and $\mathfrak{d}(\mathbf{R})\geq|\delta|$.*

Proof. See [GMS16, Cor. 3.6]. □

3. Coherent systems of FS iterations

Definition 3.1 (Relative embeddability)**.** Let M be a transitive model of ZFC (or a finite large fragment of it), $\mathbb{P}\in M$ and $\mathbb{Q}$ posets (the latter not necessarily in M). Say that $\mathbb{P}$ *is a complete subposet of* $\mathbb{Q}$ *with respect to* M, denoted by $\mathbb{P}\lessdot_M\mathbb{Q}$, if $\mathbb{P}$ is a suborder of $\mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ that belongs to M is also a maximal antichain in $\mathbb{Q}$.

Recall that in this case, if $N\supseteq M$ is another transitive model of ZFC with $\mathbb{Q}\in N$ and G is $\mathbb{Q}$-generic over N then $G\cap\mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G\cap\mathbb{P}]\subseteq N[G]$. Moreover, if $\dot{\mathbb{P}}'\in M$ is a $\mathbb{P}$-name of a poset, $\dot{\mathbb{Q}}'\in N$ is a $\mathbb{Q}$-name of a poset and $\Vdash_{\mathbb{Q},N}\dot{\mathbb{P}}'\lessdot_{M^{\mathbb{P}}}\dot{\mathbb{Q}}'$, then $\mathbb{P}*\dot{\mathbb{P}}'\lessdot_M\mathbb{Q}*\dot{\mathbb{Q}}'$. In particular, if $M=N=V$ (the universe), then $\mathbb{P}*\dot{\mathbb{P}}'\lessdot\mathbb{Q}*\dot{\mathbb{Q}}'$ whenever $\mathbb{P}\lessdot\mathbb{Q}$ and $\Vdash_{\mathbb{Q}}\dot{\mathbb{P}}'\lessdot_{V^{\mathbb{P}}}\dot{\mathbb{Q}}'$.

Definition 3.2 (Coherent system of FS iterations)**.** A *coherent system (of FS iterations)* $\mathbf{s}$ is composed by the following objects:

(I) a partial ordered set $I^{\mathbf{s}}$ and an ordinal $\pi^{\mathbf{s}}$,

(II) a system of posets $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi \leq \pi^{\mathbf{s}} \rangle$ such that

  (i) $\mathbb{P}^{\mathbf{s}}_{i,0} \lessdot \mathbb{P}^{\mathbf{s}}_{j,0}$ whenever $i \leq j$ in $I^{\mathbf{s}}$, and

  (ii) $\mathbb{P}^{\mathbf{s}}_{i,\eta}$ is the direct limit of $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : \xi < \eta \rangle$ for each limit $\eta \leq \pi^{\mathbf{s}}$,

(III) a sequence $\langle \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi < \pi^{\mathbf{s}} \rangle$ where each $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ is a $\mathbb{P}^{\mathbf{s}}_{i,\xi}$-name for a poset, $\mathbb{P}^{\mathbf{s}}_{i,\xi+1} = \mathbb{P}^{\mathbf{s}}_{i,\xi} * \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ and $\mathbb{P}^{\mathbf{s}}_{j,\xi}$ forces $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} \lessdot_{V^{\mathbb{P}^{\mathbf{s}}_{i,\xi}}} \dot{\mathbb{Q}}^{\mathbf{s}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{s}}$ and $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{j,\xi}$.

Note that, for a fixed $i \in I^{\mathbf{s}}$, the posets $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : \xi \leq \pi^{\mathbf{s}} \rangle$ are generated by a FS iteration $\langle \mathbb{P}'_{i,\xi}, \dot{\mathbb{Q}}'_{i,\xi} : \xi < 1 + \pi^{\mathbf{s}} \rangle$ where $\dot{\mathbb{Q}}'_{i,0} = \mathbb{P}^{\mathbf{s}}_{i,0}$ and $\dot{\mathbb{Q}}'_{i,1+\xi} = \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ for all $\xi < 1 + \pi^{\mathbf{s}}$. Therefore (by induction) $\mathbb{P}'_{i,1+\xi} = \mathbb{P}_{i,\xi}$ for all $\xi \leq \pi^{\mathbf{s}}$ and, thus, $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{i,\eta}$ whenever $\xi \leq \eta \leq \pi^{\mathbf{s}}$.

On the other hand, by Lemma 3.6, $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{s}}$ and $\xi \leq \pi^{\mathbf{s}}$.

For $j \in I^{\mathbf{s}}$ and $\eta \leq \pi^{\mathbf{s}}$ we write $V^{\mathbf{s}}_{j,\eta}$ for the $\mathbb{P}^{\mathbf{s}}_{j,\eta}$-generic extensions. Concretely, if G is $\mathbb{P}^{\mathbf{s}}_{j,\eta}$-generic over V, $V^{\mathbf{s}}_{j,\eta} := V[G]$ and $V^{\mathbf{s}}_{i,\xi} := V[\mathbb{P}^{\mathbf{s}}_{i,\xi} \cap G]$ for all $i \leq j$ in $I^{\mathbf{s}}$ and $\xi \leq \eta$. Note that $V^{\mathbf{s}}_{i,\xi} \subseteq V^{\mathbf{s}}_{j,\eta}$.

We say that the coherent system $\mathbf{s}$ is *ccc* if, additionally, $\mathbb{P}^{\mathbf{s}}_{i,0}$ has ccc and $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ forces that $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ has ccc for each $i \in I^{\mathbf{s}}$ and $\xi < \pi^{\mathbf{s}}$. This implies that $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ has ccc for all $i \in I^{\mathbf{s}}$ and $\xi \leq \pi^{\mathbf{s}}$.

We consider the following particular cases.

(1) When $I^{\mathbf{s}}$ is a well-ordered set, we say that $\mathbf{s}$ is a *2D-coherent system (of FS iterations).*

(2) If $I^{\mathbf{s}}$ is of the form $\{i_0, i_1\}$ ordered as $i_0 < i_1$, we say that $\mathbf{s}$ is a *coherent pair (of FS iterations).*

(3) If $I^{\mathbf{s}} = \gamma^{\mathbf{s}} \times \delta^{\mathbf{s}}$ where $\gamma^{\mathbf{s}}$ and $\delta^{\mathbf{s}}$ are ordinals and the order of $I^{\mathbf{s}}$ is defined as $(\alpha, \beta) \leq (\alpha', \beta')$ iff $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we say that $\mathbf{s}$ is a *3D-coherent system (of FS iterations).*

For a coherent system $\mathbf{s}$ and a set $J \subseteq I^{\mathbf{s}}$, $\mathbf{s}|J$ denotes the coherent system with $I^{\mathbf{s}|J} = J$, $\pi^{\mathbf{s}|J} = \pi^{\mathbf{s}}$ and the posets and names corresponding to (II) and (III) defined as for $\mathbf{s}$. On the other hand, if $\eta \leq \pi^{\mathbf{s}}$, $\mathbf{s}{\restriction}\eta$ denotes the coherent system with $I^{\mathbf{s}\restriction\eta} = I^{\mathbf{s}}$, $\pi^{\mathbf{s}\restriction\eta} = \eta$ and the posets for (II) and (III) defined as for $\mathbf{s}$. Note that, if $i_0 < i_1$ in $I^{\mathbf{s}}$, then $\mathbf{s}|\{i_0, i_1\}$ is a coherent pair and $\mathbf{s}|\{i_0\}$ corresponds just to the FS iteration $\langle \mathbb{P}'_{i_0,\xi}, \dot{\mathbb{Q}}'_{i_0,\xi} : \xi < 1 + \pi^{\mathbf{s}} \rangle$ (see the comment after (III)).

If $\mathbf{t}$ is a 3D-coherent system, for $\alpha < \gamma^{\mathbf{t}}$, $\mathbf{t}_\alpha := \mathbf{t}|\{(\alpha, \beta) : \beta < \delta^{\mathbf{t}}\}$ is a 2D-coherent system where $I^{\mathbf{t}_\alpha}$ has order type $\delta^{\mathbf{t}}$. For $\beta < \delta^{\mathbf{t}}$, $\mathbf{t}^\beta := \mathbf{t}|\{(\alpha, \beta) : \alpha < \delta^{\mathbf{t}}\}$ is a 2D-coherent system where $I^{\mathbf{t}^\beta}$ has order type $\gamma^{\mathbf{t}}$.

In particular, the super indexes $\mathbf{s}$ are omitted when there is no place for ambiguity.

Concerning consistency results about cardinal characteristics of the real line, Blass and Shelah [BS89] produced the first 2D-coherent system to obtain that $\mathfrak{u} < \mathfrak{d}$ is consistent with large continuum. This was followed by new consistency results from Brendle and Fischer [BF11] and Mejía [Mej13a] where Blass and Shelah's construction (which consists, basically, on 2D-coherent systems as formalized in Definition 3.2(1)) is theorized and improved. For their results, the main features of the produced matrix of generic extensions $\langle V_{\alpha,\xi} : \alpha \leq \gamma, \xi \leq \pi \rangle$ from a 2D-coherent system $\mathbf{m}$, as illustrated in Figure 1, are:

(F1) For $\alpha < \gamma$, there is a real $c_\alpha \in V_{\alpha+1,0}$ which "diagonalizes" $V_{\alpha,0}$ (e.g., $\mathbf{R}$-unbounded over $V_{\alpha,0}$ for a fixed Polish relational system $\mathbf{R}$, or diagonalizes it in the sense of Definition 4.2) and, through the coherent pair $\mathbf{m}|\{\alpha, \alpha+1\}$, c_α also diagonalizes all the models in the α-th row, that is, $V_{\alpha,\xi}$ for all $\xi \leq \pi$ (Lemmas 3.6 and 4.13).

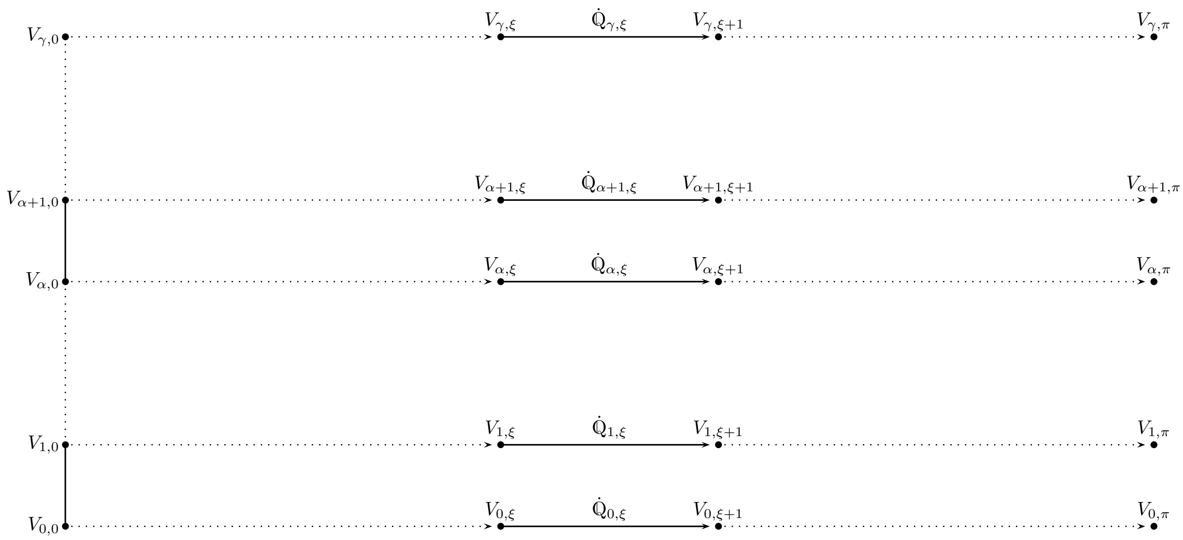

FIGURE 1. Matrix of generic extensions (2D-coherent system).

(F2) Provided that γ (the top level of the matrix) has uncountable cofinality. Given any column of the matrix, any real in the model of the top is actually in some of the models below, that is, $\mathbb{R} \cap V_{\gamma,\xi} = \bigcup_{\alpha<\gamma} \mathbb{R} \cap V_{\alpha,\xi}$ for every $\xi \leq \pi$ (Lemma 3.7 and Corollary 3.9).

To prove the main results of this paper, we extend this approach to 3D rectangles of generic extensions which help us separate more cardinal invariants at the same time. In a similar fashion as a matrix above, such a construction starts with a matrix of posets and "coherent" FS iterations are shot from each poset, which is formalized in Definition 3.2(3) as 3D-coherent systems. Figure 2 illustrates this idea. More generally, Definition 3.2 can be used to define multidimensional rectangles of generic extensions, though applications are unknown from 4 dimensions.

The feature (F1) can also be applied in general to coherent systems of FS iterations since any such system is composed by several coherent pairs of FS iterations. For coherent pairs, (F1) for "$\mathbf{R}$-unbounded over a model" had been quite understood in [BS89, BF11, Mej13a] whose results we review below. For the remaining of this section, fix $M \subseteq N$ transitive models of ZFC and a Polish relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ coded in M (in the sense that all its components are coded in M).

Recall that $\mathbb{S}$ is a *Suslin ccc poset* if it is a $\mathbf{\Sigma}^1_1$ subset of ω^ω (or another uncountable Polish space in general) and both its order and incompatibility relations are $\mathbf{\Sigma}^1_1$. Note that, if $\mathbb{S}$ is coded in M, then $\mathbb{S}^M \lessdot_M \mathbb{S}^N$.

