DEFINABLE MAXIMAL COFINITARY GROUPS

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Abstract. Using countable support iteration of S-proper posets, for some appropriate stationary set S, we obtain a generic extension of the constructible universe, in which $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there is a maximal cofinitary group with a Π^1_2-definable set of generators.

1. Introduction

Following standard notation, we denote by S_∞ the set of all permutations of the natural numbers. A function $f \in S_\infty$ is said to be *a cofinitary permutation*, if it has only finitely many fixed points. A subgroup $\mathcal{G}$ of S_∞ is said to be *a cofinitary group* if each of its non-identity elements has only finitely many fixed points, i.e. is a cofinitary permutation. A maximal cofinitary group, abbreviated mcg, is a cofinitary group, which is maximal with respect to these properties, under inclusion. The minimal size of a maximal cofinitary group is denoted $\mathfrak{a}_g$. It is known that $\mathfrak{b} \leq \mathfrak{a}_g$ (see [6]).

There has been significant interest towards the existence of maximal cofinitary groups which are low in the projective hierarchy. The existence of a closed maximal cofinitary group is still open, while S. Gao and Y. Zhang (see [7]) showed that the axiom of constructibility implies the existence of a maximal cofinitary group with a co-analytic generating set. The result was improved by B. Kastermans, who showed that in the constructible universe L there is a co-analytic maximal cofinitary group (see [6]).

There is little known about the existence of nicely definable maximal cofinitary groups in models of $\mathfrak{c} > \aleph_1$. Our main result can be formulated as follows:

Theorem. *There is a generic extension of the constructible universe in which $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there is a maximal cofinitary group with a Π^1_2-definable set of generators.*

The extension is obtained via a countable support iteration of S-proper posets, for some appropriate stationary set S. Along the iteration cofinally often we add generic permutations which using a ground model set of almost disjoint functions provide codes for themselves. Of use for this construction is on the one hand the poset for adding a maximal cofinitary group of desired cardinality, developed in [5], and on the other hand the coding techniques of [2] and [4].

The paper is organized as follows: in section 2 we give an outline of a poset which adjoins a cofinitary permutation to a given co-fnitary group and describe our main coding techniques; section

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3 contains a detailed proof of our main theorem and in section 4 we conclude with the discussion of some remaining open questions.

2. Maximal Cofinitary Groups and Coding

2.1. **Adding generic permutations.** Our methods for adding a generic permutation are based on [5], where the first and third authors provide a poset which given an arbitrary index set A and a (freely generated) cofinitary group G, generically adjoins a family of permutations $\{g_a\}_{a\in A}$ such that the group generated by $G\cup\{g_a\}_{a\in A}$ is cofinitary. We will be interested in the particular case in which $|A|=1$. Following the terminology of [5], given a non-empty set B, a mapping $\rho : B\to S_\infty$ is said to *induce a cofinitary representation* if the natural extension of ρ to a mapping $\hat{\rho} : \mathbb{F}_B \to S_\infty$, where $\mathbb{F}_B$ denotes the free group on the set B, has the property that its image is a cofinitary group. For $A\neq\emptyset$, we denote by W_A the set of all reduced words on the alphabet A and by $\widehat{W_A}$ the set of all words on the same alphabet which start and end with a different letter, or are a power of a single letter. We refer to the elements of $\widehat{W_A}$ as *good words*. Note that every word is a conjugate of a good word, that is $\forall w\in W_A \exists w_0\in\widehat{W_A}\exists u\in W_A$ such that $w=uw_0u^{-1}$. The empty word is not a good word.

Whenever a is an index, which does not belong to the set B, s is a finite partial injection from ω to ω, $\rho : B\to S_\infty$ is a mapping which induces a cofinitary representation and w is a reduced word on the alphabet $\{a\}\cup B$, we denote by $e_w[s,\rho]$ the (partial) function obtained by substituting every appearance of a letter b from B with $\rho(b)$, and every appearance of the letter a with the partial mapping s. By definition, let $e_\emptyset[s,\rho]$ be the identity. For the exact recursive definition see [5]. Note that if s is injective, then so is $e_w[s,\rho]$ (see [5]).

Definition 2.1. Let B be a non-empty set, $a\notin B$ and $\rho : B\to S_\infty$ a mapping which induces a cofinitary representation. The poset $\mathbb{Q}_{\{a\},\rho}$ consists of all pairs (s,F) where $s\in {}^{<\omega}\omega$ is a finite partial injection, F is a finite set of words in $\widehat{W_{\{a\}\cup B}}$. The extension relation states that $(t,H)\leq(s,F)$ if and only if t end-extends s, $F\subseteq H$ and $\forall w\in F\forall n\in\omega$ if $e_w[t,\rho](n)=n$ then $e_w[s,\rho](n)$ is already defined (and so $e_w[s,\rho](n)=n$).

