Combinatorics and Projective Wellorders on the Reals

Vera Fischer

Kurt Gödel Research Center
University of Vienna

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- definable wellorder of the reals
- cardinal characteristics of the reals

- To what extent the combinatorial properties of the real line (expressed in terms of cardinal characteristics) are compatible with the existence of a projective wellorder of the reals?
- What other 'natural' combinatorial objects on the reals are consistent with the existence of a projective wellorder of the reals?

Eventual dominance

If $f, g \in {}^{\omega}\omega$ then $f \leq^* g$ (g dominates f) if $\exists n \in \omega$ s.t. $\forall m \geq n(f(m) \leq g(m))$.

Bounding number

$\mathcal{B} \subseteq {}^{\omega}\omega$ is unbounded if there is no single function in ${}^{\omega}\omega$ which simultaneously dominates the elements of $\mathcal{B}$.

$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is unbounded}\}$

Dominating number

$\mathcal{D} \subseteq {}^\omega\omega$ is dominating if $\forall f \in {}^\omega\omega \exists g \in \mathcal{D}$ s.t. g dominates f.
$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ is dominating}\}$

Splitting number

$S \subseteq [\omega]^\omega$ is splitting if $\forall A \in [\omega]^\omega \exists B \in S$ s.t.
$|A \cap B| = |A \cap B^c| = \omega$.
$\mathfrak{s} = \min\{|S| : S \text{ is splitting}\}$

- All cardinal characteristics have values between $\aleph_1$ and $\mathfrak{c}$. That is if $\mathfrak{f}$ is a cardinal characteristics then $\aleph_1 \leq \mathfrak{f} \leq \mathfrak{c}$.
- ZFC relations between the card. char. (e.g. $\mathfrak{b} \leq \mathfrak{d}$)
- Independence (e.g. $\mathfrak{b}$, $\mathfrak{s}$)

If $a, b \in [\omega]^\omega$, then a, b are almost disjoint if $a \cap b$ is finite.

mad families

An infinite $\mathcal{A} \subseteq [\omega]^\omega$ is almost disjoint (a.d.) if its elements are pairwise almost disjoint; $\mathcal{A} \subseteq [\omega]^\omega$ is maximal almost disjoint (m.a.d.) if it is maximal with respect to inclusion among a.d. families.

ω-mad families

If $\mathcal{A}$ is a.d., let $\mathcal{L}(\mathcal{A}) = \{b \in [\omega]^\omega : b$ is not covered by finitely many $a \in \mathcal{A}\}$. A m.a.d. family $\mathcal{A}$ is ω-mad if $\forall B \in [\mathcal{L}(\mathcal{A})]^\omega$ there is $a \in A$ such that $|a \cap b| = \omega$ for all $b \in \mathcal{B}$.

L. Harrington

The existence of $\mathbf{\Delta}^1_3$-definable wellorder of the reals is consistent with $\mathfrak{c}$ being as large as desired and MA.

S. Friedman

The existence of Δ^1_3-definable wellorder of the reals is consistent with $\mathfrak{c} = \omega_2$ and MA.

Note that under MA all cardinal characteristics are equal to $\mathfrak{c}$.

Develop iteration techniques which allows one to separate certain cardinal characteristics in the presence of a projective wellorder.

V.F. - S.D. Friedman, 2009

- The existence of a Δ^1_3-wellorder of the reals is relatively consistent with $\mathfrak{d} < \mathfrak{c} = \omega_2$.
- The existence of a Δ^1_3-definable wellorder of the reals is relatively consistent with $\mathfrak{b} < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \omega_2$.
- The existence of a Δ^1_3-definable wellorder of the reals is relatively consistent with $\mathfrak{b} < \mathfrak{g} = \mathfrak{c} = \omega_2$.

Conjecture

Each admissible assignment of $\aleph_1$ and $\aleph_2$ to the cardinal invariants (associated with measure and category) in the Cichón diagram, is relatively consistent with the existence of a projective wellorder of the reals.

There is general interest, however also major difficulties, in obtaining models in which the real line has desireable combinatorial properties and $\mathfrak{c} \geq \omega_3$.

V.F., S.D. Friedman, L. Zdomskyy, 2010

The existence of a Δ^1_3-definable wellorder of the reals is consistent with $\mathfrak{b} = \mathfrak{c} = \omega_3$ and the existence of a Π^1_2-definable ω-mad subfamily of infinite subsets of ω.

We expect that an application of Jensen's coding technique will lead to the same result with essentially arbitrary values for $\mathfrak{c}$.