Lemma 3.3 ([Mej13a, Thm. 7]). *Let $\mathbb{S}$ be a Suslin ccc poset coded in M. If $M \models$ "$\mathbb{S}$ is $\mathbf{R}$-good" then, in N, $\mathbb{S}^N$ forces that every real in $X \cap N$ which is $\mathbf{R}$-unbounded over M is $\mathbf{R}$-unbounded over $M^{\mathbb{S}^M}$.*

Corollary 3.4. *Let $\Gamma \in M$ be a non-empty set. If $M \models$ "$\mathbb{B}_\Gamma$ is $\mathbf{R}$-good" then $\mathbb{B}^N_\Gamma$, in N, forces that every real in $X \cap N$ which is $\mathbf{R}$-unbounded over M is $\mathbf{R}$-unbounded over $M^{\mathbb{B}^M_\Gamma}$.*

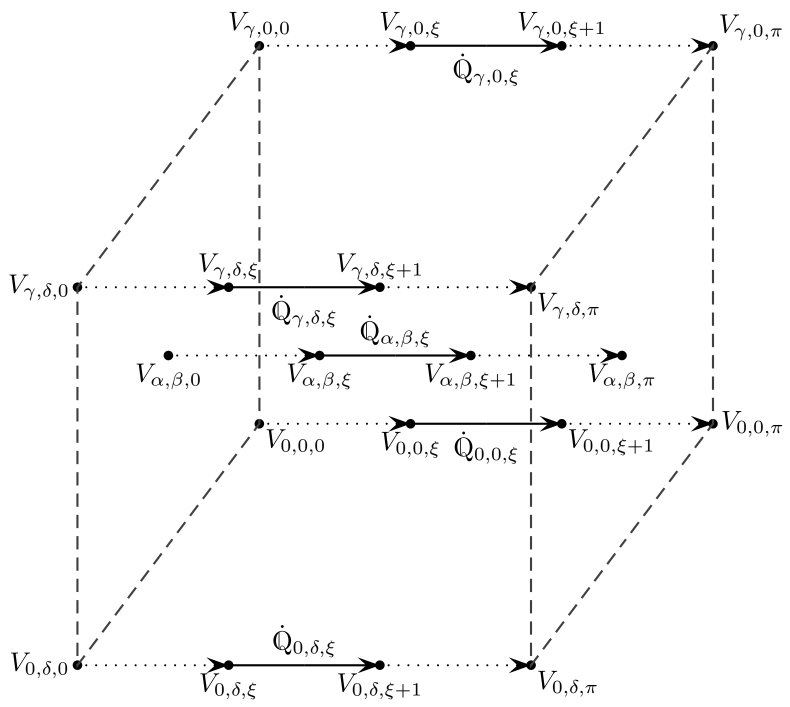

FIGURE 2. 3D rectangle of generic extensions (3D-coherent system).

Lemma 3.5 ([BF11, Lemma 11], see also [Mej15, Lemma 5.13])**.** *Assume $\mathbb{P} \in M$ is a poset. Then, in N, $\mathbb{P}$ forces that every real in $X \cap N$ which is $\mathbf{R}$-unbounded over M is $\mathbf{R}$-unbounded over $M^{\mathbb{P}}$.*

Lemma 3.6 (Blass and Shelah [BS89], [BF11, Lemmas 10, 12 and 13])**.** *Let $\mathbf{s}$ be a coherent pair of FS iterations as in Definition 3.2(2). Then, $\mathbb{P}_{i_0,\xi} \lessdot \mathbb{P}_{i_1,\xi}$ for all $\xi \leq \pi$.*

Even more, if $\dot{c}$ is a $\mathbb{P}_{i_1,0}$-name of a real in X, π is limit and $\mathbb{P}_{i_1,\xi}$ forces that $\dot{c}$ is $\mathbf{R}$-unbounded over $V_{i_0,\xi}$ for all $\xi < \pi$, then $\mathbb{P}_{i_1,\pi}$ forces that $\dot{c}$ is $\mathbf{R}$-unbounded over $V_{i_0,\pi}$.

Note that if c is a Cohen real over M then c is $\mathbf{R}$-unbounded over M by Definition 2.1(iii). In fact, all the unbounded reals used in our applications are actually Cohen.

Now we turn to discuss feature (F2). We aim to have such property for 3D-coherent systems but, as they are composed by several 2D-coherent systems, it is enough to understand (F2) for 2D-coherent systems. This was already understood in [BS89] and formalized in [BF11, Lemma 15] (see Corollary 3.9), which we generalize as follows.

Lemma 3.7. *Let $\mathbf{m}$ be a ccc 2D-coherent system with $I^{\mathbf{m}} = \gamma + 1$ an ordinal and $\pi^{\mathbf{m}} = \pi$. Assume that*

(i) γ has uncountable cofinality,
(ii) $\mathbb{P}_{\gamma,0}$ is the direct limit of $\langle \mathbb{P}_{\alpha,0} : \alpha < \gamma \rangle$, and
(iii) for any $\xi < \pi$, $\mathbb{P}_{\gamma,\xi}$ forces "$\dot{\mathbb{Q}}_{\gamma,\xi} = \bigcup_{\alpha<\gamma} \dot{\mathbb{Q}}_{\alpha,\xi}$" whenever $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $\langle \mathbb{P}_{\alpha,\xi} : \alpha < \gamma \rangle$.

Then, for any $\xi \leq \pi$, $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $\langle \mathbb{P}_{\alpha,\xi} : \alpha < \gamma \rangle$.

Proof. We proceed by induction on ξ. The case when ξ is not successor is clear, so we just need to deal with the successor step. Assume that the conclusion holds for ξ. If

$p \in \mathbb{P}_{\gamma,\xi+1}$ then $p = (r, \dot{q})$ where $r \in \mathbb{P}_{\gamma,\xi}$ and $\dot{q}$ is a $\mathbb{P}_{\gamma,\xi}$-name of a member of $\dot{\mathbb{Q}}_{\gamma,\xi}$. By (iii), there is a maximal antichain $\{p_n : n < \omega\}$ in $\mathbb{P}_{\gamma,\xi}$ such that p_n decides $\dot{q} = \dot{q}_n \in \dot{\mathbb{Q}}_{\alpha_n,\xi}$ for some $\alpha_n < \gamma$ and some $\mathbb{P}_{\alpha_n,\xi}$-name $\dot{q}_n$.[1] By (i), (ii) and the induction hypothesis, there is an $\alpha < \gamma$ above all α_n such that $\{p_n : n < \omega\} \subseteq \mathbb{P}_{\alpha,\xi}$ and $r \in \mathbb{P}_{\alpha,\xi}$. Therefore, $\dot{q}$ is a $\mathbb{P}_{\alpha,\xi}$-name of a member of $\dot{\mathbb{Q}}_{\alpha,\xi}$ and $p \in \mathbb{P}_{\alpha,\xi+1}$. □

The 2D and 3D-coherent systems constructed to prove our main results can be classified in terms of the following notion.

Definition 3.8 (Standard coherent system of FS iterations)**.** A ccc coherent system of FS iterations $\mathbf{s}$ is *standard* if

- (I) it consists, additionally, of:
  - (i) a partition $\langle S^{\mathbf{s}}, C^{\mathbf{s}} \rangle$ of $\pi^{\mathbf{s}}$,
  - (ii) a function $\Delta^{\mathbf{s}} : C^{\mathbf{s}} \to I^{\mathbf{s}}$ so that $\Delta^{\mathbf{s}}(i)$ is not maximal in $I^{\mathbf{s}}$ for all $i \in C^{\mathbf{s}}$,
  - (iii) a sequence $\langle \mathbb{S}^{\mathbf{s}}_\xi : \xi \in S^{\mathbf{s}} \rangle$ where each $\mathbb{S}^{\mathbf{s}}_\xi$ is either a Suslin ccc poset or a random algebra, and
  - (iv) a sequence $\langle \dot{\mathbb{Q}}^{\mathbf{s}}_\xi : \xi \in C^{\mathbf{s}} \rangle$ such that each $\dot{\mathbb{Q}}^{\mathbf{s}}_\xi$ is a $\mathbb{P}^{\mathbf{s}}_{\Delta^{\mathbf{s}}(\xi),\xi}$-name of a poset which is forced to be ccc by $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ for all $i \geq \Delta^{\mathbf{s}}(\xi)$ in $I^{\mathbf{s}}$, and
- (II) it is satisfied, for any $i \in I^{\mathbf{s}}$ and $\xi < \pi^{\mathbf{s}}$, that

$$\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} = \begin{cases} (\mathbb{S}^{\mathbf{s}}_\xi)^{V^{\mathbf{s}}_{i,\xi}} & \text{if } \xi \in S^{\mathbf{s}} \\ \dot{\mathbb{Q}}^{\mathbf{s}}_\xi & \text{if } \xi \in C^{\mathbf{s}} \text{ and } i \geq \Delta^{\mathbf{s}}(\xi), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

As in Definition 3.2, the super index $\mathbf{s}$ may be omitted when it is clear from the context.

All the standard coherent systems in this paper are constructed by recursion on $\xi < \pi$. To be more precise, we start with some partial order of ccc posets $\langle \mathbb{P}_{i,0} : i \in I \rangle$ as in Definition 3.2(II)(i), fix the partition in (I)(i) and, by recursion, the posets $\mathbb{P}_{i,\xi}$ and names $\dot{\mathbb{Q}}_{i,\xi}$ for all $i \in I$, along with the function Δ and the sequence of Suslin ccc posets in (I)(iii) (though in some cases Δ and the sequence of Suslin ccc posets are fixed before the recursion), are defined as follows: when $\mathbb{P}_{i,\xi}$ has been constructed for all $i \in I$, we look at the case whether $\xi \in S$ or $\xi \in C$. In the first case, $\mathbb{S}_\xi$ is chosen; in the second, we choose $\Delta(\xi)$ and then we define the ($\mathbb{P}_{\Delta(\xi),\xi}$-name of a) poset $\dot{\mathbb{Q}}_\xi$ as in (I)(iv). After this, the iterations continue with $\mathbb{P}_{i,\xi+1} = \mathbb{P}_{i,\xi} * \dot{\mathbb{Q}}_{i,\xi}$ as indicated in (II). It is clear that the requirements in Definition 3.2 for a ccc coherent system are satisfied.

In practice, a standard coherent system as above is constructed by using posets adding generic reals and the cases whether $\xi \in S$ or $\xi \in C$ indicate how generic is the real. Namely, when $\xi \in S$, $\mathbb{S}_\xi$ adds a real that is generic over $V_{i,\xi}$ for all $i \in I$, which means that we add a *full generic real* at stage ξ; on the other hand, when $\xi \in C$ we just add a *restricted generic* in the sense that $\dot{\mathbb{Q}}_\xi$ adds a real which is generic over $V_{\Delta(\xi),\xi}$ but <u>not</u> necessarily over $V_{i,\xi}$ when $i \not\leq \Delta(\xi)$, for instance, if $\dot{\mathbb{Q}}_\xi$ is a name for $\mathbb{D}^{V_{\Delta(\xi),\xi}}$, at the ξ-step the Hechler real added is generic only over $V_{\Delta(\xi),\xi}$. This approach of adding full and restricted generic reals is useful for controlling many cardinal invariants at the same time like in [BS89, BF11, Mej13a] and this work.

It is clear that any standard 2D-coherent system satisfies the hypothesis (iii) of Lemma 3.7 whenever (i) and (ii) are satisfied. Therefore,

[1] It is implicit in this proof that the names considered for the members of $\dot{\mathbb{Q}}_{\alpha,\gamma}$ are *canonical* in the sense described by the mentioned maximal antichains.

Corollary 3.9 ([BF11, Lemma 15]). *If $\mathbf{m}$ is a standard 2D-coherent system with $I^{\mathbf{m}} = \gamma+1$ and ordinal and $\pi^{\mathbf{m}} = \pi$ satisfying (i) and (ii) of Lemma 3.7 then, for any $\xi \leq \pi$, $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $\langle \mathbb{P}_{\alpha,\xi} : \alpha < \gamma \rangle$.*

The elements of this section can be summarized in the following result.

Theorem 3.10 ([Mej13a, Thm. 10 & Cor. 1]). *Let $\mathbf{m}$ be a standard 2D-coherent system with $I^{\mathbf{m}} = \gamma+1$ (an ordinal), $\pi^{\mathbf{m}} = \pi$ and $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ a Polish relational system coded in V. Assume that*

(i) for any $\xi \in S$ and $\alpha \leq \gamma$, $\mathbb{P}_{\alpha,\xi}$ forces that $\dot{\mathbb{Q}}_{\alpha,\xi} = \mathbb{S}_\xi^{V_{\alpha,\xi}}$ is $\mathbf{R}$-good and

(ii) for any $\alpha < \gamma$ there is a $\mathbb{P}_{\alpha+1,0}$-name $\dot{c}_\alpha$ of a $\mathbf{R}$-unbounded member of X over $V_{\alpha,0}$.

Then, for any $\xi \leq \pi$ and $\alpha < \gamma$, $\mathbb{P}_{\alpha+1,\xi}$ forces that $\dot{c}_\alpha$ is $\mathbf{R}$-unbounded over $V_{\alpha,\xi}$. In addition, if $\mathbf{m}$ satisfies (i) and (ii) of Lemma 3.7 then $\mathbb{P}_{\gamma,\pi}$ forces $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathbf{R})$.

Proof. The first statement is a direct consequence of Lemmas 3.3, 3.5 and 3.6. For the second statement, note that Corollary 3.9 implies that, in $V_{\gamma,\pi}$, $\{c_{\alpha_\eta} : \eta < \mathrm{cf}(\gamma)\}$ is a $\mathrm{cf}(\gamma)$-$\mathbf{R}$-unbounded family where $\langle \alpha_\eta : \eta < \mathrm{cf}(\gamma) \rangle \in V$ is an increasing cofinal sequence of γ, so $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathbf{R})$ follows. □

4. Preservation of Hechler mad families

We review from [BF11] the theory of preserving, through coherent pairs of FS iterations, a mad family added by Hechler's poset for adding an a.d. family (see Definition 4.1). This theory is quite similar to the approach in Section 3. Additionally, we show in Lemmas 4.8 and 4.10 that random forcing $\mathbb{B}$ and the eventually different forcing $\mathbb{E}$ fit well in this framework.

Definition 4.1 (Hechler [Hec72]). For a set Ω define the poset $\mathbb{H}_\Omega := \{p : F_p \times n_p \to 2 : F_p \in [\Omega]^{<\aleph_0} \text{ and } n_p < \omega\}$. The order is given by $q \leq p$ iff $p \subseteq q$ and, for any $i \in n_q \smallsetminus n_p$, there is at most one $z \in F_p$ such that $q(z,i) = 1$.

If G is $\mathbb{H}_\Omega$-generic over V then $A = A_G := \{a_z : z \in \Omega\}$ is an a.d. family where $a_z \subseteq \omega$ is defined as $i \in a_z$ iff $p(z,i) = 1$ for some $p \in G$. Moreover, $V[G] = V[A]$ and, when Ω is uncountable, A is mad in $V[G]$.