Recall that a poset $\mathbb{P}$ is said to be σ-centered, if $\mathbb{P}=\bigcup_{n\in\omega}\mathbb{P}_n$ where for each n, $\mathbb{P}_n$ is centered, that is whenever p,q are conditions in $\mathbb{P}_n$ then there is $r\in\mathbb{P}_n$ which is their common extension. Note that $\mathbb{Q}_{\{a\},\rho}$ is σ-centered. If G is $\mathbb{Q}_{\{a\},\rho}$-generic, then $g=\bigcup\{s : \exists F(s,F)\in G\}$ is a cofinitary permutation such that the mapping $\rho_G : \{a\}\cup B\to S_\infty$ defined by $\rho_G(a)=g$ and $\rho_G\restriction B=\rho$, induces a cofinitary representation in $V[G]$. For the proofs of both of these statements see [5].

2.2. **Coding with a ground model almost disjoint family of functions.** We work over the constructible universe L. Recall that a ZF^- model M is said to be *suitable* iff

$$M\vDash(\omega_2 \text{ exists and } \omega_2=\omega_2^L).$$

In our construction, we will use a family $\mathcal{F}=\{f_{\langle\zeta,\xi\rangle} : \zeta\in\omega\cdot 2, \xi\in{\omega_1}^L\}\in L$ of almost disjoint bijective functions such that $\mathcal{F}\cap M=\{f_{\langle\zeta,\xi\rangle} : \zeta\in\omega\cdot 2,\xi\in(\omega_1^L)^M\}$ for every transitive model M of ZF^- (see [4, Proposition 3]).

For our purposes, we will need the following Lemma, which is analogous to [4, Proposition 4].

Lemma 2.2. *There is a sequence $\bar{S} = \langle S_\beta : \beta < \omega_2 \rangle$ of almost disjoint stationary subsets of ω_1, which is Σ_1 definable over L_{ω_2} with parameter ω_1, and whenever M, N are suitable models of ZF^- such that $\omega_1^M = \omega_1^N$, then $\bar{S}^M$ agrees with $\bar{S}^N$ on $\omega_2^M \cap \omega_2^N$.*

Proof. Let $\langle D_\gamma : \gamma < \omega_1 \rangle$ be the canonical L_{ω_1} definable $\diamondsuit$ sequence (see [1]) and for each $\alpha < \omega_2$ let A_α be the L-least subset of ω_1 coding α. Now, let $S_\alpha := \{i < \omega_1 : D_i = A_\alpha \cap i\}$. $\square$

Let $\bar{S}$ be as in the preceding Lemma and let S be a stationary subset of ω_1 which is almost disjoint from every element of $\bar{S}$. We will use the following coding of an ordinal $\alpha < \omega_2$ by a subset of ω_1 (see [4, Fact 5]).

Lemma 2.3. *There is a formula $\phi(x, y)$ and for every $\alpha < \omega_2^L$ a set $X_\alpha \in ([\omega_1]^{\omega_1})^L$ such that*

- *for every suitable model M containing $X_\alpha \cap \omega_1^M$, $\phi(x, X_\alpha \cap \omega_1^M)$ has a unique solution in M, and this solution equals α provided $\omega_1 = \omega_1^M$.*
- *for arbitrary suitable models M, N with $\omega_1^M = \omega_1^N$ and $X_\alpha \cap \omega_1^M \in M \cap N$, the solutions of $\phi(x, X_\alpha \cap \omega_1^M)$ in M, N coincide.*

3. Π^1_2-definable set of generators

In this section we will provide a generic extension of the constructible universe L in which $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there is a maximal cofinitary group with a Π^1_2-definable set of generators. Fix a recursive bijection $\psi : \omega \times \omega \to \omega$. Recursively define a countable support iteration of S-proper posets $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ as follows. If $\alpha < \omega_1$ let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for Hechler forcing for adding a dominating real.[1] Suppose $\mathbb{P}_\alpha$ has been defined and

- for every $\beta \in \mathrm{Lim}(\alpha \backslash \omega_1)$ the poset $\mathbb{Q}_\beta$ adds a cofinitary permutation g_β, and
- the mapping $\rho_\beta : \mathrm{Lim}(\alpha \backslash \omega_1) \to S_\infty$ where $\rho_\alpha(\beta) = g_\beta$ induces a cofinitary representation.