- Is the existence of a Δ^1_3-projective wellorder of the reals relatively consistent with MA in the presence of $\mathfrak{c} \geq \aleph_3$? (The iteration techniques from the previous theorem can take care only of Suslin posets).
- How about models, in which desired inequalities between cardinal characteristics of the real line hold, in the presence of a projective wellorder and $\mathfrak{c} \geq \aleph_3$? (In the model from the last theorem there is a major problem in bookkeeping families of reals of size $> \aleph_0$.)
- Definable cardinal characteristics.

Definition

A transitive ZF^- model $\mathcal{M}$ is suitable if $\mathcal{M} \vDash \omega_2 = \omega_2^L$ exists.

Throughout this section work in some generic extension $L[G^*]$ of L in which cofinalities have not been changed.

Definition

Let $X \subseteq \omega_1$ and let $\phi(\omega_1, X)$ be a Σ_1-sentence with parameters ω_1, X which is true in all suitable models containing ω_1 and X as elements. Let $\mathcal{L}(\phi)$ be the poset of all $r : |r| \to 2$ where $|r|$ is a countable limit ordinal such that:
1. $\forall \gamma \in |r| (\gamma \in X \text{ iff } r(2\gamma) = 1)$
2. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \upharpoonright \gamma$ as an element, where $\omega_1^{\mathcal{M}} = \gamma$, then $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$.
The extension relation is end-extension.

$\mathcal{L}(\phi)$ is proper and does not add new reals. In fact $\mathcal{L}(\phi)$ has a countably closed dense suborder.

Let $Y \subseteq \omega_1$ be generic over L such that in $L[Y]$ cofinalities have not been changed. Inductively define $\bar{\mu} = \{\mu_i\}_{i\in\omega_1}$ of L-countable ordinals as follows: μ_i is least $\mu > \sup_{j<i} \mu_j$ such that $L_\mu[Y \cap i] \models ZF^-$ and $L_\mu \models$ (ω is the largest cardinal).

A real R codes Y below i if for all $j < i$

$$j \in Y \text{ iff } L_{\mu_j}[Y \cap j, R] \models ZF^-.$$

For $T \subseteq 2^{<\omega}$ a perfect tree, let $|T| = \min\{i : T \in L_{\mu_i}[Y \cap i]\}$.

Definition

Let $\mathcal{C}(Y)$ be the poset of all perfect trees T such that every branch R through T codes Y below $|T|$. Whenever T_0, T_1 are conditions in $\mathcal{C}(Y)$ let $T_0 \leq T_1$ iff $T_0 \subseteq T_1$.

$\mathcal{C}(Y)$ is proper and ${}^{\omega}\omega$-bounding.

Definition

Let $T \subseteq \omega_1$ be a stationary set. A poset $\mathbb{Q}$ is *T-proper*, if for every countable elementary submodel $\mathcal{M}$ of $H(\Theta)$, where Θ is a sufficiently large cardinal, such that $\mathcal{M} \cap \omega_1 \in T$, every condition $p \in \mathbb{Q} \cap \mathcal{M}$ has an $(\mathcal{M}, \mathbb{Q})$-generic extension q.

- Let $S \subseteq \omega_1$ be a stationary, co-stationary set. Then $Q(S)$ is the poset of all countable closed subsets of $\omega_1 \backslash S$, with the end-extension as the extension relation. $Q(S)$ is $\omega_1 \backslash S$-proper.
- S-proper posets preserve ω_1 and the stationarity of all stationary subsets of S. The countable support iteration of S-proper posets is S-proper.

Lemma

Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper posets of size ω_1. Then $\mathbb{P}_\delta$ is $\aleph_2$-c.c.

Lemma

Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ be a countable support iteration of length $\delta < \omega_2$ of S-proper posets of size ω_1. Then $V^{\mathbb{P}_\delta} \models CH$.

Lemma

There is $F : \omega_2 \to L_{\omega_2}$ definable over L_{ω_2} via a formula ϕ and a sequence $\bar{S} = (S_\beta : \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1 definable over L_{ω_2} via a formula ψ such that $F^{-1}(a)$ is unbounded in ω_2 for every $a \in L_{\omega_2}$, and

- *If $\mathcal{M}, \mathcal{N}$ are suitable models and $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ then $F^{\mathcal{M}}, F^{\mathcal{N}}$ agree on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$.*
- *If $\mathcal{M}$ is suitable and $\omega_1^{\mathcal{M}} = \omega_1$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ equal the restrictions of F, $\bar{S}$ to the ω_2 of $\mathcal{M}$.*

Proof.