If $\Omega \subseteq \Omega'$ it is clear that $\mathbb{H}_\Omega \lessdot \mathbb{H}_{\Omega'}$ and even the ($\mathbb{H}_\Omega$-name of the) quotient $\mathbb{H}_{\Omega'}/\mathbb{H}_\Omega$ is nicely expressed (see, e.g., [BF11, §2]). On the other hand, if $\mathcal{C}$ is a $\subseteq$-chain of sets then $\mathbb{H}_{\bigcup \mathcal{C}} = \mathrm{limdir}_{\Omega \in \mathcal{C}} \mathbb{H}_\Omega$. Therefore, if γ is an ordinal, $\mathbb{H}_\gamma$ can be obtained by a FS iteration of length γ where $\mathbb{H}_\alpha$ is the poset obtained in the α-th stage of the iteration and $\mathbb{H}_{\alpha+1}/\mathbb{H}_\alpha$, which is σ-centered, is the α-th iterand. Since $\mathbb{H}_\Omega$ only depends on the size of Ω then $\mathbb{H}_\Omega$ has precaliber ω_1 (though this can be proved directly by a Δ-system argument). Moreover, if Ω is non-empty and countable then $\mathbb{H}_\Omega \simeq \mathbb{C}$ and, if $|\Omega| = \aleph_1$, then $\mathbb{H}_\Omega \simeq \mathbb{C}_{\omega_1}$.

From now on, fix $M \subseteq N$ transitive models of ZFC. We define below a diagonalization property to preserve mad families like the one added by Hechler's poset.

Definition 4.2 ([BF11, Def. 2]). Let $A = \langle a_z \rangle_{z \in \Omega} \in M$ be a family of infinite subsets of ω and $a^* \in [\omega]^{\aleph_0}$ (not necessarily in M). Say that a^* *diagonalizes M outside A* if, for all $h \in M$, $h : \omega \times [\Omega]^{<\aleph_0} \to \omega$ and for any $m < \omega$, there are $i \geq m$ and $F \in [\Omega]^{<\aleph_0}$ such that $[i, h(i,F)) \smallsetminus \bigcup_{z \in F} a_z \subseteq a^*$.

Given A a collection of subsets of ω, *the ideal generated by A* is defined as

$$\mathcal{I}(A) := \{x \subseteq \omega : x \subseteq^* \bigcup_{a \in F} a \text{ for some finite } F \subseteq A\}.$$

Lemma 4.3 ([BF11, Lemma 3]). *If a^* diagonalizes M outside A then $|a^* \cap x| = \aleph_0$ for any $x \in M \smallsetminus \mathcal{I}(A)$.*

Lemma 4.4 ([BF11, Lemma 4]). *Let Ω be a set, $z^* \in \Omega$ and $A := \{a_z : z \in \Omega\}$ the a.d. family added by $\mathbb{H}_\Omega$. Then, $\mathbb{H}_\Omega$ forces that a_{z^*} diagonalizes $V^{\mathbb{H}_{\Omega \smallsetminus \{z^*\}}}$ outside $A{\restriction}(\Omega \smallsetminus \{z^*\})$*

Corollary 4.5. *Let γ be an ordinal of uncountable cofinality and let $\langle M_\alpha \rangle_{\alpha \leq \gamma}$ be an increasing sequence of transitive ZFC models such that $[\omega]^{\aleph_0} \cap M_\gamma = \bigcup_{\alpha<\gamma} [\omega]^{\aleph_0} \cap M_\alpha$. Assume that $A = \{a_\alpha : \alpha < \gamma\} \in M_\gamma$ is a family of infinite subsets of ω such that, for any $\alpha < \gamma$, $A{\restriction}\alpha \in M_\alpha$ and $a_\alpha \in M_{\alpha+1}$ diagonalizes M_α outside $A{\restriction}\alpha$. Then, for any $x \in [\omega]^{\aleph_0} \cap M_\gamma$, there exists an $\alpha < \gamma$ such that $|x \cap a_\alpha| = \aleph_0$. If, additionally, A is almost disjoint, then A is mad in M_γ.*

The previous corollary implies that the a.d. family added by $\mathbb{H}_\Omega$ for Ω uncountable is actually mad (since $\mathbb{H}_\Omega \cong \mathbb{H}_\gamma$ for some ordinal γ of uncountable cofinality).

The main idea for mad preservation in [BF11] is that, when ccc 2D-coherent systems are constructed, the first column, along with a mad family $A = \{a_\alpha : \alpha < \gamma\}$, satisfies the hypothesis of Corollary 4.5 (e.g. $\mathbb{P}_{\alpha,0} = \mathbb{H}_\alpha$ for all $\alpha \leq \gamma$) and each a_α is preserved to diagonalize the models in the α-th row outside $A{\restriction}\alpha$ (that is, the second case of (F1) at the beginning of Section 3). For this purpose, we present the following results related to the preservation of the property in Definition 4.2 through coherent pairs of iterations.

Lemma 4.6 ([BF11, Lemma 11]). *Let $\mathbb{P} \in M$ be a poset. If $N \models$ "a^* diagonalizes M outside A" then*

$$N^{\mathbb{P}} \models \text{“}a^* \text{ diagonalizes } M^{\mathbb{P}} \text{ outside } A\text{”}.$$

Corollary 4.7. *If $N \models$ "a^* diagonalizes M outside A" then*

$$N^{\mathbb{C}^N} \models \text{“}a^* \text{ diagonalizes } M^{\mathbb{C}^M} \text{ outside } A\text{”}.$$

Lemma 4.8. *If $N \models$ "a^* diagonalizes M outside A" then*

$$N^{\mathbb{E}^N} \models \text{“}a^* \text{ diagonalizes } M^{\mathbb{E}^M} \text{ outside } A\text{”}.$$

Proof. Let $\dot{h} \in M$ be an $\mathbb{E}$-name for a function from $\omega \times [\Omega]^{<\aleph_0}$ into ω. Work within M and fix a non-principal ultrafilter D on ω (in M). For $s \in \omega^{<\omega}$ and $n < \omega$ define $h_{s,n} : \omega \times [\Omega]^{<\aleph_0} \to \omega + 1$ as

$$h_{s,n}(i, F) = \min\{j < \omega : (\forall \varphi,\ \mathrm{width}(\varphi) \leq n)((s, \varphi) \not\Vdash \dot{h}(i, F) > j)\}.$$

Claim 4.9. *$h_{s,n}(i, F) \in \omega$ for all $i < \omega$ and $F \in [\Omega]^{<\aleph_0}$.*

Proof. Assume not, so there is a sequence of slaloms $\langle \varphi_j \rangle_{j<\omega}$ of width $\leq n$ such that $(s, \varphi_j) \Vdash \dot{h}(i, F) > j$. Define the slalom φ^* as

$$\varphi^*(i) = \{m < \omega : \{j < \omega : m \in \varphi_j(i)\} \in D\}.$$

By a compactness argument, $\mathrm{width}(\varphi^*) \leq n$, so $(s, \varphi^*) \in \mathbb{E}$. Now, there are $(t, \psi) \leq (s, \varphi^*)$ and $j_0 < \omega$ such that $(t, \psi) \Vdash \dot{h}(i, F) = j_0$. By the definition of φ^*,

$$\{j < \omega : \forall i \in |t| \smallsetminus |s| (t(i) \notin \varphi_j(i))\} \in D$$

so that set is infinite. For any $j > j_0$ in that set, (t, ψ) is compatible with (s, φ_j) and, therefore, any common stronger condition forces $j_0 = \dot{h}(i, F) > j$, a contradiction. $\square$

Now, in N, fix $m<\omega$ and $p=(s,\varphi)\in\mathbb{E}^N$ with $n:=\mathrm{width}(\varphi)$. As a^* diagonalizes M outside A, there are $i\geq m$ and $F\in[\Omega]^{<\aleph_0}$ such that $[i,h_{s,n}(i,F))\smallsetminus\bigcup_{z\in F}a_z\subseteq a^*$. By definition of $h_{s,n}$, $(\forall\varphi,\ \mathrm{width}(\varphi)\leq n)((s,\varphi)\nVdash\dot{h}(i,F)>h_{s,n}(i,F))$ is a true $\mathbf{\Pi}^1_1$-statement in M so, by absoluteness, it is also true in N. Therefore, there is a $q\in\mathbb{E}^N$ stronger than p that forces $\dot{h}(i,F)\leq h_{s,n}(i,F)$ and then we conclude that q forces $[i,\dot{h}(i,F))\smallsetminus\bigcup_{z\in F}a_z\subseteq a^*$. □

Lemma 4.10. *If $N\models$ "a^* diagonalizes M outside A" then*

$$N^{\mathbb{B}^N}\models \text{“}a^* \text{ diagonalizes } M^{\mathbb{B}^M} \text{ outside } A\text{”}.$$

Proof. In the standard proof that $\mathbb{B}$ is ω^ω-bounding (see for example [BJ95]) it is shown that, for any $p\in\mathbb{B}$, $\epsilon\in(0,1)$ and $\dot{x}$ a $\mathbb{B}$-name for a real in ω^ω, there are $q\leq p$ and $g\in\omega^\omega$ such that $q\Vdash\dot{x}\leq g$ and $\lambda(p\smallsetminus q)\leq\epsilon\lambda(p)$ where λ is the Lebesgue measure. We are going to use this fact to prove the lemma.

Fix $\dot{h}\in M$ a $\mathbb{B}$-name for a function from $\omega\times[\Omega]^{<\aleph_0}$ to ω, $p\in\mathbb{B}^N$ and $m<\omega$. By the Lebesgue density Theorem there is a clopen non-empty set C such that $\lambda(C\smallsetminus p)<\frac{1}{4}\lambda(C)$. Now, in M, find $g:\omega\times[\Omega]^{<\aleph_0}\to\omega$ such that, for any $F\in[\Omega]^{<\aleph_0}$, there is a $q_F\leq C$ in $\mathbb{B}$ with $\lambda(C\smallsetminus q_F)\leq\frac{1}{4}\lambda(C)$ that forces $\forall i<\omega(\dot{h}(i,F)\leq g(i,F))$. Then, in N, there are $i\geq m$ and $F\in[\Omega]^{<\aleph_0}$ such that $[i,g(i,F))\smallsetminus\bigcup_{z\in F}a_z\subseteq a^*$, so q_F forces $[i,\dot{h}(i,F))\smallsetminus\bigcup_{z\in F}a_z\subseteq a^*$. As $\lambda(p\cap q_F)>\frac{1}{2}\mu(C)$, $p\cap q_F\in\mathbb{B}^N$ is stronger than p and forces $[i,\dot{h}(i,F))\smallsetminus\bigcup_{z\in F}a_z\subseteq a^*$. □

Corollary 4.11. *Let $\Gamma\in M$ be a non-empty set. If $N\models$ "a^* diagonalizes M outside A" then*

$$N^{\mathbb{B}_\Gamma^N}\models \text{“}a^* \text{ diagonalizes } M^{\mathbb{B}_\Gamma^M} \text{ outside } A\text{”}.$$

Proofs of both Lemmas 4.8 and 4.10 use a similar argument to that of the proof that the respective posets are $\mathbf{D}$-good (the compactness argument for $\mathbb{E}$ and ω^ω-bounding for $\mathbb{B}$).

Question 4.12. *Assume $\mathbb{S}$ is a Suslin ccc poset coded in M such that $M\models$ "$\mathbb{S}$ is $\mathbf{D}$-good" and $N\models$ "a^* diagonalizes M outside A". Does*

$$N^{\mathbb{S}^N}\models \text{“}a^* \text{ diagonalizes } M^{\mathbb{S}^M} \text{ outside } A\text{”?}$$

Lemma 4.13 ([BF11, Lemma 12])**.** *Let $\mathbf{s}$ be a coherent pair of FS iterations, $A\in V$ a family of infinite subsets of ω and $\dot{a}^*$ a $\mathbb{P}_{i_1,0}$-name for an infinite subset of ω such that*

$$\Vdash_{\mathbb{P}_{i_1,\xi}} \text{“}\dot{a}^* \text{ diagonalizes } V_{i_0,\xi} \text{ outside } A\text{”}$$

for all $\xi<\pi$. Then, $\mathbb{P}_{i_0,\pi}\lessdot\mathbb{P}_{i_1,\pi}$ and $\Vdash_{\mathbb{P}_{i_1,\pi}}$ "$\dot{a}^*$ diagonalizes $V_{i_0,\pi}$ outside A".

The results above are summarized as follows when considering standard 2D-coherent systems.

Theorem 4.14. *Let $\mathbf{m}$ be a standard 2D-coherent system with $I^{\mathbf{m}}=\gamma+1$ an ordinal and $\pi^{\mathbf{m}}=\pi$ satisfying (i) and (ii) of Lemma 3.7 and, for each $\alpha<\gamma$, let $\dot{a}_\alpha$ be a $\mathbb{P}_{\alpha+1,0}$-name of an infinite subset of ω such that $\mathbb{P}_{\alpha+1}$ forces that $\dot{a}_\alpha$ diagonalizes $V_{\alpha,0}$ outside $\{\dot{a}_\varepsilon:\varepsilon<\alpha\}$ and $\mathbb{P}_{\gamma,0}$ forces $\dot{A}=\{\dot{a}_\alpha:\alpha<\gamma\}$ to be an a.d. family. If $\mathbb{S}_\xi\in\{\mathbb{C},\mathbb{E}\}\cup\mathfrak{R}$ for all $\xi\in S$ then $\mathbb{P}_{\gamma,\pi}$ forces that $\dot{A}$ is mad and $\mathfrak{a}\leq\mathrm{cf}(\gamma)$.*

Proof. Lemmas 3.9, 4.6, 4.8, 4.10 and 4.13 imply that $\langle V_{\alpha,\pi}:\alpha\leq\gamma\rangle$ and A satisfy the hypothesis of Corollary 4.5, so A and $\{a_{\alpha_\eta}:\eta<\mathrm{cf}(\gamma)\}$ are mad in $V_{\gamma,\pi}$ where $\langle\alpha_\eta:\eta<\mathrm{cf}(\gamma)\rangle\in V$ is an increasing cofinal sequence of γ. □

Remark 4.15. A version of the previous theorem was originally proved by Brendle and Fischer [BF11] for a special case where Mathias-Prikry posets with ultrafilters are considered, though the ultrafilters are built very carefully. Concretely, when $\langle M_\alpha : \alpha \leq \gamma\rangle$ and an a.d. family A are as in Corollary 4.5 and $M_0 \models$"U_0 is an ultrafilter on ω", they elaborated a method to construct an increasing chain of filters $\langle U_\alpha : \alpha \leq \gamma\rangle \in M_\gamma$ such that, for all $\beta \leq \gamma$,

(a) $\langle U_\alpha : \alpha \leq \beta\rangle \in M_\beta$ and U_β is an ultrafilter in M_β,
(b) $\mathbb{M}(U_\alpha) \lessdot_{M_\alpha} \mathbb{M}(U_\beta)$ for all $\alpha < \beta$ and
(c) if $\beta = \alpha + 1$ then $\mathbb{M}(U_\beta)$ forces that a_α diagonalizes $M_\alpha^{\mathbb{M}(U_\alpha)}$ outside $A{\restriction}\alpha$.