In $L^{\mathbb{P}_\alpha}$ define $\mathbb{Q}_\alpha$ as follows. *If α is a successor*, then $\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for Hechler forcing for adding a dominating real. *If $\alpha \geq \omega_1$ is a limit*, then $\alpha = \omega_1 \cdot \nu + \omega \cdot \eta$ for some $\nu \neq 0$, $\nu < \omega_2$, $\eta < \omega_1$ and the conditions of $\mathbb{Q}_\alpha$ are pairs $\langle \langle s, F, s^* \rangle, \langle c_k, y_k \rangle_{k \in \omega} \rangle$ where

(1) $(s, F) \in \mathbb{Q}_{\{\alpha\}, \rho_\alpha}$;
(2) $\forall k \in \omega$, c_k is a closed bounded subset of $\omega_1 \backslash \eta$ such that $c_k \cap S_{\alpha+k} = \emptyset$;
(3) $\forall k \in \omega$, y_k is a 0,1-valued function whose domain $|y_k|$ is a countable limit ordinal, such that $\eta \leq |y_k|$, $y_k \restriction \eta = 0$ and for every γ such that $\eta \leq \gamma < |y_k|$, $y_k(2\gamma) = 1$ if and only if $\gamma \in \eta + X_\alpha = \{\eta + \mu : \mu \in X_\alpha\}$;
(4) for every $k \in \psi[s]$ and every countable suitable model M of ZF^- such that $\xi = \omega_1^M \leq |y_k|$, ξ is a limit point of c_k and $y_k \restriction \xi$, $c_k \cap \xi$ are elements of M, we have that

$$M \models y_k \restriction \xi \text{ codes a limit ordinal } \bar{\alpha} \text{ such that } S_{\bar{\alpha}+k} \text{ is non-stationary.}$$

(5) s^* is a finite subset of $\{f_{m,\xi} : m \in \psi[s], \xi \in c_m\} \cup \{f_{\omega+m,\xi} : m \in \psi[s], y_m(\xi) = 1\}$.

The extension relation states that $\bar{q} = \langle \langle t, H, t^* \rangle, \langle d_k, z_k \rangle_{k \in \omega} \rangle$ extends $\bar{p} = \langle \langle s, F, s^* \rangle, \langle c_k, y_k \rangle_{k \in \omega} \rangle$ iff

(1) $(t, H) \leq_{\mathbb{Q}_{\{\alpha\}, \rho_\alpha}} (s, F)$,
(2) $\forall f \in s^*$, $t \backslash s \cap f = \emptyset$,

[1] For bookkeeping reasons it is more convenient to introduce the generators of the maximal cofinitary group at limit stages greater or equal ω_1.

(3) $\forall k \in \psi[s]$, d_k end-extends c_k and $y_k \subseteq z_k$

With this the recursive definition of $\mathbb{P}_{\omega_2}$ is complete. If $\bar{p} \in \mathbb{Q}_\alpha$, where $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle$ we write $\mathrm{fin}(\bar{p})$ for $\langle s, F, s^*\rangle$ and $\inf(\bar{p})$ for $\langle c_k, y_k\rangle_{k\in\omega}$. In particular $\mathrm{fin}(\bar{p})_0 = s$.

Lemma 3.1. *For every condition $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle \in \mathbb{Q}_\alpha$ and every $\gamma \in \omega_1$ there exists a sequence $\langle d_k, z_k\rangle_{k\in\omega}$ such that $\bar{q} = \langle\langle s, F, s^*\rangle, \langle d_k, z_k\rangle_{k\in\omega}\rangle \in \mathbb{Q}_\alpha$, $\bar{q} \leq \bar{p}$ and for all $k \in \omega$ we have that $|z_k|, \max d_k \geq \gamma$.*

Proof. As in [2, Lemma 1.1]. □

Lemma 3.2. *For every $p \in \mathbb{Q}_\alpha$ and every dense open set $D \subseteq \mathbb{Q}_\alpha$, there is $q \leq p$ such that fin(q) = fin(p) and for every $p_1 \in D$, $p_1 \leq q$ there is $p_2 \in D$, $p_2 \leq q$ such that fin$(p_2)_0$ = fin$(p_1)_0$ and* $\inf(p_2) = \inf(q)$.

