# MAD families, splitting families and large continuum

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- con($\mathfrak{s} < \mathfrak{b}$) (Baumgartner, Dordal, 1984)
- con($\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{a} = \aleph_2$) (Shelah, 1985)
- con($\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+$) (Brendle, 1998)
- con($\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$) (F., Steprāns, 2008)

### Theorem (Brendle, F., 2011)

*Let $\kappa < \lambda$ be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.*

### Theorem (Brenlde, F., 2011)

*Let $\mu$ be a measurable cardinal, $\kappa < \lambda$ regular such that $\mu < \kappa$. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$.*

For $\gamma$ an ordinal, $\mathbb{P}_\gamma$ is the poset of all finite partial functions $p : \gamma \times \omega \to 2$ such that $\text{dom}(p) = F_p \times n_p$ where $F_p \in [\gamma]^{<\omega}$, $n_p \in \omega$. The order is given by $q \leq p$ if $p \subseteq q$ and $|q^{-1}(1) \cap F^p \times \{i\}| \leq 1$ for all $i \in n_q \backslash n_p$ .

Let $G$ be a $\mathbb{P}_\gamma$-generic filter and for $\delta \in \gamma$ let $A_\alpha = \{i : \exists p \in G(p(\alpha, i) = 1)\}$. Then

- $\{A_\alpha : \alpha \in \gamma\}$ is an a.d. family (maximal for $\gamma \geq \omega_1$),
- if $p \in \mathbb{P}_\gamma$ then for all $\alpha \in F_p(p \Vdash \dot{A}_\alpha \restriction n_p = p \restriction \{\alpha\} \times n_p)$,
- for all $\alpha, \beta \in F_p(p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq n_p)$.

Let $\gamma < \delta$, $G$ a $\mathbb{P}_\gamma$-generic filter. In $V[G]$, let $\mathbb{P}_{[\gamma,\delta)}$ consist of all $(p, H)$ such that $p \in \mathbb{P}_\delta$ with $F_p \in [\delta \backslash \gamma]^{<\omega}$ and $H \in [\gamma]^{<\omega}$. The order is given by $(q, K) \leq (p, H)$ if $q \leq_{\mathbb{P}_\delta} p$, $H \subseteq K$ and for all $\alpha \in F_p$, $\beta \in H$, $i \in n_q \backslash n_p$ if $i \in A_\beta$, then $q(\alpha, i) = 0$.

- That is for all $\alpha \in F_p, \beta \in H$, $p \Vdash \dot{A}_\alpha \cap \check{A}_\beta \subseteq n_p$.
- $\mathbb{P}_\delta$ is forcing equivalent to $\mathbb{P}_\gamma * \mathbb{P}_{[\gamma,\delta)}$.

## Property $\star$

Let $M \subseteq N$, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq [\omega]^\omega \cap M$, $A \in N \cap [\omega]^\omega$. Then $(\star^{M,N}_{\mathcal{B},A})$ holds if for every $h : \omega \times [\gamma]^{<\omega} \to \omega$, $h \in M$ and $m \in \omega$ there are $n \geq m$, $F \in [\gamma]^{<\omega}$ such that $[n, h(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$.

## Lemma A

If $G_{\gamma+1}$ is $\mathbb{P}_{\gamma+1}$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{P}_\gamma$, $\mathcal{A}_\gamma = \{A_\alpha\}_{\alpha<\gamma}$, where $A_\alpha = \{i : \exists p \in G(p(\alpha, i) = 1)\}$. Then $(\star^{V[G_\gamma], V[G_{\gamma+1}]}_{\mathcal{A}_\gamma, A_\gamma})$ holds.

## Lemma B

Let $(\star^{M,N}_{\mathcal{B},A})$ hold, where $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma}$, let $\mathcal{I}(\mathcal{B})$ be the ideal generated by $\mathcal{B}$ and the finite sets and let $B \in M \cap [\omega]^\omega$, $B \notin \mathcal{I}(\mathcal{B})$. Then $|A \cap B| = \aleph_0$.

## Lemma C

Let $M \subseteq N$, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$. Let $\mathcal{U}$ be an ultrafilter in $M$. Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in $N$ such that

1. every maximal antichain of $\mathbb{M}_\mathcal{U}$ which belongs to $M$ is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in $N$,
2. $(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds where $G$ is $\mathbb{M}_\mathcal{V}$-generic over $N$ (and thus, by (1), $\mathbb{M}_\mathcal{U}$-generic over $M$).