The idea of this construction is originated from a similar method of Blass and Shelah [BS89] to preserve unbounded reals. Instead of an a.d. family A, they consider a sequence of reals $\langle c_\alpha : \alpha < \gamma\rangle \in M_\gamma$ such that each $c_\alpha \in M_{\alpha+1}$ is unbounded (i.e., $\mathbf{D}$-unbounded) over M_α and the ultrafilters are constructed such that (a) and (b) above are satisfied and $\mathbb{M}(U_{\alpha+1})$ forces that c_α is unbounded over $M_\alpha^{\mathbb{M}(U_\alpha)}$ for all $\alpha < \gamma$.

This form of extending a 2D-coherent iteration can be incorporated in standard 2D-iterations as in Definition 3.8, so π is partitioned into three sets S, C and F where, in addition, the previous constructions are considered for $\xi \in F$. This was done in [Mej13b] to obtain consistency results about the cardinal invariants $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{u}$ in relation with those in Cichoń's diagram. But thanks to Lemmas 4.8 and 4.10, some of the constructions there can be modified to obtain, additionally, $\mathfrak{b} = \mathfrak{a}$ (like in Theorem 5.8).

Remark 4.16. (1) Other mad families can be considered in this theory of preservation, for instance, the mad family added by a FS iteration of Mathias-Prikry posets. Given $A \subseteq [\omega]^{\aleph_0}$, the Mathias-Prikry poset $\mathbb{M}(\mathcal{I}(A)^*)$ (for an ideal $\mathcal{I}$, $\mathcal{I}^*$ denotes its dual filter) adds a real $a^* \in [\omega]^{\aleph_0}$ which is almost disjoint from all the members of A. Moreover, $\mathbb{M}(\mathcal{I}(A)^*)$ forces that a^* diagonalizes V outside A. Thus, for an ordinal γ with uncountable cofinality, the FS iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\gamma}$ with $\dot{\mathbb{Q}}_\alpha = \mathbb{M}(\mathcal{I}(A{\restriction}\alpha)^*)$ adds an a.d. family $A = \{a_\alpha : \alpha < \gamma\}$ where each a_α is the Mathias real added by $\dot{\mathbb{Q}}_\alpha$. By Corollary 4.5, $\mathbb{P}_\gamma$ forces that A is mad.
(2) Any FS iteration of length ω_1 of non-trivial ccc posets adds a mad family of size $\aleph_1$ (so it forces $\mathfrak{a} = \aleph_1$), actually, the mad family is formed by the Cohen reals added at limit stages. To understand this, it is enough to note that, if $A \in V$ is a countable subset of $[\omega]^{\aleph_0}$ and $c \in [\omega]^{\aleph_0}$ is Cohen over V, then c diagonalizes V outside A and c is almost disjoint from all the members of A (because $\mathbb{C} \simeq \mathbb{M}(\mathcal{I}(A)^*)$).

The following is a generalization of a result of Steprans [Ste93] which shows that the maximal almost disjoint family added by the forcing $\mathbb{H}_\kappa$ is indestructible after forcing with some particular posets. Steprans' result can be then deduced when $\kappa = \omega_1$ (so $\mathbb{H}_{\omega_1} = \mathbb{C}_{\omega_1}$) and $\dot{\mathbb{Q}}_\xi = \mathbb{C}$ for all $\xi < \pi$.

Theorem 4.17. *Let κ be an uncountable regular cardinal. After forcing with $\mathbb{H}_\kappa$, any FS iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi\rangle_{\xi<\pi}$ where each iterand is either*

(i) in $\{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$ or
(ii) a ccc poset of size $< \kappa$

preserves the mad family added by $\mathbb{H}_\kappa$.

Proof. We reconstruct the iteration $\mathbb{H}_\kappa$ followed by $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi\rangle_{\xi<\pi}$ as a standard 2D-coherent system $\mathbf{m}$ so that $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi} = \mathbb{H}_\kappa * \mathbb{P}_\xi$ for all $\xi \leq \pi$. The construction goes as follows (see Definition 3.8):

(1) $I^{\mathbf{m}} = \kappa + 1$ and $\pi^{\mathbf{m}} = \pi$.
(2) For each $\alpha \leq \kappa$, $\mathbb{P}^{\mathbf{m}}_{\alpha,0} = \mathbb{H}_\alpha$.
(3) The partition $\langle S^{\mathbf{m}}, C^{\mathbf{m}} \rangle$ of $\pi^{\mathbf{m}}$ corresponds to the set of ordinals in the iteration where a poset coming from (i) or (ii) is used. In other words, $\xi \in S^{\mathbf{m}}$ if (i) holds for $\dot{\mathbb{Q}}_\xi$, and $\xi \in C^{\mathbf{m}}$ otherwise.
(4) The functions $\Delta^{\mathbf{m}} : C^{\mathbf{m}} \to \kappa$ and the sequences $\langle \dot{\mathbb{S}}^{\mathbf{m}}_\xi : \xi \in S^{\mathbf{m}} \rangle$ and $\langle \dot{\mathbb{Q}}^{\mathbf{m}}_\xi : \xi \in C^{\mathbf{m}} \rangle$ are constructed by recursion on $\xi < \pi$ along with the FS iterations of the 2D-coherent system. We split into the following cases:
   - If $\xi \in S^{\mathbf{m}}$ define $\dot{\mathbb{S}}^{\mathbf{m}}_\xi$ to be one of the posets in the set $\{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$ depending on what $\mathbb{P}_\xi$ forces $\dot{\mathbb{Q}}_\xi$ to be.
   - If $\xi \in C^{\mathbf{m}}$ we define both $\Delta^{\mathbf{m}}(\xi)$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$, the latter as a $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),0}$-name. Since $\xi \in C^{\mathbf{m}}$ we have that $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi}$-name for a ccc poset of size $< \kappa$, hence without loss of generality we can assume that the domain of $\dot{\mathbb{Q}}_\xi$ is an ordinal $\gamma_\xi < \kappa$ (not just a name). By Lemma 3.7, $\dot{\mathbb{Q}}_\xi$ is (forced by $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi}$ to be equal to) a $\mathbb{P}^{\mathbf{m}}_{\alpha,\xi}$-name $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$ for some $\alpha < \kappa$. So put $\Delta^{\mathbf{m}}(\xi) = \alpha + 1$.

Notice that $\mathbf{m}$ satisfies the assumptions of Theorem 4.14 for the mad family A added by $\mathbb{H}_\kappa$, so A is still mad in $V^{\mathbf{m}}_{\kappa,\pi}$.

□

Remark 4.18. When $\kappa = \omega_1$ in Theorem 4.17, by Remark 4.16(2) the result still holds when $\mathbb{H}_{\omega_1}$ is replaced by any FS iteration of length with cofinality ω_1. This is an alternative (and also a generalization) of Zhang's result [Zha99] which states that, under CH, there is a mad family in the ground model which stays mad after a FS iteration of $\mathbb{E}$.

5. Consistency results on Cichoń's diagram

In this section, we prove the consistency of certain constellations in Cichoń's diagram where, additionally, the almost disjointness number can be decided (equal to $\mathfrak{b}$). For all the results, we fix uncountable regular cardinals $\theta_0 \leq \theta_1 \leq \kappa \leq \mu \leq \nu$ and a cardinal $\lambda \geq \nu$. We denote the ordinal product between cardinals by, e.g., $\lambda \cdot \mu$.

The following summarizes the results in [Mej13a, Sect. 3] but in addition we get that $\mathfrak{b} = \mathfrak{a}$ can be forced.

Theorem 5.1. *Assume $\lambda = \lambda^{<\kappa}$ and $\lambda' \geq \lambda$ with $(\lambda')^{\aleph_0} = \lambda'$. For each of the items below, there is a ccc poset forcing the corresponding statement.*

(a) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \kappa$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$.
(b) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$ *and* $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.
(c) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ *and* $\mathfrak{d} = \mathfrak{c} = \lambda$.
(d) $\mathrm{non}(\mathcal{N}) = \aleph_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathfrak{d} = \lambda$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \lambda'$.

Proof. The proofs are basically the same as in [Mej13a] combined with the methods of preservation of mad families developed in Section 4 which we include in this paper for completeness. For all the items, start adding a mad family with $\mathbb{H}_\kappa$.

(a) Construct an iteration as in the last part of [Mej13a, Thm. 2]. To be more precise, perform a FS iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\lambda}$ where each $\dot{\mathbb{Q}}_\alpha$ is either
   (i) a σ-linked subposet of $\mathbb{LOC}$ of size $< \theta_0$,
   (ii) a subalgebra of $\mathbb{B}$ of size $< \theta_1$ or
   (iii) a σ-centered subposet of $\mathbb{D}$ of size $< \kappa$.

To be more specific, in (i) $\dot{\mathbb{Q}}_\alpha$ is of the form $\mathbb{LOC}^{\dot{N}_\alpha}$ where $\dot{N}_\alpha$ is a $\mathbb{P}_\alpha$-name of a transitive model of size $<\theta_0$ of a finite large enough fragment of ZFC, so that the slalom added by $\dot{\mathbb{Q}}_\alpha$ is generic over $\dot{N}_\alpha$. This similarly applies to (ii) and (iii). By a book-keeping device, the iteration satisfies for every $\alpha<\lambda$:

(i') if $\dot{K}$ is a $\mathbb{P}_\alpha$-name of a subset of ω^ω of size $<\theta_0$, then there is an $\alpha'\in[\alpha,\lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in (i) and the slalom it adds localizes all the reals in $\dot{K}$,

(ii') if $\dot{B}$ is a $\mathbb{P}_\alpha$-name of a family of size $<\theta_1$ of Borel-null sets (coded in $V^{\mathbb{H}_\kappa*\mathbb{P}_\alpha}$) then there is an $\alpha'\in[\alpha,\lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in (ii) and the random real it adds is outside the Borel sets in $\dot{B}$ and

(iii') if $\dot{F}$ is a $\mathbb{P}_\alpha$-name of a subset of ω^ω of size $<\kappa$, then there is an $\alpha'\in[\alpha,\lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in (iii) and the generic real it adds dominates the reals in $\dot{F}$.

For instance, in (i'), considering $\dot{\mathbb{Q}}_{\alpha'}=\mathbb{LOC}^{\dot{N}_{\alpha'}}$ as explained above, α' is found such that $\dot{K}\subseteq\dot{N}_{\alpha'}$, so the generic slalom localizes the reals in $\dot{K}$. This similarly applies to (ii') and (iii').

In order to finish the proof we present the arguments that show why each cardinal characteristic takes the desired value in the generic extension given by $\mathbb{P}_\lambda$.

$\underline{\mathrm{add}(\mathcal{N})=\theta_0}$. The inequality $\mathrm{add}(\mathcal{N})\leq\theta_0$ follows from both the fact that $\mathrm{add}(\mathcal{N})=\mathfrak{b}(\mathbf{Lc})$ (see Example 2.5(4)) and that all the posets we are using in the iteration are θ_0-$\mathbf{Lc}$-good, so Theorem 2.7 applies and we get $\mathfrak{b}(\mathbf{Lc})\leq\theta_0$. On the other hand, (i') implies $\mathrm{add}(\mathcal{N})\geq\theta_0$.

$\underline{\mathrm{cov}(\mathcal{N})=\theta_1}$. For $\mathrm{cov}(\mathcal{N})\leq\theta_1$ note that $\mathrm{cov}(\mathcal{N})\leq\mathfrak{b}(\mathbf{Ed}_b)$ (see Example 2.5,(3)). Thus, since the posets in the iteration are θ_1-$\mathbf{Ed}_b$-good, Theorem 2.7 applies to obtain $\mathfrak{b}(\mathbf{Ed}_b)\leq\theta_1$. Conversely, The inequality $\mathrm{cov}(\mathcal{N})\geq\theta_1$ follows from (ii').

$\underline{\mathfrak{b}=\mathfrak{a}=\mathrm{non}(\mathcal{M})=\kappa}$. It is enough to show $\kappa\leq\mathfrak{b}$, $\mathrm{non}(\mathcal{M})\leq\kappa$ and $\mathfrak{a}\leq\kappa$. For the latter note that the mad family added at the beginning by the forcing $\mathbb{H}_\kappa$ will stay mad after the iteration thanks to Theorem 4.17. Item (iii') implies $\mathfrak{b}\geq\kappa$ and $\mathrm{non}(\mathcal{M})\leq\kappa$ follows from both $\mathrm{non}(\mathcal{M})=\mathfrak{b}(\mathbf{Ed})$ and the fact that the posets in the iteration are κ-$\mathbf{Ed}$-good.

$\underline{\mathrm{cov}(\mathcal{M})=\mathfrak{c}=\lambda}$. The inequality $\mathrm{cov}(\mathcal{M})\geq\lambda$ is a simple consequence from the equality $\mathrm{cov}(\mathcal{M})=\mathfrak{d}(\mathbf{Ed})$ together with Theorem 2.7; on the other hand, $\mathfrak{c}\leq\lambda$ because, in the ground model, $|\mathbb{H}_\kappa*\mathbb{P}_\lambda|\leq\lambda$.

(b) Like in (a), perform a FS iteration $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\lambda\cdot\mu}$ as in [Mej13a, Thm. 3] where each $\dot{\mathbb{Q}}_\alpha$ is either

(i) a σ-linked subposet of $\mathbb{LOC}$ of size $<\theta_0$,

(ii) a subalgebra of $\mathbb{B}$ of size $<\theta_1$,

(iii) a σ-centered subposet of $\mathbb{D}$ of size $<\kappa$ or

(iv) $\mathbb{E}$.