Proof. Let $p = \langle\langle t_0, F_0, t_0^*\rangle, \langle d_k^0, z_k^0\rangle_{k\in\omega}\rangle$. Let $\mathcal{M}$ be a countable elementary submodel of L_Θ, for Θ a sufficiently large regular cardinal, which contains $\mathbb{Q}_\alpha$, $\bar{p}$, X_α, D as elements and such that $j = \mathcal{M} \cap \omega_1 \notin \bigcup_{k\in\psi[t_0]} S_{\alpha+k}$. Let $\langle \bar{r}_n, s_n\rangle_{n\in\omega}$ enumerate all pairs $\langle \bar{r}_n, s_n\rangle$ where $\bar{r}_n \in \mathbb{Q}_\alpha \cap \mathcal{M}$, s_n is a finite partial injective function from ω to ω and each pair is enumerated cofinally often. Let $\{j_n\}_{n\in\omega}$ be an increasing sequence which is cofinal in j. Inductively we will construct a decreasing sequence $\langle \bar{p}_n\rangle_{n\in\omega} \subseteq \mathbb{Q} \cap \mathcal{M}$ such that for all n, $\mathrm{fin}(\bar{p}_n) = \mathrm{fin}(\bar{p})$.

Let $\bar{p}_0 = \bar{p}$. Suppose $\bar{p}_n$ has been defined. If there is $\bar{r}_{1,n} \in \mathcal{M} \cap \mathbb{Q}$ such that $\bar{r}_{1,n} \leq \bar{p}_n, \bar{r}_n$ and $\mathrm{fin}(\bar{r}_{1,n}) = s_n$ then extend $\inf(\bar{r}_{1,n})$ to a sequence $\langle d_k^{n+1}, z_k^{n+1}\rangle_{k\in\omega}$ in $\mathcal{M}$ in such a way that for all $k \in \omega$, $\max d_k^{n+1} \geq j_n$, $|z_k^{n+1}| \geq j_n$. Then let $\bar{p}_{n+1} = \langle \mathrm{fin}(\bar{p}_0), \langle d_k^{n+1}, z_k^{n+1}\rangle_{k\in\omega}\rangle$. If there is no such $\bar{r}_{1,n}$, then extend $\inf(\bar{p}_n)$ to a sequence $\langle d_k^{n+1}, z_k^{n+1}\rangle_{k\in\omega}$ in $\mathcal{M}$ such that for all $k \in \omega$, $\max d_k^{n+1} \geq j_n$, $|z_k^{n+1}| \geq j_n$. With this the inductive construction is complete. For every $k \in \omega$, let $d_k = \bigcup_{n\in\omega} d_k^n \cup \{j\}$ and $z_k = \bigcup_{n\in\omega} z_k^n$. Let $q = \langle \mathrm{fin}(\bar{p}), \langle d_k, z_k\rangle_{k\in\omega}\rangle$.

We will show that q is indeed a condition. For this we only need to verify part (4) of being a condition, since the other clauses are clear. Fix $k \in \psi[t_0]$. Let $\mathcal{M}_0$ be a countable suitable model of ZF^- such that $\omega_1^{\mathcal{M}_0} = j$ and z_k, d_k are elements of $\mathcal{M}_0$. Let $\bar{\mathcal{M}}$ be the Mostowski collapse of the model $\mathcal{M}$ and let $\pi : \mathcal{M} \to \bar{\mathcal{M}}$ be the corresponding isomorphism. Note that $j = \omega_1 \cap \mathcal{M} = \omega_1^{\bar{\mathcal{M}}}$. Since $X_\alpha \in \mathcal{M}$ and $\mathcal{M}$ is an elementary submodel of L_Θ, α is the unique solution of $\phi(x, X_\alpha)$ in $\mathcal{M}$. Therefore $\bar{\alpha} = \pi(\alpha)$ is the unique solution of $\phi(x, X_\alpha \cap j) = \phi(x, \pi(X_\alpha))$ in $\bar{\mathcal{M}}$. Note also that $S_{\bar{\alpha}+k}^{\bar{\mathcal{M}}} = \pi(S_{\alpha+k}) = S_{\alpha+k} \cap j$. Since $\omega_1^{\bar{\mathcal{M}}} = \omega_1^{\mathcal{M}_0}$ and $X_\alpha \cap j \in \bar{\mathcal{M}} \cap \mathcal{M}_0$, the solutions of $\phi(x, X_\alpha \cap j)$ in $\bar{\mathcal{M}}$ and $\mathcal{M}_0$ coincide. That is, the solution of $\phi(x, X_\alpha \cap j)$ in $\mathcal{M}_0$ is $\bar{\alpha}$. By the properties of the sequence of stationary sets which we fixed in the ground model, we have $S_{\bar{\alpha}+k}^{\mathcal{M}_0} = S_{\bar{\alpha}+k}^{\bar{\mathcal{M}}} = \pi(S_{\alpha+k}^{\mathcal{M}}) = S_{\alpha+k} \cap j$. Since $d_k \in \mathcal{M}_0$ and d_k is unbounded in j, we obtain that $S_{\bar{\alpha}+k}^{\mathcal{M}_0}$ is not stationary in $\mathcal{M}_0$. Therefore q is indeed a condition.