Define $F(\alpha) = a$ iff via Gödel pairing α codes a pair (α_0, α_1) where a has rank α_0 in the natural wellorder of the sets in L. For the almost disjoint stationary sets, let $(D_\gamma : \gamma < \omega_1)$ be the canonical L_{ω_1} definable $\diamondsuit$ sequence, for each $\alpha < \omega_2$ let A_α be the L-least subset of ω_1 coding α and define S_α to be the set of all $i < \omega_1$ such that $D_i = A_\alpha \cap i$. □

Recursively define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ such that $\mathbb{P} = \mathbb{P}_{\omega_2}$ will be the desired poset.

- For $\alpha < \beta < \omega_2$ we can assume that all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals which are not $\mathbb{P}_\alpha$ names.
- For $\alpha < \omega_2$, define a wellorder $<_\alpha$ on the reals of $L[G_\alpha]$, where G_α is a $\mathbb{P}_\alpha$-generic as follows. If x is a real in $L[G_\alpha]$ let σ^α_x be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \leq \alpha$. Then let $x <_\alpha y$ if and only if $\sigma^\alpha_x <_L \sigma^\alpha_y$.

- Note that $<_\alpha$ is an initial segment of $<_\beta$.

Then if G is a $\mathbb{P}$-generic filter, $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ will be the desired wellorder of the reals. Also, for x, y reals in $L[G_\alpha]$ such that $x <_\alpha y$ let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$. Let S be a stationary set almost disjoint from every element of $\bar{S}$.

Proceed with the definition of $\mathbb{P}_{\omega_2}$. Let $\mathbb{P}_0$ be the trivial poset. Suppose $\mathbb{P}_\alpha$ has been defined. Let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha$-name for a poset such that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing notion of size at most $\aleph_1$ and $\dot{\mathbb{Q}}^1_\alpha$ is defined as follows.

- If $F(\alpha)$ is not of the form $\{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals x, y in $V^{\mathbb{P}_\alpha}$ then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the trivial poset.
- Otherwise $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals $x <_\alpha y$ in $V^{\mathbb{P}_\alpha}$. Let $x_\alpha = x$, $y_\alpha = y$. Then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for $\mathbb{K}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$ where:

Destroying stationary sets ($\mathbb{K}^0_\alpha$)

In $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha}$ let $\mathbb{K}^0_\alpha$ be the direct limit $\langle \mathbb{P}^0_{\alpha,n}, \dot{\mathbb{K}}^0_{\alpha,n} : n \in \omega \rangle$, where $\dot{\mathbb{K}}^0_{\alpha,n}$ is a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n})$ for $n \in x_\alpha * y_\alpha$, and $\dot{\mathbb{K}}^0_{\alpha,n}$ is a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n+1})$ for $n \notin x_\alpha * y_\alpha$.

Localization $(\mathbb{K}^1_\alpha)$

Let G^0_α be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-generic filter and let H_α be a $\mathbb{K}^0_\alpha$-generic over $L[G^0_\alpha]$. In $L[G^0_\alpha * H_\alpha]$ let X_α be a subset of ω_1, coding α, coding (x_α, y_α), coding a level of L in which α has size at most ω_1 and coding the generic $G^0_\alpha * H_\alpha$ which we can regard as a subset of an element of L_{ω_2}.

Then let $\mathbb{K}^1_\alpha = \mathcal{L}(\phi_\alpha)$ where $\phi_\alpha = \phi_\alpha(\omega_1, X_\alpha)$ is the Σ_1-sentence which holds iff X_α codes an ordinal $\bar{\alpha} < \omega_2$ and a pair (x, y) such that $S_{\bar{\alpha}+2n}$ is nonstationary for $n \in x * y$, $S_{\bar{\alpha}+2n+1}$ is nonstationary for $n \notin x * y$. Let $\dot{\mathbb{K}}^1_\alpha$ be a $\mathbb{P}^0_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha$-name for $\mathbb{K}^1_\alpha$.

Coding with Perfect Tress ($\mathbb{K}^2_\alpha$)

Let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G^0_\alpha * H_\alpha]$. Since Y_α codes X_α, $L[G^0_\alpha * H_\alpha * Y_\alpha] = L[Y_\alpha]$. Let $\mathbb{K}^2_\alpha = \mathcal{C}(Y_\alpha)$. Let $\dot{\mathbb{K}}^2_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha$-name for $\mathbb{K}^2_\alpha$.