By counting arguments, the FS iteration is constructed such that, for any $\alpha<\mu$,

(i') if $\dot{K}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\theta_0$, then there is a $\xi<\lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (i) and the slalom it adds localizes all the reals in $\dot{K}$,

(ii') if $\dot{B}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a family of size $<\theta_1$ of Borel-null sets (coded in $V^{\mathbb{H}_\kappa*\mathbb{P}_{\lambda\cdot\alpha}}$) then there is a $\xi<\lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (ii) and the random real it adds is outside the Borel sets in $\dot{B}$ and

(iii') if $\dot{F}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\kappa$, then there is a $\xi<\lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (iii) and the generic real it adds dominates the reals in $\dot{F}$.

The arguments for $\mathrm{add}(\mathcal{N})=\theta_0$, $\mathrm{cov}(\mathcal{N})=\theta_1$ and $\mathfrak{b}=\mathfrak{a}=\kappa$ are similar to the ones in (a). We present here the remaining ones.

$\underline{\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{M})=\mu}$. Both inequalities $\mathrm{cov}(\mathcal{M})\leq\mu$ and $\mathrm{non}(\mathcal{M})\geq\mu$ are witnessed by the cofinal μ-many eventually different reals added by $\mathbb{E}$ (note that the cofinality of the iteration is μ). On the other hand, $\mathrm{non}(\mathcal{M})\leq\mu\leq\mathrm{cov}(\mathcal{M})$ follows by the cofinal μ-many Cohen reals added at limit stages.

$\underline{\mathfrak{d}=\mathrm{non}(\mathcal{N})=\mathfrak{c}=\lambda}$. Follows from Theorem 2.7, just recall that $\mathrm{non}(\mathcal{N})\geq\mathfrak{d}(\mathbf{Ed}_b)$.

(c) Perform a FS iteration $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\lambda\cdot\mu}$ as in [Mej13a, Thm. 3]. In this case, each $\dot{\mathbb{Q}}_\alpha$ is either:
   (i) a σ-linked subposet of $\mathbb{LOC}$ of size $<\theta_0$,
   (ii) a σ-centered subposet of $\mathbb{D}$ of size $<\kappa$ or
   (iii) $\mathbb{B}$.

   As in (b), the iteration is build up (using counting arguments) so that (i') and (iii') from the previous proof hold.
(d) After the iteration in (a) force with $\mathbb{B}_{\lambda'}$.

□

Now we turn to prove some consistency results with standard 3D-coherent systems (recall Definitions 3.2(3) and 3.8). Recall that, if $\mathbf{t}$ is such a system with $I^{\mathbf{t}}=(\gamma+1)\times(\delta+1)$, it can be extracted standard 2D-coherent systems $\mathbf{t}_\alpha$ for each $\alpha\leq\gamma$ and $\mathbf{t}^\beta$ for each $\beta\leq\delta$. When referring to Figure 2, we call the vertical axis the α-*axis*, the axis pointing "perpendicular to the sheet of paper" is the β-*axis* and the horizontal axis is the ξ-*axis*. To get a picture of these 2D-systems, in Figure 2, $\mathbf{t}_\alpha$ is the 2D-system obtained by restricting the 3D rectangle to the horizontal plane on α (i.e., fixing α on the α-axis), while $\mathbf{t}^\beta$ is the restriction to the vertical plane on β (i.e., fixing β on the β-axis). These 2D-coherent systems allow us to directly apply the results in the previous sections to 3D-coherent systems. In consequence, we have the following general result for standard 3D-coherent systems.

Theorem 5.2. *Let $\mathbf{t}$ be a standard 3D-coherent system with $I^{\mathbf{t}}=(\gamma+1)\times(\delta+1)$ and $\mathbf{m}$ a standard 2D-coherent system with $I^{\mathbf{m}}=\gamma+1$ and $\pi^{\mathbf{m}}=\delta$ such that $\mathbb{P}_{\alpha,\beta,0}=\mathbb{P}^{\mathbf{t}}_{\alpha,\beta,0}=\mathbb{P}^{\mathbf{m}}_{\alpha,\beta}$ for all $\alpha\leq\gamma$ and $\beta\leq\delta$. Let $\mathbf{R}=\langle X,Y,\sqsubset\rangle$ be a Polish relational system coded in V. Assume*

(I) $\mathbf{m}$ satisfies the hypotheses of either
   (i) Lemma 3.7(i) and (ii) and Theorem 3.10 with $\langle\dot{c}_\alpha:\alpha<\gamma\rangle$ and $\mathbf{R}$, or
   (ii) Theorem 4.14 with $\dot{A}=\{\dot{a}_\alpha:\alpha<\gamma\}$
   (note that, in either case, γ has uncountable cofinality),

(II) all the posets that conform $\mathbf{m}$ are non-trivial (see Definition 3.8(iii) and (iv)),
(III) all the posets that conform $\mathbf{t}$ are non-trivial (see Definition 3.8(iii) and (iv)),
(IV) δ and π have uncountable cofinality,
(V) for $\xi\in S=S^{\mathbf{t}}$, $\dot{\mathbb{Q}}_{\alpha,\beta,\xi}$ is forced to be $\mathbf{R}$-good by $\mathbb{P}_{\alpha,\beta,\xi}$ for all $\alpha\leq\gamma$ and $\beta\leq\delta$, and
(VI) if (I)(ii) is assumed then $\mathbb{S}_\xi\in\{\mathbb{C},\mathbb{E}\}\cup\mathcal{R}$ for all $\xi\in S$.

Then, $\mathbb{P}_{\gamma,\delta,\pi}$ forces
(a) $\mathrm{non}(\mathcal{M})\leq\mathrm{cf}(\pi)\leq\mathrm{cov}(\mathcal{M})$,
(b) $\mathfrak{b}(\mathbf{R})\leq\min\{\mathrm{cf}(\delta),\mathrm{cf}(\pi)\}\leq\max\{\mathrm{cf}(\delta),\mathrm{cf}(\pi)\}\leq\mathfrak{d}(\mathbf{R})$,
(c) $\mathfrak{b}(\mathbf{R})\leq\min\{\mathrm{cf}(\gamma),\mathrm{cf}(\delta),\mathrm{cf}(\pi)\}\leq\max\{\mathrm{cf}(\gamma),\mathrm{cf}(\delta),\mathrm{cf}(\pi)\}\leq\mathfrak{d}(\mathbf{R})$ *<u>when</u> (I)(i) is assumed and*
(d) $\mathfrak{a}\leq\mathrm{cf}(\delta)$ *<u>when</u> (I)(ii) is assumed.*

Proof. (a) Any FS iteration of length π of uncountable cofinality adds cofinal $\mathrm{cf}(\pi)$-many Cohen reals which witness $\mathrm{non}(\mathcal{M}) \leq \mathrm{cf}(\pi) \leq \mathrm{cov}(\mathcal{M})$. Also note that the FS iteration $\langle \mathbb{P}_{\gamma,\delta,\xi}, \dot{\mathbb{Q}}_{\gamma,\delta,\xi} : \xi < \pi \rangle$ generates the final extension $V_{\gamma,\delta,\pi}$ of the coherent system $\mathbf{t}$.

(b) We look at the 2D-coherent system t_γ. As the chain of posets $\langle \mathbb{P}_{\gamma,\beta,0} : \beta \leq \delta \rangle$ is generated by a FS iteration of ccc posets, for a fixed cofinal sequence $\langle \beta_\zeta : \zeta < \mathrm{cf}(\delta) \rangle$ in δ of limit ordinals, for each $\zeta < \mathrm{cf}(\delta)$ there is a $\mathbb{P}_{\gamma,\beta_{\zeta+1},0}$-name $\dot{c}'_\zeta$ for a Cohen real over $V_{\gamma,\beta_\zeta,0}$. Thus, t_γ and $\langle \dot{c}'_\zeta : \zeta < \mathrm{cf}(\delta) \rangle$ satisfy the hypotheses of Theorem 3.10 by (V), so $\mathbb{P}_{\gamma,\delta,\pi}$ forces $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(\delta) \leq \mathfrak{d}(\mathbf{R})$. Besides, since $\mathfrak{b}(\mathbf{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathbf{R})$, (a) immediately implies $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(\pi) \leq \mathfrak{d}(\mathbf{R})$.

(c) We first look at the 2D-coherent system $\mathbf{m}$. By Theorem 3.10, $\mathbb{P}_{\alpha+1,\delta,0}$ forces that $\dot{c}_\alpha$ is $\mathbf{R}$-unbounded over $V_{\alpha,\delta,0}$ for every $\alpha < \gamma$. Now, we apply Theorem 3.10 to $\mathbf{t}^\delta$ to conclude that $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathbf{R})$.

(d) By Theorem 4.14 applied to the 2D-coherent system $\mathbf{m}$, each $\dot{a}_\alpha$ is forced by $\mathbb{P}_{\alpha+1,\delta,0}$ to diagonalize $V_{\alpha,\delta,0}$ outside $\dot{A}{\restriction}\alpha$ for each $\alpha < \gamma$ and furthermore, using the same theorem one more time for the coherent system $\mathbf{t}^\delta$, $\mathbb{P}_{\alpha+1,\delta,\pi}$ forces that $\dot{a}_\alpha$ diagonalizes $V_{\alpha,\delta,\pi}$ outside $\dot{A}{\restriction}\alpha$. Thus, the maximality of A is preserved in $V_{\gamma,\delta,\pi}$ and so $\mathfrak{a} \leq \mathrm{cf}(\gamma)$. □

In our applications and in accordance with the previous result, we consider standard 3D-coherent systems where $\langle \mathbb{P}_{\alpha,\beta,0} : \alpha \leq \gamma, \beta \leq \delta \rangle$ is generated by a standard 2D-coherent system.

Definition 5.3. Given ordinals γ and δ, define the following standard 2D-coherent systems.

(1) The system $\mathbf{m}^{\mathbb{C}}(\gamma,\delta)$ where
  - (i) $I^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)} = \gamma + 1$,
  - (ii) $\mathbb{P}^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)}_{\alpha,0} = \mathbb{C}_\alpha$ for each $\alpha \leq \gamma$, and
  - (iii) $\pi^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)} = \delta$, $S = \delta$, $C = \emptyset$ and $\mathbb{S}_\beta = \mathbb{C}$ for all $\beta < \delta$.

(2) The system $\mathbf{m}^*(\gamma,\delta)$ where
  - (i) $I^{\mathbf{m}^*(\gamma,\delta)} = \gamma + 1$,
  - (ii) $\mathbb{P}^{\mathbf{m}^*(\gamma,\delta)}_{\alpha,0} = \mathbb{H}_\alpha$ for each $\alpha \leq \gamma$, and
  - (iii) $\pi^{\mathbf{m}^*(\gamma,\delta)} = \delta$, $S = \delta$, $C = \emptyset$ and $\mathbb{S}_\beta = \mathbb{C}$ for all $\beta < \delta$.

If both γ and δ have uncountable cofinality, it is clear that both $\mathbf{m}^{\mathbb{C}}(\gamma,\delta)$ and $\mathbf{m}^*(\gamma,\delta)$ satisfy (I) and (II) of Theorem 5.2, moreover, the former satisfies (I)(i) and the latter satisfies (I)(ii). These standard 2D-coherent systems are the start point for the 3D-coherent systems constructed to prove the main results below.

Note that in Theorems 5.6(b), 5.7(c) and (d) we cannot say anything about $\mathfrak{a}$ because full Hechler generics are added (see the discussion about full and restricted generics after Definition 3.8) so mad families are not preserved anymore in the way proposed in Section 4. For these results we start with $\mathbf{m}^{\mathbb{C}}(\cdot,\cdot)$. For the results where we can force $\mathfrak{b} = \mathfrak{a}$ we start with $\mathbf{m}^*(\cdot,\cdot)$ (we can start with $\mathbf{m}^{\mathbb{C}}(\cdot,\cdot)$ as well, but $\mathfrak{a}$ should be ignored in that case). Observe that the results below are "three-dimensionalizations" of the 2D-coherent systems constructed in [Mej13a, Sect. 6].

We first prove that there is a constellation of Cichoń's diagram with 7 different values as illustrated in Figure 3.

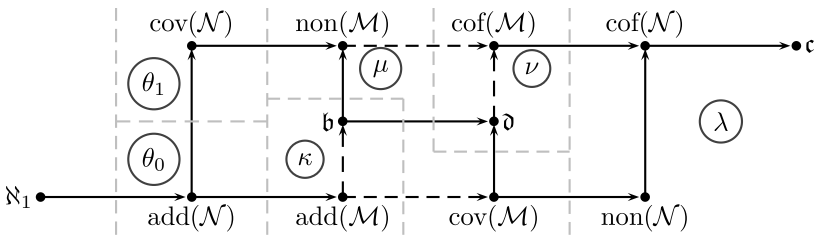

FIGURE 3. Cichoń's diagram as in Theorem 5.4.

Theorem 5.4. *Assume* $\lambda^{<\theta_1} = \lambda$. *Then, there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Proof. Let V be the ground model where we perform a FS iteration which comes from the standard 3D-coherent system $\mathbf{t}$ constructed as follows. Fix a bijection $g = \langle g_0, g_1, g_2\rangle : \lambda \to \kappa \times \nu \times \lambda$.