Consider an arbitrary extension $p_1 = \langle \mathrm{fin}(p_1), \inf(p_1)\rangle$ of $\bar{q}$ from the dense open set D and let $\mathrm{fin}(p_1)_0 = r_1$. Then $\langle r_1, F_0, t_0^*\rangle \in \mathcal{M}$, and so for some m, $\bar{r}^* = \langle\langle r_1, F_0, t_0^*\rangle, \langle d_k^m, z_k^m\rangle_{k\in\omega}\rangle \in \mathbb{Q}_\alpha \cap \mathcal{M}$. Then there is some $n \geq m$ such that $s_n = r_1$, $\bar{r}_n = \bar{r}^*$. Note that $p_1 \leq q, \bar{r}_n$ and so p_1 is a common extension of $\bar{p}_n$, $\bar{r}_n$. By elementarity there is $\bar{r}_{1,n} \in \mathcal{M} \cap D$ which is a common extension of $\bar{p}_n$, $\bar{r}_n$, such that $\mathrm{fin}(\bar{r}_{1,n}) = \langle r_1 = s_n, F_2, r_2^*\rangle$. Let $p_2 := \langle\langle r_1, F_2, r_2^*\rangle, \langle d_k, z_k\rangle_{k\in\omega}\rangle$. Note that $\inf(\bar{p}_{n+1})$ extends $\inf(\bar{r}_{1,n})$ and so $p_2 \leq \bar{r}_{1,n}$, which implies that $p_2 \in D$. Clearly $p_2 \leq q$ and so p_2 is as desired. □

Lemma 3.3. *Let $\mathcal{M}$ be a countable elementary submodel of L_Θ for sufficiently large Θ containing all relevant parameters, $i = \mathcal{M} \cap \omega_1$, $\bar{p} = \langle\langle s, F, s^*\rangle, \langle d^0_k, z^0_k\rangle_{k\in\omega}\rangle$ an element of $\mathcal{M} \cap \mathbb{Q}_\alpha$. If $i \notin \bigcup_{k\in\psi[s]} S_{\alpha+k}$, then there exists an $(\mathcal{M}, \mathbb{Q}_\alpha)$-generic condition $\bar{q} \leq \bar{p}$ such that $fin(\bar{q}) = fin(\bar{p})$.*

Proof. Let $\{D_n\}_{n\in\omega}$ be an enumeration of all dense open subsets of $\mathbb{Q}_\alpha$ from $\mathcal{M}$ and let $\{i_n\}_{n\in\omega}$ be an increasing sequence which is cofinal in i. Inductively, construct a sequence $\langle \bar{q}_n\rangle_{n\in\omega} \subseteq \mathcal{M} \cap \mathbb{Q}_\alpha$ such that $\bar{q}_0 = \bar{p}$, and

(1) for every $n \in \omega$, $\bar{q}_{n+1} \leq \bar{q}_n$, $\mathrm{fin}(q_n) = \mathrm{fin}(\bar{p})$;
(2) if $\inf(q_n) = \langle d^n_k, z^n_k\rangle_{k\in\omega}$ then for all $k \in \omega$, $\max d^n_k \geq i_n$, $|z^n_k| \geq i_n$;
(3) for every $\bar{p}_1 \in D_n$ extending $\bar{q}_n$, there is $\bar{p}_2 \in D_n$ which extends $\bar{q}_n$ and such that $\mathrm{fin}(\bar{p}_2)_0 = \mathrm{fin}(\bar{p}_1)_0$, $\inf(\bar{p}_2) = \inf(\bar{q}_n)$.