With this the definition of $\dot{\mathbb{Q}}_\alpha$ and so $\mathbb{P} = \mathbb{P}_{\omega_2}$ is complete.

Lemma

$\mathbb{P}$ is S-proper and ω_2-c.c.

Lemma A

Let G be a $\mathbb{P}$-generic and let x, y be reals in $L[G]$. If $x < y$, then there is a real R such that for every countable suitable $\mathcal{M}$, $R \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$.

Proof

Pick α such that $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$. Then $x_\alpha = x$, $y_\alpha = y$. Let G^0_α be $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-generic, let H_α be $\mathbb{K}^0_\alpha$-generic over $L[G^0_\alpha]$, let Y_α be the $\mathbb{K}^1_\alpha$-generic over $L[G^0_\alpha * H_\alpha]$, let R_α be the $\mathbb{K}^2_\alpha$-generic over $L[Y_\alpha]$.

Let $\mathcal{M}$ be countable suitable, $R_\alpha \in \mathcal{M}$. However R_α codes Y_α and so $Y_\alpha \restriction \gamma \in \mathcal{M}$, where $\gamma = \omega_1^{\mathcal{M}}$. Then in particular $X_\alpha \cap \gamma \in \mathcal{M}$. By the properties of localization $\phi_\alpha(\gamma, X_\alpha \cap \gamma)$ holds in $\mathcal{M}$ and so $\exists \bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$. $\square$

Lemma B

Let G be $\mathbb{P}$-generic. Then for β not of the form $\alpha + 2n$, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha + 2n + 1$, for $n \notin x^G_\alpha * y^G_\alpha$, the set S_β is stationary in $L[G]$.

Proof

Let $p \in \mathbb{P}$ be a condition forcing that $\beta < \omega_2$ is not of the form $\alpha + 2n$, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha + 2n + 1$, for $n \notin x^G_\alpha * y^G_\alpha$. Consider the forcing notion $\mathbb{P} \restriction p$ which consists of all conditions in $\mathbb{P}$ which extend p. Note that G is also $\mathbb{P} \restriction p$-generic. However $\mathbb{P} \restriction p$ is S_β-proper and so S_β remains stationary in $L[G]$.

Let G be $\mathbb{P}$-generic and let x, y be reals in $L[G]$. Then

(1) $x < y$ iff for some $\alpha < \omega_2$, $S_{\alpha+2n}$ is nonstationary for n in $x * y$ and $S_{\alpha+2n+1}$ is nonstationary for n not in $x * y$.

(2) If $x < y$ then there is a real R such that for every countable suitable $\mathcal{M}$, $R \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$.

Observation

(1) implies the converse of (2).

Let R be given. The conclusion of (2) holds for arbitrary suitable models and so it holds for $L_\Theta[R] = \mathcal{M}$ where Θ is large. Let $\alpha < \omega_2$ be the corresponding ordinal. As $\bar{S}$ is definable over L_{ω_2} and $\Theta > \omega_2$, $S^{\mathcal{M}}_\beta = S_\beta$ for all $\beta < \omega_2$. Thus $S^{\mathcal{M}}_{\alpha+2n} = S_{\alpha+2n}$ is nonstationary in $\mathcal{M}$ for n in $x * y$ and $S^{\mathcal{M}}_{\alpha+2n+1} = S_{\alpha+2n+1}$ is nonstationary in $\mathcal{M}$ for n not in $x * y$. These sets are nonstationary in the larger model $L[G]$ and so by (1), we have $x < y$.

Therefore in $L[G]$, $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ has a Σ^1_3 definition.

$x < y$ iff there is a real R such that for every countable suitable $\mathcal{M}$, $R \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$

It remains to observe that since $x \not<^G y$ is Π^1_3 and $<^G$ is a linear order, $<^G$ indeed has a Δ^1_3 definition.

Lemma

Let $S \subseteq \omega_1$ be a stationary set and let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \delta \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, ${}^\omega\omega$-bounding posets. Then $\mathbb{P}_\delta$ is ${}^\omega\omega$-bounding and S-proper.

Observation

For all $\alpha < \omega_2$,

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}^1_\alpha \text{ is } S\text{-proper and } {}^\omega\omega\text{-bounding.}$$

Theorem

It is consistent with $\mathfrak{d} < \mathfrak{c}$ that there is a Δ^1_3 wellorder of the reals.

Proof.