(1) $\gamma = \kappa + 1$, $\delta = \nu + 1$ and $\pi = \lambda \cdot \nu \cdot \mu$.
(2) $\langle \mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu\rangle$ is obtained from $\mathbf{m}^*(\kappa,\nu)$.
(3) Consider $\lambda \cdot \nu \cdot \mu$ as the disjoint union of the $\nu \cdot \mu$-many intervals $I_\zeta = [l_\zeta, l_{\zeta+1})$ (for $\zeta < \nu \cdot \mu$) of order type λ. Let $S := \{l_\zeta : \zeta < \nu\cdot\mu\}$ and $C = \pi \smallsetminus S$ (note that $l_\zeta = \lambda \cdot \zeta$).
(4) A function $\Delta = \langle \Delta_0, \Delta_1\rangle : C \to \kappa \times \nu$ such that the following properties are satisfied:
   (i) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals,[2]
   (ii) $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta + 1 : \zeta < \nu \cdot \mu\}$ is cofinal in π for any $(\alpha, \beta) \in \kappa \times \nu$, and
   (iii) for fixed $\zeta < \nu \cdot \mu$ and $e < 2$, $\Delta(l_\zeta + 2 + 2\cdot\varepsilon + e) = (g_0(\varepsilon)+1, g_1(\varepsilon)+1)$ for all $\varepsilon < \lambda$.
(5) $\dot{\mathbb{S}}_\xi = \mathbb{E}$ for all $\xi \in S$.
(6) Fix, for each $\alpha < \kappa$, $\beta < \nu$ and $\zeta < \nu\cdot\mu$, two sequences $\langle \dot{\mathbb{LOC}}^\zeta_{\alpha,\beta,\eta}\rangle_{\eta<\lambda}$ and $\langle \dot{\mathbb{B}}^\zeta_{\alpha,\beta,\eta}\rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of the localization forcing $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $< \theta_0$ and all subalgebras of random forcing $\mathbb{B}^{V_{\alpha,\beta,l_\zeta}}$ of size $< \theta_1$, respectively.
   Given $\xi \in C$, define $\dot{\mathbb{Q}}_\xi$ according to the following cases.
   (i) If $\xi = l_\zeta + 1$ then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$, the Hechler poset adding a dominating real $\dot{d}_\zeta$ over the model $V_{\Delta(\xi),\xi}$.
   (ii) If $\xi = l_\zeta + 2 + 2\varepsilon$ then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}^\zeta_{g(\varepsilon)}$.
   (iii) If $\xi = l_\zeta + 2 + 2\varepsilon + 1$ then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}^\zeta_{g(\varepsilon)}$.

We prove that $V_{\kappa,\nu,\pi}$ satisfies the statements of this theorem.

Claim 5.5. *If* $X \in V_{\kappa,\nu,\pi}$ *is a set of reals of size* $< \mu$, *then there are* $(\beta, \zeta) \in \nu \times (\nu\cdot\mu)$ *so that* $X \in V_{\kappa,\beta,l_\zeta}$. *Furthermore, if* $|X| < \kappa$, *then there is also an* $\alpha < \kappa$ *such that* $X \in V_{\alpha,\beta,l_\zeta}$.

Proof. As $\mathrm{cf}(\pi) = \mu$ and $V_{\kappa,\nu,\pi}$ is obtained by a FS iteration of length π, there is a $\zeta < \nu\cdot\mu$ such that $X \in V_{\kappa,\nu,l_\zeta}$ (because $\{l_\zeta : \zeta < \nu\cdot\mu\}$ is cofinal in π). Now, look at the 2D-coherent

[2]Both ordinals $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor because, if they are limits of uncountable cofinality and we force with $\mathbb{D}^V_{\Delta(\xi),\xi}$ above $(\Delta(\xi), \xi)$ and trivial otherwise, then $\mathbb{R} \cap V_{\Delta(\xi),\xi+1}$ may not be $\mathbb{R} \cap \bigcup_{\alpha<\Delta_0(\xi),\beta<\Delta_1(\xi)} V_{\alpha,\beta,\xi+1}$.

system $\mathbf{t}_\kappa$ and apply Corollary 3.9 to find a $\beta<\nu$ so that $X\in V_{\kappa,\beta,l_\zeta}$. In the case that $|X|<\kappa$, apply Corollary 3.9 to $\mathbf{t}^\beta$ to find an $\alpha<\kappa$ so that X belongs to V_{α,β,l_ζ}. $\square$

$\underline{\mathrm{add}(\mathcal{N})=\theta_0}$. For the inequality $\mathrm{add}(\mathcal{N})\geq\theta_0$ take an arbitrary set X of reals in $V_{\kappa,\nu,\pi}$ of size $<\theta_0$ so, by Claim 5.5, there is a triple of ordinals $(\alpha,\beta,\zeta)\in\kappa\times\nu\times(\nu\cdot\mu)$ such that $X\in V_{\alpha,\beta,l_\zeta}$. In V_{α,β,l_ζ}, there is a transitive model N of (a large enough finite fragment of) ZFC such that $X\subseteq N$ and $|N|<\theta_0$. Then, there exists an $\eta<\lambda$ such that $\mathbb{LOC}^\zeta_{\alpha,\beta,\eta}=\mathbb{LOC}^N$. Put $\varepsilon=g^{-1}(\alpha,\beta,\eta)$ and $\xi'=l_\zeta+2+2\varepsilon$, so $\mathbb{Q}_{\xi'}=\mathbb{LOC}^\zeta_{\alpha,\beta,\eta}=\mathbb{LOC}^N$ adds a generic slalom over N and, therefore, it localizes all the reals in X.

To obtain the converse inequality, apply Theorem 2.7 to $\langle\mathbb{P}_{\kappa,\nu,\xi},\dot{\mathbb{Q}}_{\kappa,\nu,\xi}\rangle_{\xi<\pi}$.

$\underline{\mathrm{cov}(\mathcal{N})=\theta_1}$. This case is similar to the one above. To get $\mathrm{cov}(\mathcal{N})\geq\theta_1$ take an arbitrary family Z of Borel null sets coded in $V_{\kappa,\nu,\pi}$ of size $<\theta_1$ so, by Claim 5.5, there exists $(\alpha,\beta,\zeta)\in\kappa\times\nu\times(\nu\cdot\mu)$ such that the sets in Z are already coded in V_{α,β,l_ζ}. Hence, as in the previous argument, there exists an ordinal $\eta<\lambda$ such that the generic random real added by $\mathbb{B}^\zeta_{\alpha,\beta,\eta}$ avoids all the Borel sets in Z. Put $\varepsilon=g^{-1}(\alpha,\beta,\eta)$ and $\xi'=l_\zeta+2+2\varepsilon+1$, so $\mathbb{Q}_{\xi'}=\mathbb{B}^\zeta_{\alpha,\beta,\eta}$ and the random real it adds is already in $V_{\alpha+1,\beta+1,\xi'+1}$.

Conversely, since the posets we use in the FS iteration $\langle\mathbb{P}_{\kappa,\nu,\xi},\dot{\mathbb{Q}}_{\kappa,\nu,\xi}\rangle_{\xi<\pi}$ are θ_1-$\mathbf{Ed}_b$-good posets and $\mathrm{cov}(\mathcal{N})\leq\mathfrak{b}(\mathbf{Ed}_b)$, Theorem 2.7 implies that, in $V_{\kappa,\nu,\pi}$, $\mathfrak{b}(\mathbf{Ed}_b)\leq\theta_1$.

$\underline{\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{M})=\mu}$. The inequalities $\mathrm{non}(\mathcal{M})\leq\mu\leq\mathrm{cov}(\mathcal{M})$ follow from Theorem 5.2(a). Conversely, from the cofinal μ-many eventually different reals added by the iteration $\langle\mathbb{P}_{\kappa,\nu,\xi},\dot{\mathbb{Q}}_{\kappa,\nu,\xi}\rangle_{\xi<\pi}$, we force the inequalities $\mathrm{cov}(\mathcal{M})\leq\mu$ and $\mathrm{non}(\mathcal{M})\geq\mu$.

$\underline{\mathrm{add}(\mathcal{M})=\mathfrak{b}=\mathfrak{a}=\kappa}$. Given a family F of reals in $V_{\kappa,\nu,\pi}$ of size $<\kappa$, we can find a $(\alpha,\beta,\zeta)\in\kappa\times\nu\times(\nu\cdot\mu)$ such that $F\in V_{\alpha,\beta,l_\zeta}$. We use now the restricted dominating reals $\{\dot{d}_\zeta:\zeta<\nu\cdot\mu\}$. Since $(\Delta)^{-1}(\alpha+1,\beta+1)\cap\{l_\zeta+1:\zeta<\nu\cdot\mu\}$ is cofinal in $\nu\cdot\mu$, there exists a $\zeta'\in[\zeta,\nu\cdot\mu)$ such that $\Delta(l_{\zeta'}+1)=(\alpha+1,\beta+1)$ and then the real $\dot{d}_{\zeta'}$ added by $\mathbb{Q}_{\alpha+1,\beta+1,\xi'}$, where $\xi'=l_{\zeta'}+1$, dominates all the reals in F.

On the other hand, $\mathfrak{a}\leq\kappa$ follows from Theorem 5.2 which guarantees that the mad family added along the α-axis, which lives in the model $V_{\kappa,0,0}$, still remains mad in the final extension $V_{\kappa,\nu,\pi}$.

$\underline{\mathfrak{d}=\mathrm{cof}(\mathcal{M})=\nu}$. For $V_{\kappa,\nu,\pi}\models\mathfrak{d}\geq\nu$ we just use Theorem 5.2. Conversely, to see $V^{\mathbb{P}}\models\mathfrak{d}\leq\nu$ note that the argument above shows that the family of (restricted) dominating reals $\{\dot{d}_\zeta:\zeta<\nu\cdot\mu\}$ is dominating in $V_{\kappa,\nu,\pi}$.

$\underline{\mathrm{non}(\mathcal{N})=\mathrm{cof}(\mathcal{N})=\mathfrak{c}=\lambda}$. As $\mathfrak{d}(\mathbf{Ed}_b)\leq\mathrm{non}(\mathcal{N})$, from Theorem 2.7 we have that, in $V_{\kappa,\nu,\pi}$, $\mathfrak{d}(\mathbf{Ed}_b)\geq|\pi|=\lambda$. Certainly, $\mathfrak{c}\leq\lambda$ holds because $|\mathbb{P}_{\kappa,\nu,\pi}|=\lambda$.

$\square$

Theorem 5.6. *Assume $\lambda^{<\theta_0}=\lambda$. Then, for any of the statements below, there is a ccc poset forcing it.*

(a) $\mathrm{add}(\mathcal{N})=\theta_0$, $\mathfrak{b}=\mathfrak{a}=\kappa$, $\mathrm{cov}(\mathcal{I})=\mathrm{non}(\mathcal{I})=\mu$ *for* $\mathcal{I}\in\{\mathcal{M},\mathcal{N}\}$, $\mathfrak{d}=\nu$ *and* $\mathrm{cof}(\mathcal{N})=\mathfrak{c}=\lambda$.

(b) $\mathrm{add}(\mathcal{N})=\theta_0$, $\mathrm{cov}(\mathcal{N})=\kappa$, $\mathrm{add}(\mathcal{M})=\mathrm{cof}(\mathcal{M})=\mu$, $\mathrm{non}(\mathcal{N})=\nu$ *and* $\mathrm{cof}(\mathcal{N})=\mathfrak{c}=\lambda$.

(c) $\mathrm{add}(\mathcal{N})=\theta_0$, $\mathrm{cov}(\mathcal{N})=\mathfrak{b}=\mathfrak{a}=\kappa$, $\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{M})=\mu$, $\mathfrak{d}=\mathrm{non}(\mathcal{N})=\nu$ *and* $\mathrm{cof}(\mathcal{N})=\mathfrak{c}=\lambda$.

Proof. Fix bijection $g:\lambda\to\kappa\times\nu\times\lambda$. All the 3D-coherent systems we use in this proof are of the form $\mathbf{t}$ where

(1) $\gamma=\kappa+1$, $\delta=\nu+1$ and $\pi=\lambda\cdot\nu\cdot\mu$, the latter which is the disjoint union of $\nu\cdot\mu$-many intervals $\{I_\zeta:=[l_\zeta,l_{\zeta+1}):\zeta<\nu\cdot\mu\}$ of length λ where each $l_\zeta:=\lambda\cdot\zeta$.

(2) $S = \{l_\zeta : \zeta < \nu \cdot \mu\}$ and $C = \pi \smallsetminus S$.
(3) For (a) and (c) $\langle \mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu\rangle$ comes from $\mathbf{m}^*(\gamma, \delta)$ and, for (b), it comes from from $\mathbf{m}^{\mathbb{C}}(\kappa, \nu)$.
(4) A function $\Delta = \langle \Delta_0, \Delta_1\rangle : C \to \kappa \times \nu$ such that the following properties are satisfied:
   (i) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals,
   (ii) $\Delta^{-1}(\alpha + 1, \beta + 1) \cap \{l_\eta + 1 : \eta < \nu \cdot \mu\}$ is cofinal in π for each $(\alpha, \beta) \in \kappa \times \nu$; additionally, for (c), $\Delta^{-1}(\alpha + 1, \beta + 1) \cap \{l_\eta + 2 : \eta < \nu \cdot \mu\}$ is cofinal in π and
   (iii) for fixed $\zeta < \nu \cdot \mu$, $\Delta(l_\zeta + n_0 + \varepsilon) = (g_0(\varepsilon) + 1, g_1(\varepsilon) + 1)$ for all $\varepsilon < \lambda$, where $n_0 = 2$ for (a) and (b), and $n_0 = 3$ for (c).

For each of the item below, $\mathbf{t}$ is properly defined.

(a) For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{B}$. Fix, for each $\alpha < \kappa$, $\beta < \nu$ and $\zeta < \nu \cdot \mu$, a sequence $\langle \dot{\mathbb{LOC}}^\zeta_{\alpha,\beta,\eta}\rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $< \theta_0$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according to the following cases.
   (i) If $\xi = l_\zeta + 1$ then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$ which adds a dominating real $\dot{d}_\zeta$ over $V_{\Delta(\xi),\xi}$.
   (ii) If $\xi = l_\zeta + 2 + \varepsilon$ for some $\varepsilon < \lambda$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}^\zeta_{g(\varepsilon)}$.

   Most of the arguments for each of the cardinals characteristics are identical as the ones presented in Theorem 5.4, so we just present the missing ones.

   $\underline{\mathrm{non}(\mathcal{N}) \leq \mu \leq \mathrm{cov}(\mathcal{N})}$. It holds because we add cofinal μ-many random reals (corresponding to the coordinates $\xi \in S$).

   $\underline{\mathrm{cof}(\mathcal{N}) \geq \lambda}$. It is a consequence of both the fact that $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\mathbf{Lc})$ and Theorem 2.7 which gives us $\mathfrak{d}(\mathbf{Lc}) \geq |\pi| = \lambda$.

(b) For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{D}$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined as in (a) but, in (i), we consider $\mathbb{B}^{V_{\Delta(\xi),\xi}}$ instead.