Now define a condition $\bar{q}$ such that $\mathrm{fin}(\bar{q}) = \mathrm{fin}(\bar{p})$, $\inf(\bar{q}) = \langle d_k, z_k\rangle_{k\in\omega}$ where $d_k = \bigcup_{n\in\omega} d^n_k \cup \{i\}$, $z_k = \bigcup_{n\in\omega} z^n_k$. To verify that $\bar{q}$ is indeed a condition, proceed as in the proof of q being a condition from Lemma 3.2. Then $\bar{q} \leq \bar{p}$ and we will show that $\bar{q}$ is $(\mathcal{M}, \mathbb{Q}_\alpha)$-generic. For this it is sufficient to show that for every $n \in \omega$, the set $D_n \cap \mathcal{M}$ is predense below $\bar{q}$. Thus fix some $n \in \omega$ and $\bar{p}_1 = \langle\langle t_1, F_1, t^*_1\rangle, \inf(\bar{p}_1)\rangle$ an arbitrary extension of $\bar{q}$. Without loss of generality $\bar{p}_1 \in D_n$. Since $\bar{p}_1 \leq \bar{q}_n$ we obtain the existence of $F_2, t^*_2 \in \mathcal{M}$ such that $\bar{p}_2 = \langle\langle t_1, F_2, t^*_2\rangle, \langle d^n_k, z^n_k\rangle_{k\in\omega}\rangle \leq \bar{q}_n$ and $\bar{p}_2 \in \mathcal{M} \cap D_n$. Then $\bar{p}_3 = \langle\langle t_1, F_1 \cup F_2, t^*_1 \cup t^*_2\rangle, \inf(\bar{p}_1)\rangle$ is a common extension of $\bar{p}_1$ and $\bar{p}_2$. □

Corollary 3.4. For every $\alpha < \omega_2$, the poset $\mathbb{Q}_\alpha$ is S-proper. Consequently, $\mathbb{P}_{\omega_2}$ is S-proper and hence preserves cardinals. More precisely, for every condition $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle\rangle \in \mathbb{Q}^1_\alpha$ the poset $\{\bar{r} \in \mathbb{Q}_\alpha : \bar{r} \leq \bar{p}\}$ is $\omega_1 \backslash \bigcup_{n\in\psi[s]} S_{\alpha+n}$-proper.

3.1. **Properties of $\mathbb{Q} = \mathbb{Q}_\alpha$.** Throughout the subsection, let α be a limit ordinal such that $\omega_1 \leq \alpha < \omega_2$. We study the properties of $\mathbb{Q} := \mathbb{Q}_\alpha$ in $L^{\mathbb{P}_\alpha}$.

Claim 3.5 (Domain Extension)**.** *For every condition $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_m, y_m\rangle_{m\in\omega}\rangle$, natural number n such that $n \notin \mathrm{dom}(s)$ there are co-finitely many $m \in \omega$ such that $\langle\langle s\cup\{(n, m)\}, F, s^*\rangle, \langle c_m, y_m\rangle_{m\in\omega}\rangle$ is a condition extending $\bar{p}$.*

Proof. Fix $\bar{p}$, n as above. By [5, Lemma 2.7] there is a co-finite set I such that for all $m \in I$ $(s \cup \{(n, m)\}, F) \leq_{\mathbb{Q}_{\{\alpha\},\rho_\alpha}} (s, F)$. Since s^* is finite, we can define $N_0 = \max\{f(n) : n \in s^*\}$. Then for every $m \in I\backslash N_0$,

$$\langle\langle s \cup \{(n, m)\}, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle \leq \bar{p}.$$

□

Claim 3.6 (Range Extension)**.** *For any condition $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_m, y_m\rangle_{m\in\omega}\rangle$, natural number $m \notin \mathrm{ran}(s)$ there are co-finitely many $n \in \omega$ such that $\langle\langle s \cup \{(n, m)\}, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle$ is a condition, extending $\bar{p}$.*

Proof. Fix $\bar{p}$, m as above. By [5, Lemma 2.7] there is a co-finite set I such that for all $n \in I$, $(s \cup \{(n, m)\}, F) \leq_{\mathbb{Q}_{\{\alpha\},\rho_\alpha}} (s, F)$. Now for every n, consider the set $A_n = \{f(n)\}_{f\in s^*}$. If there are infinitely many n such that $m \in A_n$ then $\exists f \in s^*\exists^\infty n$ such that $f(n) = m$, which is a contradiction to f being a bijection. That is $\forall^\infty n(m \notin A_n)$. Choose N such that $\forall n \geq N(m \notin A_n)$. Then $\forall n \in I\backslash N(\langle\langle s \cup \{(n, m)\}, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle)$ is an extension of $\bar{p}$ with the desired properties. □

The following claim is straightforward.