Let $\mathbb{P}_{\mathbb{S}}$ be defined just as $\mathbb{P} = \mathbb{P}_{\omega_2}$ with the additional requirement that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Let G be $\mathbb{P}_{\mathbb{S}}$-generic. Since destroying stationary sets, localization and coding with perfect trees are ${}^\omega\omega$-bounding, $\mathbb{P}_{\mathbb{S}}$ is weakly bounding. Then $L[G] \vDash \mathfrak{d} = \omega_1 < \mathfrak{c} = \omega_2$. □

Theorem

It is consistent with $\mathfrak{b} < \mathfrak{g}$ that there is a Δ^1_3 wellorder of the reals.

Proof.

Let $\mathbb{P}_{\mathbb{M}}$ be defined just as $\mathbb{P} = \mathbb{P}_{\omega_2}$ with the additional requirement that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for Miller forcing $\mathbb{M}$. Since $\mathbb{M}$ is almost ${}^\omega\omega$-bounding, $\mathbb{P}_{\mathbb{M}}$ is weakly bounding. The Miller real has supersets in all groupwise dense families from the ground model, and so if G is $\mathbb{P}_{\mathbb{S}}$-generic, $L[G] \vDash \mathfrak{b} = \omega_1 < \mathfrak{g} = \omega_2$. □

Theorem

It is consistent with $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$ that there is a Δ^1_3 definable wellorder of the reals.

Proof

Let Q be an almost ${}^\omega\omega$-bounding poset which adds a real not split by the ground model reals. By a result of S. Shelah if $V \vDash CH$ and $\mathcal{A}$ is a mad family in V, then in $V_1 = V^{\mathbb{C}(\omega_1)}$ there is an almost ${}^\omega\omega$-bounding poset which destroys the maximality of $\mathcal{A}$.

Let F_0 be a bookkeeping function, $\mathrm{dom}(F_0) = \omega_2$ such that every relevant name for a mad family is enumerated cofinally often. Let $\mathbb{P}_Q$ be defined just as $\mathbb{P}$ with the additional requirement that $\mathbb{Q}^0_\alpha = \mathbb{H}^0_\alpha * \dot{\mathbb{H}}^1_\alpha * \dot{\mathbb{H}}^2_\alpha$ where

- $\mathbb{H}^0_\alpha$ adds ω_1 Cohen reals.
- If $F_0(\alpha)$ is a $\mathbb{P}_\alpha$-name for a mad family then $\mathbb{H}^1_\alpha$ is an almost ${}^\omega\omega$-bounding poset which destroys its maximality. If $F_0(\alpha)$ is not a $\mathbb{P}_\alpha$-name for a mad family then $\mathbb{H}^1_\alpha$ is the trivial poset.
- $\mathbb{H}^2_\alpha$ is Shelah's poset Q.

Let G be $\mathbb{P}_Q$-generic.

- Cohen forcing, Q and the posets used to kill mad families are almost ${}^\omega\omega$-bounding. Thus $\mathbb{P}_Q$ is weakly bounding and so $L[G] \vDash \mathfrak{b} = \omega_1$.
- Let $W \subseteq L[G] \cap [\omega]^\omega$, $|W| = \omega_1$. Then $W \subseteq L[G_\alpha]$ for some $\alpha < \omega_2$. However $\mathbb{H}^2_\alpha$ adds a real not split by W and so $L[G] \vDash \mathfrak{s} = \omega_2$

- Suppose $\mathcal{A}$ is a mad family in $L[G]$, $|\mathcal{A}| = \omega_1$. Since $F_0^{-1}(\dot{\mathcal{A}})$ is unbounded there is $\beta \geq \alpha$ with $F_0(\beta) = \dot{\mathcal{A}}$. Then $\mathbb{H}^1_\alpha$ destroys the maximality of $\mathcal{A}$ and so $L[G_{\beta+1}] \vDash \mathcal{A}$ is not mad, which is a contradiction. Thus $L[G] \vDash \mathfrak{a} = \omega_2$.

1. Which other inequalities between the standard cardinal characteristics of the real line are consistent with the existence of a projective wellorder of the reals?

2. What is the complexity in the projective hierarchy of the witnesses of the corresponding cardinal characteristics in these models?

A family $D \subseteq [\omega]^\omega$ is *groupwise dense* if

1. if $X \in D$ and $Y \backslash X$ is finite, then $Y \in D$
2. if Π is a family of infinitely many pairwise disjoint finite subsets of ω, the union of some subfamily of Π is in D.

The groupwise density number $\mathfrak{g}$ is the minimal κ such that for some family $\mathcal{D}$ of κ-many groupwise dense families, $\bigcap \mathcal{D} = \emptyset$