   Recall that, in this construction, our base 2D-coherent system comes from $\mathbf{m}^{\mathbb{C}}(\kappa, \nu)$. The argument to prove that $V_{\kappa,\nu,\pi}$ satisfy (b) is similar to (a) and to the proof of Theorem 5.4. For instance,

   $\underline{\mathrm{cov}(\mathcal{N}) = \kappa \text{ and } \mathrm{non}(\mathcal{N}) = \nu}$. Given a family X of Borel-null sets coded in $V^{\mathbb{P}}$ of size $< \kappa$, we can find $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that all the sets in X are already coded in V_{α,β,l_ζ}. Since $\Delta^{-1}(\alpha + 1, \beta + 1) \cap \{l_\zeta + 1 : \zeta < \nu \cdot \mu\}$ is cofinal in $\nu \cdot \mu$, there exists $\zeta' \in [\zeta, \lambda)$ such that $\Delta(l_{\zeta'} + 1) = (\alpha + 1, \beta + 1)$ and then the random real $\dot{r}_{\zeta'}$ added by $\dot{\mathbb{Q}}_{\alpha,\beta,\xi'}$ with $\xi' = l_{\zeta'} + 1$ avoids all the sets in X. Note that this same argument also proves that the set $\{\dot{r}_\zeta : \zeta < \nu \cdot \mu\}$ is not null, so $\mathrm{non}(\mathcal{N}) \leq \nu$.

   Conversely, $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathbf{Ed}_b) \leq \kappa$ and $\nu \leq \mathfrak{d}(\mathbf{Ed}_b) \leq \mathrm{non}(\mathcal{N})$ are direct consequences of Theorem 5.2.

   $\underline{\mathfrak{b} = \mathfrak{d} = \mu}$. Because the cofinal μ-many dominated reals added by $\langle \mathbb{P}_{\kappa,\nu,\xi}, \dot{\mathbb{Q}}_{\kappa,\nu,\xi}\rangle_{\xi<\pi}$ forms a scale of length μ.

(c) For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{E}$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according to the following cases
   (i) If $\xi = l_\zeta + 1$, then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$.
   (ii) If $\xi = l_\zeta + 2$, then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{B}^{V_{\Delta(\xi),\xi}}$.
   (iii) Otherwise, like (ii) of the proof of (a).

□

Theorem 5.7. *Assume $\lambda^{\aleph_0} = \lambda$. Then, for any of the statements below there is a ccc poset forcing it.*

(a) $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
(b) $\mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, $\mathfrak{d} = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
(c) $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \kappa$, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$, $\mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
(d) $\mathrm{add}(\mathcal{N}) = \kappa$, $\mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mu$, $\mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.

Proof. The 3D-coherent systems we use in this proof are of the form $\mathbf{t}$ where

(1) $\gamma = \kappa+1$, $\delta = \nu+1$ and $\pi = \lambda\cdot\nu\cdot\mu$ is a disjoint union of $\{I_\xi = [l_\zeta, l_{\zeta+1}) : \zeta < \nu\cdot\mu\}$ as in Theorem 5.4.
(2) $C = \{l_\zeta : \zeta < \nu\cdot\mu\}$ and $S = \pi \smallsetminus C$.
(3) For items (a) and (b) $\langle \mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu\rangle$ comes from $\mathbf{m}^*(\gamma,\delta)$; for (c) and (d), it comes from $\mathbf{m}^{\mathbb{C}}(\kappa,\nu)$.
(4) A function $\Delta = \langle \Delta_0, \Delta_1\rangle : C \to \kappa \times \nu$ such that the following properties are satisfied:
  (i) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals and
  (ii) $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta : \zeta < \nu\cdot\mu\}$ is cofinal in π.

(a) Put $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi = \mathbb{LOC}^{V_{\Delta(\xi),\xi}}$.
We just prove $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \kappa$ and $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \nu$. If X is a set of reals in $V_{\kappa,\nu,\pi}$ of size $< \kappa$, there is a $(\alpha,\beta,\zeta) \in \kappa\times\nu\times(\nu\cdot\mu)$ such that $X \in V_{\alpha,\beta,l_\zeta}$. Since $\Delta^{-1}(\alpha+1,\beta+1) \cap \{l_\zeta : \zeta < \mu\}$ is cofinal in $\nu\cdot\mu$, there exists a $\zeta' \in [\zeta, \lambda)$ such that $\Delta(l_{\zeta'}) = (\alpha+1, \beta+1)$ and then the slalom $\dot{\varphi}_{\zeta'}$ added by $\dot{\mathbb{Q}}_{\alpha,\beta,l_{\zeta'}}$ localizes all the reals in X. Note that $\{\dot{\varphi}_\zeta : \zeta < \nu\cdot\mu\}$ witnesses $\mathrm{cof}(\mathcal{N}) \leq \nu$.
The inequalities $\mathfrak{b}, \mathrm{cov}(\mathcal{N}) \leq \kappa$ and $\nu \leq \mathfrak{d}, \mathrm{non}(\mathcal{N})$ follow directly from Theorem 5.2.
(b) Put $\mathbb{S}_\xi = \mathbb{B}$ for all $\xi \in S$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is as in (a).
(c) Put $\mathbb{S}_\xi = \mathbb{D}$ for all $\xi \in S$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is as in (a)
(d) For $\xi \in S$, if it is odd then $\mathbb{S}_\xi = \mathbb{D}$, but when it is even then $\mathbb{S}_{\xi+1} = \mathbb{B}$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined as in (a).

□

We present some other models of constellations of the Cichoń diagram known from [Mej13a], where additionally $\mathfrak{b} = \mathfrak{a}$ holds.

Theorem 5.8. *(a) If* $\lambda^{<\theta_1} = \lambda$ *then there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \kappa$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.
(b) If $\lambda^{<\theta_0} = \lambda$ *then there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \kappa$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.
(c) If $\lambda^{\aleph_0} = \lambda$ *then there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.

Proof. For (a) use the construction in [Mej13a, Thm. 20], for (b) see [Mej13a, Thm. 16] and for (c) see [Mej13a, Thm. 11] but, for the 2D-coherent systems, obtain the first column by forcing with $\mathbb{H}_\kappa$ instead. □

Remark 5.9. In Theorems 5.1, 5.4, 5.6 and 5.8(a) and (b) we can slightly modify the constructions to force, additionally, $\mathrm{MA}_{<\theta_0}$. For instance, in (6) of the proof of Theorem 5.4 we use, instead of $\langle \mathbb{LOC}^\zeta_{\alpha,\beta,\eta}\rangle_{\eta<\lambda}$, an enumeration $\langle \dot{\mathbb{Q}}^\zeta_{\alpha,\beta,\eta}\rangle_{\eta<\lambda}$ of all the (nice) $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all the ccc posets with domain an ordinal $< \theta_0$. In (6)(ii), $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{Q}}^\zeta_{g(\epsilon)}$ whenever

$\mathbb{P}_{\kappa,\nu,\xi}$ forces $\dot{\mathbb{Q}}^{\zeta}_{g(\epsilon)}$ to be ccc, otherwise, $\dot{\mathbb{Q}}_{\xi}$ is just a name for the trivial poset. In a similar way, we can additionally force $\mathrm{MA}_{<\kappa}$ in Theorems 5.7 and 5.8(c).

6. Δ^1_3 WELL-ORDERS OF THE REALS

There has been significant interest towards the study of possible constellations among the classical cardinal characteristics of the continuum in the presence of projective, in fact Δ^1_3-definable well-order of the reals (see [FF10, FFZ11, FFK14]). Answering a question of [FFK14], we show that each of the constellations described in the previous section is consistent with the existence of such projective well-order. Since the proofs are very similar, we will only outline the proof of the following theorem.

Theorem 6.1. *In L, let $\theta_0 < \theta_1 < \kappa < \mu < \nu < \lambda$ be uncountable regular cardinals and, in addition, $\lambda < \aleph_\omega$. Then there is a cardinals preserving forcing extension of the constructible universe, L, in which there is a Δ^1_3 well-order of the reals and in addition* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

In order to prove the above theorem, we will use the method of almost disjoint coding as it is developed in [FFZ11]. The forcing construction can be viewed as a two stage construction: a preliminary stage in which we prepare the universe, followed by a coding stage in which we will not only add the Δ^1_3-definition, but also provide the desired constellations of the cardinal characteristics. We will work over the constructible universe L. The preparatory stage of the construction is very similar to the preparatory stage of the construction in [FFZ11], with the sole difference that now the size of the continuum in the desired generic extension is λ (rather than ω_3 as in the construction from [FFZ11]). The prepared universe (a generic extension of L) will be denoted V. The coding stage will be a 3D-iterated forcing construction which is a modification of the construction providing Theorem 5.4, modification which allows to adjoin the desired Δ^1_3-definition.

Now we turn to the preliminary stage of the construction. As most of the ideas (and proofs) are identical to the ones in [FFZ11] we will only give a brief outline. For convenience we fix natural numbers $n_1 < \cdots < n_6$ such that $\theta_0 = \omega_{n_1}$, $\theta_1 = \omega_{n_2}$, $\kappa = \omega_{n_3}$, $\mu = \omega_{n_4}$, $\nu = \omega_{n_5}$, $\lambda = \omega_{n_6}$. Let $\pi = \lambda \cdot \nu \cdot \mu$ and let $f : \pi \to \lambda$ be a canonical bijection. For each $\alpha < \pi$, let W_α be the L-least subset of ω_{n_6-1} coding $f(\alpha)$. In the following, we will refer to W_α as the *L-least code of α modulo f*, or simply the *L-least code of α*. We will say that a transitive ZF^- model $\mathcal{M}$ is *suitable* if $\omega^{\mathcal{M}}_{n_6}$ exists and $\omega^{\mathcal{M}}_{n_6} = \omega^{L^{\mathcal{M}}}_{n_6}$ (here ZF^- denotes ZF^- minus the power set axiom). We start with a nicely definable sequence $\bar{S} = \langle S_\alpha : \alpha < \pi \rangle$ of stationary, co-stationary subsets of ω_{n_6-1}. Using bounded approximations, for each $\alpha < \pi$, we add a closed unbounded subset C_α of ω_{n_6-1} which is disjoint from S_α. Our intended Δ^1_3-definition will depend on the existence of reals which code this stationary kill of some of the S_α's. Following the notation of [FFZ11], for a set of ordinals X, $\mathrm{Even}(X)$ denotes the subset of all even ordinals in X. Now, reproducing the ideas of [FFZ11], we can find subsets $Z_\alpha \subseteq \omega_{n_6-1}$ such that

$(*)_\alpha$: If $\beta < \omega_{n_6-1}$ and $\mathcal{M}$ is a suitable model such that $\omega_{n_6-2} \subseteq \mathcal{M}$, $\omega^{\mathcal{M}}_{n_6-1} = \beta$, $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \psi(\omega_{n_6-1}, Z_\alpha \cap \beta)$, where $\psi(\omega_{n_6-1}, X)$ is the formula "$\mathrm{Even}(X)$ codes a triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$ where $\bar{W}, \bar{\bar{W}}$ are the L-least codes modulo f of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \pi$ respectively such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_{n_6-1} disjoint from $S_{\bar{\bar{\alpha}}}$".

For each $m = 1, \cdots, n_6-2$, let $\bar{S}^m = \langle S^m_\alpha : \alpha < \omega_{n_6-m}\rangle$ be a nicely definable in L_{n_6-m-1} sequence of almost disjoint subsets of ω_{n_6-m-1}. Successively using almost disjoint coding with respect to the sequences $\bar{S}^m$ (see [FFZ11]), we can code the sets Z_α into subsets X_α of ω_1 with the following property.

$(**)_\alpha$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model with $\omega_2^{\mathcal{M}} = \beta$, $\{X_\alpha\} \cup \omega_1 \subseteq \mathcal{M}$, then $\mathcal{M} \vDash \varphi(\omega_{n_6-1}, X_\alpha)$, where $\varphi(\omega_{n_6-1}, X)$ is the formula: "Using the sequences $\{\bar{S}^m\}_{m=1}^{m=n_6-1}$, the set X almost disjointly codes a subset Z of ω_{n_6-1} whose even part codes the triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$ where $\bar{W}$, $\bar{\bar{W}}$ are the L-least codes modulo f of ordinals $\bar{\alpha}$, $\bar{\bar{\alpha}} < \pi$, respectively, such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_{n_6-1} disjoint from $S_{\bar{\alpha}}$".

Finally, using the posets $\mathcal{L}(X_{\alpha+m}, X_\alpha)$ for $\alpha \in \mathrm{Lim}(\pi)$ (for a set of ordinals C, $\mathrm{Lim}(C)$ denotes the set of limit ordinals in C), $m \in \omega$ from [FFZ11, Definition 1], we can add the characteristics functions of subsets $Y_{\alpha+m}$ of ω_1 such that:

$(***)$: If $\beta < \omega_1$, $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \beta$, $Y_{\alpha+m} \cap \beta \in \mathcal{M}$, then $\mathcal{M} \vDash \varphi(\omega_{n_6-1}, X_{\alpha+m} \cap \beta) \wedge \varphi(\omega_{n_6-1}, X_\alpha \cap \beta)$.

With this, the preliminary stage of the construction is complete. We denote by $\mathbb{P}_0$ the finite iteration of forcing notions described above. Note that the poset $\mathbb{P}_0$ is ω-distributive (the proof is almost identical to [FFZ11, Lemma 1]) so in particular $\mathbb{P}_0$ does not add new reals. Let $V = L^{\mathbb{P}_0}$ and let $\bar{\mathcal{B}} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega\rangle \in L$ be a nicely definable sequence of almost disjoint subsets of ω. As in the proof of Theorem 5.4 partition π into intervals $I_\zeta = [l_\zeta, l_{\zeta+1})$ for $\zeta < \nu \cdot \mu$, where $l_\zeta = \lambda \cdot \zeta$, and let

$$C_0 = \{2 \cdot \zeta' + 1 : \zeta' < \nu \cdot \mu\}.$$

Furthermore let $C_0^* = \bigcup\{[l_\zeta, l_{\zeta+1}) : \zeta \in C_0\}$, let $S^* = \{l_\zeta : \zeta \in \nu \cdot \mu \smallsetminus C_0\}$ and let $C_1^* = \pi \smallsetminus (S^* \cup \mathrm{Lim}(C_0^*))$.