Claim 3.7. *For every $w_0 \in \widehat{W}_{\{\alpha\}\cup Lim(\alpha\backslash\omega_1)}$ the set $D_{w_0} = \{\bar{p} \in \mathbb{Q} : w_0 \in fin(\bar{p})_1\}$ is dense.*

Claim 3.8. *Suppose $\bar{q} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle \Vdash_{\mathbb{Q}_\alpha} e_w[\rho_G](n) = n$ for some $w \in \widehat{W}_{\{\alpha\}\cup Lim(\alpha\backslash\omega_1)}$. Then $e_w[s, \rho_\alpha](n)$ is defined and $e_w[s, \rho_\alpha](n) = n$.*

Proof. Let G be $\mathbb{Q}_\alpha$ generic over $L^{\mathbb{P}_\alpha}$ such that $q \in G$. By definition of the extension relation there is a condition $\bar{r} = \langle\langle t, H, t^*\rangle, \langle d_k, z_k\rangle_{k\in\omega}\rangle$ in G such that $e_w[\rho_G](n) = e_w[t, \rho_\alpha](n) = n$. Then $(t, H) \leq_{\mathbb{Q}_{\{\alpha\},\rho_\alpha}} (s, F)$ and since the extension of $\mathbb{Q}_{\{\alpha\},\rho_\alpha}$ does not allow new fixed points we obtain $e_w[s, \rho_\alpha](n) = n$. □

Lemma 3.9. *Let G be $\mathbb{Q}_\alpha$-generic over $L^{\mathbb{P}_\alpha}$ and let $g_\alpha = \bigcup_{\bar{p}\in G} fin(\bar{p})_0$. Then g_α is a cofinitary permutation and $\langle g_\beta\rangle_{\beta\leq\alpha}$ is a cofinitary group.*

Proof. Since for every n, m in ω, the sets $D_n = \{\bar{p} \in \mathbb{Q} : n \in \mathrm{dom}(\mathrm{fin}(\bar{p})_0)\}$, $R_m = \{\bar{p} \in \mathbb{Q}, m \in \mathrm{ran}(\mathrm{fin}(\bar{p})_0)\}$ are dense, it is easy to see than $g = g_\alpha$ is a surjective function. Injectivity follows directly from the properties of $\mathbb{Q}_{\{\alpha\},\rho_\alpha}$ (see [5]), and so g is a permutation.

We will show that the group generated by $\{g_\beta\}_{\beta\in\mathrm{Lim}(\alpha\backslash\omega_1)} \cup \{g_\alpha\}$ is a cofinitary group. Fix an arbitrary word $w \in W_{\{\alpha\}\cup\mathrm{Lim}(\alpha\backslash\omega_1)}$. Then there are $w' \in \widehat{W}_{\{\alpha\}\cup\mathrm{Lim}(\alpha\backslash\omega_1)}$ and $u \in W_{\{\alpha\}\cup\mathrm{Lim}(\alpha\backslash\omega_1)}$ such that $w = u^{-1}w'u$. Since $D_{w'}$ is dense, there is a condition $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle$ in G such that $w' \in F$. Suppose $e_{w'}[\rho_G](n) = n$. Then there is $\bar{q} \in G$, $\bar{q} \leq \bar{p}$ such that $\bar{q} \Vdash e_{w'}[\rho_G](n) = n$. By the above Lemma, $e_{w'}[t, \rho_\alpha](n) = n$, where $\bar{q} = \langle(t, F', t^*), \langle d_k, z_k\rangle_{k\in\omega}\rangle$ and so by the extension relation $e_{w'}[s, \rho_\alpha](n) = n$. Then $\mathrm{fix}(e_{w'}[\rho_G]) = \mathrm{fix}(e_{w'}[s, \rho_\alpha])$ which is finite and so $\mathrm{fix}(e_w[\rho_G])$ is also finite. □

Lemma 3.10 (Generic Hitting)**.** *In $L^{\mathbb{P}_\alpha}$ suppose $\langle\{h\} \cup \{g_\beta\}_{\beta<\alpha}\rangle$ is a cofinitary group and h is not covered by finitely many members of $\mathcal{F}$ with indices above η. Then $L^{\mathbb{P}_{\alpha+1}} \vDash \exists^\infty n \in \omega(g_\alpha(n) = h(n))$.*

Proof. We claim that for every $N \in \omega$, the set $D_N = \{\bar{q} : \exists n \geq N(s(n) = h(n))\}$ is dense in $\mathbb{Q}_\alpha$. Let $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle$ be an arbitrary condition. By [5, Lemma 2.19] there is N such that for all $n \geq N$,

$$(s \cup \{(n, h(n))\}, F) \leq_{\mathbb{Q}_{\{\alpha\},\rho_\alpha}} (s, F).$$

Since h is not covered by the members of s^*, we have that $\exists^\infty n$ such that $h(n) \notin \{f(n)\}_{f\in s^*}$. Denote this set $I_h(\bar{p})$. Let $n \in I_h(\bar{p})\backslash \max\{N_{\bar{p}}, N\}$. Then

$$\bar{q} := \langle\langle s \cup \{(n, h(n))\}, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle \leq \bar{p}$$

and $\bar{q} \in D_N$. Therefore $L^{\mathbb{P}_{\alpha+1}} \vDash \exists^\infty n(g_\alpha(n) = h(n))$. □