## Lemma D

Let $M \subseteq N$, $\mathbb{P} \in M$ a poset such that $\mathbb{P} \subseteq M$, $G$ a $\mathbb{P}$-generic filter over $M, N$. Let $\mathcal{B} = \{B_\alpha\}_{\alpha \in \gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$ holds. Then $(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds.

## Lemma E

Let $\langle \mathbb{P}_{\ell,n}, \dot{\mathbb{Q}}_{\ell,n} : n \in \omega \rangle$, $\ell \in \{0,1\}$ be finite support iterations such that $\mathbb{P}_{0,n}$ is a complete suborder of $\mathbb{P}_{1,n}$ for all $n$. Let $V_{\ell,n} = V^{\mathbb{P}_{\ell,n}}$. Let $\mathcal{B} = \{A_\gamma\}_{\gamma < \alpha} \subseteq V_{0,0} \cap [\omega]^\omega$, $A \in V_{1,0} \cap [\omega]^\omega$. If $(\star^{V_{0,n},V_{1,n}}_{\mathcal{B},A})$ holds for all $n \in \omega$, then $(\star^{V_{0,\omega},V_{1,\omega}}_{\mathcal{B},A})$ holds.

### Lemma

Let $\mathbb{P}, \mathbb{Q}$ be partial orders, such that $\mathbb{P}$ is completely embedded into $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion, $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$, and every maximal antichain of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ is a maximal antichain of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$.

Let $f : \{\eta < \lambda : \eta \equiv 1 \bmod 2\} \to \kappa$ be an onto mapping, such that for all $\alpha < \kappa$, $f^{-1}(\alpha)$ is cofinal in $\lambda$. Recursively define a system of finite support iterations

$$\langle\langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$$

as follows. For all $\alpha, \zeta$ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$.

(1) If $\zeta = 0$, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,0}$ is Hechler's poset for adding an a.d. family $\mathcal{A}_\alpha = \{A_\beta\}_{\beta<\alpha}$,

(2) If $\zeta = \eta + 1$, $\zeta \equiv 1 \bmod 2$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ where $\dot{\mathcal{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$, $\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathcal{U}}_{\alpha,\eta} \subseteq \dot{\mathcal{U}}_{\beta,\eta}$,

(3) If $\zeta = \eta + 1$, $\zeta \equiv 0 \bmod 2$, then if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion; if $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.

(4) If $\zeta$ is a limit, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$.

Furthermore the construction will satisfy the following two properties:

(a) $\forall\zeta \leq \lambda \forall\alpha < \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$,

(b) $\forall\zeta \leq \lambda \forall\alpha < \kappa$ $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds.

Proceed by recursion on $\zeta$. For $\zeta = 0$, $\alpha \leq \kappa$ let $\mathbb{P}_{\alpha,0} = \mathbb{P}_\alpha$. Then clearly properties (*a*) and (*b*) above hold. Let $\zeta = \eta + 1$ be a successor ordinal and suppose $\forall \alpha \leq \kappa$, $\mathbb{P}_{\alpha,\eta}$ has been defined.

If $\zeta \equiv 1 \bmod 2$ define $\dot{\mathbb{Q}}_{\alpha,\eta}$ by induction on $\alpha \leq \kappa$ as follows.

- *If* $\alpha = 0$, let $\dot{\mathcal{U}}_{0,\eta}$ be a $\mathbb{P}_{0,\eta}$-name for an ultrafilter, $\dot{\mathbb{Q}}_{0,\eta}$ a $\mathbb{P}_{0,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{0,\eta}}$ and let $\mathbb{P}_{0,\zeta} = \mathbb{P}_{0,\eta} * \dot{\mathbb{Q}}_{0,\eta}$.
- *If* $\alpha = \beta + 1$ and $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined, by the ind. hyp. and Lemma C there is a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$, every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ in $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and $(\star^{V_{\beta,\zeta}, V_{\beta+1,\zeta}}_{\mathcal{A}_\beta, A_\beta})$. holds. Let $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta} * \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$. In particular $\mathbb{P}_{\beta,\zeta} \lessdot \mathbb{P}_{\alpha,\zeta}$.