Modifying the 3D-coherent system from the proof of Theorem 5.4, we will define in $V = L^{\mathbb{P}_0}$ a standard 3D-coherent system $\mathbf{t}^*$ where $\gamma^{\mathbf{t}^*} = \kappa + 1$, $\delta^{\mathbf{t}^*} = \nu + 1$, $\pi^{\mathbf{t}^*} = \pi$, $S^{\mathbf{t}^*} := S^*$, $C^{\mathbf{t}^*} = C^* = \pi \smallsetminus S^*$. The sole difference between $\mathbf{t}$ of Theorem 5.4 and $\mathbf{t}^*$ is the ξ-th step of the FS iterations conforming $\mathbf{t}^*$ for $\xi \in \mathrm{Lim}(C_0^*)$. For notational simplicity, $\mathbb{P}^*_{\alpha,\beta,\xi} = \mathbb{P}^{\mathbf{t}^*}_{\alpha,\beta,\xi}$, $\dot{\mathbb{Q}}^*_{\alpha,\beta,\xi} = \dot{\mathbb{Q}}^{\mathbf{t}^*}_{\alpha,\beta,\xi}$, $V^*_{\alpha,\beta,\xi} = V^{\mathbf{t}^*}_{\alpha,\beta,\xi}$, $\Delta^* = \Delta^{\mathbf{t}^*} : \pi \to \kappa \times \nu$ and so on, while without the asterisk we refer to the components of $\mathbf{t}$, that is, $\mathbb{P}_{\alpha,\beta,\xi} = \mathbb{P}^{\mathbf{t}}_{\alpha,\beta,\xi}$ and so on.

The starting point at $\xi = 0$ for $\mathbf{t}^*$ is the same as for $\mathbf{t}$, that is, $\mathbb{P}^*_{\alpha,\beta,0} = \mathbb{P}_{\alpha,\beta,0}$ for all $\alpha \leq \kappa$ and $\beta \leq \nu$. The tasks achieved by the posets $\dot{\mathbb{Q}}_{\alpha,\beta,\xi}$ for $\xi \in S$ (in the notation of the proof of Theorem 5.4) can be achieved by the corresponding posets $\dot{\mathbb{Q}}^*_{\alpha,\beta,\xi}$ in our modified construction for $\xi \in S^*$, and similarly the tasks achieved by the posets $\dot{\mathbb{Q}}_{\alpha,\beta,\xi}$ for $\xi \in C$ can be accomplished by the posets $\dot{\mathbb{Q}}^*_{\alpha,\beta,\xi}$ for $\xi \in C_1^*$. Thus, in order to complete the proof of Theorem 6.1, we are left with describing the ξ-th step for $\xi \in \mathrm{Lim}(C_0^*)$ of this modified construction. This modified 3D-construction will have the property that for $\alpha^* \leq \kappa$, $\beta^* \leq \nu$ and $\xi^* \leq \pi$, if $G_{\alpha^*,\beta^*,\xi^*}$ is a $\mathbb{P}_0 * \mathbb{P}^*_{\alpha^*,\beta^*,\xi^*}$-generic filter over L then

$$\begin{aligned} L[G_{\alpha^*,\beta^*,\xi^*}] \cap \mathbb{R} = L[\{\dot{a}_\alpha[G_{\alpha,0,0}] : \alpha < \alpha^*\} \cup \{\dot{c}_\beta[G_{0,\beta,0}] : \beta < \beta^*\} \\ \cup \{\dot{u}_{\alpha^*,\beta^*,\xi}[G_{\alpha^*,\beta^*,\xi}] : \xi < \xi^*\}] \cap \mathbb{R}, \end{aligned}$$

where $\{\dot{a}_\alpha : \alpha < \kappa\}$ is (the set of names of) the mad family added by $\mathbb{H}_\kappa$, $\dot{c}_\beta$ is the Cohen real added by $\mathbb{P}_{\alpha,\beta+1,0}$ (which does not depend on α) and $\dot{u}_{\alpha,\beta,\xi}$ is a $\mathbb{P}_0 * \mathbb{P}^*_{\alpha,\beta,\xi+1}$-name for the generic real added by $\dot{\mathbb{Q}}_{\alpha,\beta,\xi}$. Note that, for $\xi \in S^*$, $\mathbb{P}_0 * \mathbb{P}^*_{\alpha,\beta,\xi}$ forces $\dot{u}_{\alpha,\beta,\xi} = \dot{u}_{0,0,\xi}$ and, for $\xi \in C^*$, if $\alpha \geq \Delta^*_0(\xi)$ and $\beta \geq \Delta^*_1(\xi)$ then $\mathbb{P}_0 * \mathbb{P}^*_{\alpha,\beta,\xi}$ forces $\dot{u}_{\alpha,\beta,\xi} = \dot{u}_{\Delta^*(\xi),\xi}$, otherwise, $\dot{u}_{\alpha,\beta,\xi}$ is just forced to be $\emptyset$. Thus, we only need to look at $\dot{u}_\xi := \dot{u}_{0,0,\xi}$ when $\xi \in S^*$ and to $\dot{u}_\xi := \dot{u}_{\Delta^*(\xi),\xi}$ when $\xi \in C^*$.

By recursion on $\alpha^* \leq \kappa$, $\mathbb{P}_0 * \mathbb{P}_{\alpha^*,0,0}$ forces that there is a well-order of the reals $\dot{<}_{\alpha^*,0,0}$ which depends only on $\{\dot{a}_\alpha : \alpha < \alpha^*\}$ such that it has $\dot{<}_{\alpha,0,0}$ as an initial segment for every $\alpha < \alpha^*$; by recursion on $\beta^* \leq \nu$, for every $\alpha^* \leq \kappa$, $\mathbb{P}_0 * \mathbb{P}_{\alpha^*,\beta^*,0}$ forces that there is a well-order of the reals $\dot{<}_{\alpha^*,\beta^*,0}$ which depends only on $\{\dot{a}_\alpha : \alpha < \alpha^*\} \cup \{\dot{c}_\beta : \beta < \beta^*\}$ such that it has $\dot{<}_{\alpha^*,\beta,0}$ as an initial segment for every $\beta < \beta^*$ and it contains $\dot{<}_{\alpha,\beta^*,0}$ (not necessarily as an initial segment) for every $\alpha < \alpha^*$; and by recursion on $\xi^* \leq \pi$, for all $\alpha^* \leq \kappa$ and $\beta^* \leq \nu$, $\mathbb{P}_0 * \mathbb{P}_{\alpha^*,\beta^*,\xi^*}$ forces that there is a well-order of the reals $\dot{<}_{\alpha^*,\beta^*,\xi^*}$ depending only on $\{\dot{a}_\alpha : \alpha < \alpha^*\} \cup \{\dot{c}_\beta : \beta < \beta^*\} \cup \{\dot{u}_{\alpha^*,\beta^*,\xi} : \xi < \xi^*\}$ so that it has $\dot{<}_{\alpha^*,\beta^*,\xi}$ as an initial segment for all $\xi < \xi^*$ and contains $\dot{<}_{\alpha,\beta,\xi^*}$ (not necessarily as an initial segment) for every $\alpha \leq \alpha^*$ and $\beta \leq \beta^*$. We denote $\dot{<}_{\xi^*} = \dot{<}_{\kappa,\nu,\xi^*}$. Therefore, $\mathbb{P}_0 * \mathbb{P}_{\kappa,\nu,\xi^*}$ forces that $\dot{<}_\xi$ is an initial segment of $\dot{<}_{\xi^*}$ for all $\xi < \xi^*$ and $\mathbb{P}_0 * \mathbb{P}_{\kappa,\nu,\pi}$ forces

$$\dot{<}_\pi = \bigcup\{\dot{<}_\xi : \xi < \pi\}$$

which will be the name of the desired well-order.

Now, we work in V. For each $\xi \in \mathrm{Lim}(\pi)$, we will define a $\mathbb{P}_{\kappa,\nu,\xi}$ name $\dot{A}_\xi$ for a subset of $[\xi, \xi+\omega)$. Similarly to the construction in [FFZ11], for each $\epsilon \in [\omega_{n_6}, \omega_{n_6+1})$, fix (in L) a bijection $i_\epsilon : \{\langle \xi_0, \xi_1 \rangle : \xi_0 < \xi_1 < \epsilon\} \to \mathrm{Lim}(\omega_{n_6})$. For subsets x, y of ω, let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ and let $\Delta(x) = \{2n+2 : n \in x\} \cup \{2n+1 : n \notin x\}$.

Let $\xi \in \mathrm{Lim}(C^*_0)$. Then $\xi = l_\zeta + \eta$ for some $\zeta \in C_0$ and $\eta < \lambda$. Suppose $\mathbb{P}^*_{\alpha,\beta,\xi}$ has been defined for all $\alpha \leq \kappa$, $\beta \leq \nu$. Consider the $\mathbb{P}^*_{\kappa,\nu,l_\zeta}$-names $\dot{\xi}_0, \dot{\xi}_1$ of ordinals so it is forced that $\langle \dot{\xi}_0, \dot{\xi}_1 \rangle = i^{-1}_{\mathrm{o.t.}(\dot{<}_{l_\zeta})}(\eta)$. Furthermore, let $\dot{A}_\xi$ be the $\mathbb{P}^*_{\kappa,\nu,l_\zeta}$-name of $\xi + (\omega \smallsetminus (x^\zeta_{\dot{\xi}_0} * x^\zeta_{\dot{\xi}_1}))$, where x^ζ_ρ is the ρ-th real in $L[G_{\kappa,\nu,l_\zeta}] \cap [\omega]^{\aleph_0}$ according to the well-order $\dot{<}_{l_\zeta}$. By Corollary 3.9, there are $\alpha < \kappa$ and $\beta < \nu$ such that $\dot{\xi}_0$, $\dot{\xi}_1$ and $\dot{A}_\xi$ are $\mathbb{P}^*_{\alpha,\beta,l_\zeta}$-names. So put $\Delta^*(\xi) = (\alpha+1, \beta+1)$ and

$$\mathbb{Q}^*_\xi := \left\{ \langle s_0, s_1 \rangle \in [\omega]^{<\aleph_0} \times \left[\bigcup_{m \in \Delta(x^\zeta_{\dot{\xi}_0} * x^\zeta_{\dot{\xi}_1})} Y_{\alpha+m} \times \{m\} \right]^{<\aleph_0} \right\}$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \smallsetminus s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$.

This completes the construction of the modified standard 3D-coherent system. In addition, for every $\xi \in \mathrm{Lim}(\pi) \backslash \mathrm{Lim}(C^*_0)$ define $\dot{A}_\xi$ to be the canonical $\mathbb{P}^*_{0,0,\xi}$-name for the empty set. Since all posets used to control the cardinal characteristics in Theorem 5.4 are σ-linked, one can reproduce the proof of [FFZ11, Lemma 3] to show that if G is $\mathbb{P}_0 * \mathbb{P}^*_{\kappa,\nu,\pi}$-generic over L, then for each $\eta \in \bigcup_{\xi \in \mathrm{Lim}(\pi)} \dot{A}_\xi[G]$ there is no real coding a stationary kill of S_η. The proofs of [FFZ11, Lemmas 4 and 5] can be easily modified to show that if G is $\mathbb{P}_0 * \mathbb{P}^*_{\kappa,\nu,\pi}$-generic over L and x, y are reals in $L[G]$ then $x \dot{<}_\pi[G] y$ if and only if there is a real r such that for every countable suitable model $\mathcal{M}$ such that $r \in \mathcal{M}$, there is $\bar{\bar{\alpha}} < \pi^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$,

$$(L[r])^{\mathcal{M}} \vDash (S_{\bar{\bar{\alpha}}+m} \text{ is not stationary}).$$

This completes the proof of Theorem 6.1.

7. Discussion and questions

Though the 3D-coherent systems we constructed yield models of several values in Cichoń's diagram, it is still restricted (as in [Mej13a]) to constellations where the right side of the diagram assumes at most 3 different values. So far, the only known model of more than 3 values on the right (actually 5) is constructed in [FGKS] with a proper ω^ω-bounding forcing by a large product of creatures (though it is restricted to $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$).

As discussed before Corollary 3.9, in all our constructions we only add two types of generic reals: full generic reals and restricted generic reals. Different type of generic reals could be considered (like a real which is restricted generic in some plane but full generic in the perpendicular plane), but the known attempts so far destroy the complete embeddability of the posets in the system and, therefore, the construction collapses. Success in this problem of using a different type of generic reals in 3D-coherent systems would lead to models where more than 3 different values can be obtained in the right side of Cichoń's diagram. For instance,

Question 7.1 ([Mej13a, Sect. 7])**.** *Is it consistent with ZFC that* $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$*?*

Furthermore, success in including further types of generic reals will give a key to prove

Question 7.2 ([BF11, §6])**.** *Is it consistent with ZFC (even assuming large cardinals) that*

(1) $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$*?*
(2) $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$*?*

It is even more difficult to construct a 3D-system for the latter problem. For instance, to increase $\mathfrak{s}$, we must include ccc posets that are not definable (i.e., not Suslin) to add non-restricted unsplitting reals, like a full Mathias-Prikry generic, which creates many obstacles even to understanding how to define the ultrafilters at certain stages of the system of iterations. On the other hand, the consistency of $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$ was obtained by Shelah [She04] with template iterations, a result improved to $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$ by the first and third authors [FM].

Concerning our constructions, recall that in Theorems 5.6(b) and 5.7(c),(d) we cannot decide $\mathfrak{a}$ with the methods of Section 4 because of the full Hechler generics we add. In simpler cases, even though any (non-trivial) FS iteration of ccc posets with length of cofinality ω_1 would force $\mathfrak{a} = \aleph_1$ (see Remark 4.16), it is not known, assuming $\mathfrak{c} = \aleph_3$, what an iteration of length ω_2 of $\mathbb{D}$ would force about $\mathfrak{a}$. Nevertheless, some of the models we constructed would do the job without adding full Hechler generics, if we expect to force $\mathfrak{a} = \mathfrak{b}$, e.g., Theorem 5.4 with $\kappa = \mu = \nu$. Possibly, one reason that $\mathfrak{d} < \mathfrak{a}$ is forced in Shelah's template iteration of Hechler forcing could be that the previous ω_2 length FS iteration of Hechler forcing cannot decide $\mathfrak{a}$.

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