- *If $\alpha$ is limit* and for all $\beta < \alpha$ $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined (and so $\dot{\mathbb{Q}}_{\beta,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$) consider the following two cases.
  - If $\mathrm{cf}(\alpha) = \omega$, find a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that for all $\beta < \alpha$, $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$ and every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ from $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\mathcal{U}_{\alpha,\eta}}$ (in $V_{\alpha,\eta}$) and the relevant $\star$-property is preserved.
  - If $\mathrm{cf}(\alpha) > \omega$, then let $\dot{\mathcal{U}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\bigcup_{\beta<\alpha} \mathcal{U}_{\beta,\eta}$. Let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$.

If $\zeta \equiv 0 \mod 2$, then

- for all $\alpha \leq f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial poset
- for $\alpha > f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.

Let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Note that for all $\alpha, \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$ and $(\star^{V_{\alpha,\zeta},V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha,A_\alpha})$ holds for all $\alpha$.

If $\zeta$ is a limit and for all $\eta < \zeta$, $\mathbb{P}_{\alpha,\eta}$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ have been defined, let $\mathbb{P}_{\alpha,\zeta}$ be the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$. Then $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}$ and by Lemma E $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds.

### Lemma

For $\zeta \leq \lambda$:

1. for every $p \in \mathbb{P}_{\kappa,\zeta}$ there is $\alpha < \kappa$ such that $p$ belongs to $\mathbb{P}_{\alpha,\zeta}$,
2. for every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\dot{f}$ there is $\alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.

### Lemma

$V_{\kappa,\lambda} \vDash \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

$\{A_\alpha\}_{\alpha\in\kappa}$ remains mad in $V_{\kappa,\lambda}$. Otherwise $\exists B \in V_{\kappa,\lambda} \cap [\omega]^\omega$ such that $\forall\alpha < \kappa(|B \cap A_\alpha| < \omega)$. However there is $\alpha < \kappa$ such that $B \in V_{\alpha,\lambda} \cap [\omega]^\omega$ and $B \notin \mathcal{I}(\mathcal{A}_\alpha)$. On the other hand $(\star^{V_{\alpha,\lambda},V_{\alpha+1,\lambda}}_{\mathcal{A}_\alpha,A_{\alpha+1}})$ and so $|B \cap A_{\alpha+1}| = \omega$ (Lemma B) which is a contradiction. Therefore $\mathfrak{a} \leq \kappa$.

Let $\mathcal{B} \subseteq V_{\kappa,\lambda} \cap {}^\omega\omega$ be of size $< \kappa$. Then there are $\alpha < \kappa$, $\zeta < \lambda$ such that $\mathcal{B} \subseteq V_{\alpha,\zeta}$. Since $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in $\lambda$, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$. Then $\mathbb{P}_{\alpha+1,\zeta'+1}$ adds a real dominating $V_{\alpha,\zeta'} \cap {}^\omega\omega$ (and so $V_{\alpha,\zeta} \cap {}^\omega\omega$ since $V_{\alpha,\zeta} \subseteq V_{\alpha,\zeta'}$). Thus $\mathcal{B}$ is not unbounded. Therefore $V_{\kappa,\lambda} \Vdash \mathfrak{b} \geq \kappa$.

However $\mathfrak{b} \leq \mathfrak{a}$ and so $V_{\kappa,\lambda} \Vdash \mathfrak{b} = \mathfrak{a} = \kappa$.

To see that $V_{\kappa,\lambda} \vDash \mathfrak{s} = \lambda$, note that if $S \subseteq V_{\kappa,\lambda} \cap [\omega]^\omega$ is a family of cardinality $< \lambda$, then there is $\zeta < \lambda$ such that $\zeta = \eta + 1$, $\zeta \equiv 1 \bmod 2$ and $S \subseteq V_{\kappa,\eta}$. Then $\mathcal{M}_{\mathcal{U}_{\kappa,\eta}}$ adds a real not split by $S$ and so $S$ is not splitting.

**Theorem (Brendle, F., 2011)**

*Let $\kappa < \lambda$ be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.*

- Is it relatively consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?
- Is it relatively consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$?
- It is relatively consistent that $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$ without the assumption of a measurable?
- How about $\mathfrak{b} = \mathfrak{s} = \aleph_1 < \mathfrak{a} = \aleph_2$?

Thank you!