Lemma 3.11. *The group $\mathcal{G} := \langle g_\alpha\rangle_{\alpha\in Lim(\omega_2\backslash\omega_1)}$ added by $\mathbb{P}_{\omega_2}$ is a maximal cofinitary group.*

Proof. Suppose $\mathcal{G}$ is not maximal. Then there is a cofinitary permutation h such that

$$\langle\{g_\alpha\}_{\alpha\in\mathrm{Lim}(\omega_2\backslash\omega_1)} \cup \{h\}\rangle$$

is cofinitary. Let $\alpha < \omega_2$ be the least limit ordinal such that $\alpha = \omega_1 \cdot \xi$ for some $\xi \neq 0$ and such that $h \in L^{\mathbb{P}_\alpha}$. Then there is $\eta \geq 0$ such that h is not covered by finitely many members of $\mathcal{F}$ whose second index is above η. Therefore by the Generic Hitting Lemma the poset $\mathbb{Q}_{\omega_1\cdot\xi+\omega\cdot\eta}$ adds a generic permutation $g_{\omega_1\cdot\xi+\omega\cdot\eta}$ which is infinitely often equal to h, which is a contradiction. □

3.2. **Coding.** Let G_α be $\mathbb{Q}_\alpha$-generic filter over $L^{\mathbb{P}_\alpha}$ and let $g_\alpha := \bigcup_{\bar{p}\in G} \operatorname{fin}(\bar{p})_0$. For every $k \in \psi[g_\alpha]$ define $Y^\alpha_k := \bigcup_{\bar{p}\in G_\alpha} \inf(\bar{p})_1$, $C^\alpha_k := \bigcup_{\bar{p}\in G_\alpha} \inf(\bar{p})_0$ and $S^* := \bigcup_{\bar{p}\in G_\alpha} \operatorname{fin}(\bar{p})_2$. Let $G := G_{\omega_2}$.

The following is clear using easy extendibility arguments together with Lemmas 3.1, 3.5, 3.6.

Lemma 3.12. *The sets Y^α_k, C^α_k, and S^* have the following properties:*

- $S^* = \{f_{\langle m,\xi\rangle} : m \in \psi[g_\alpha], \xi \in C^\alpha_m\} \cup \{f_{\langle \omega+m,\xi\rangle} : m \in \psi[g_\alpha], Y^\alpha_m(\xi) = 1\}$.
- *If $m \in \psi[g_\alpha]$ then* $\operatorname{dom}(Y^\alpha_m) = \omega_1$ *and C^α_m is a club in ω_1 disjoint from $S_{\alpha+m}$.*
- *If $m \in \psi[g_\alpha]$ then $|g_\alpha \cap f_{\langle m,\xi\rangle}| < \omega$ if and only if $\xi \in C^\alpha_m$.*
- *If $m \in \psi[g_\alpha]$ then $|g_\alpha \cap f_{\langle \omega+m,\xi\rangle}| < \omega$ if and only if $Y^\alpha_m(\xi) = 1$.*

Corollary 3.13. Let $n \in \omega\backslash\psi[g_\alpha]$. Then $S_{\alpha+n}$ remains stationary in $L^{\mathbb{P}_{\omega_2}}$.

Proof. Let G be $\mathbb{P}_{\omega_2}$-generic over L and let $p \in G$ such that $p \Vdash \beta \notin \{\alpha + n : n \in \psi[g_\alpha]\}$. Then G is also $\mathbb{P}_{\omega_2}(p)$-generic, where $\mathbb{P}_{\omega_2}(p) := \{q : q \leq p\}$ is the countable support iteration of $\mathbb{Q}_\gamma(p(\gamma))$ for $\gamma < \omega_2$. However for every γ, the poset $\mathbb{Q}_\gamma(p(\gamma))$ is S_β-proper and so the entire iteration is S_β-proper. $\square$

Lemma 3.14. *In $L[G]$ let $A = \{g_\alpha : \omega_1 \leq \alpha < \omega_2, \alpha$ limit$\}$. Then $g \in A$ if and only if for every countable suitable model M of ZF^- containing g as an element there exists a limit ordinal $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+k}$ is non-stationary in M for all $k \in \psi[g]$.*

Proof. The proof is analogous to that of [4, Lemma 13]. Let $g \in A$. Find $\alpha < \omega_2$ such that $g = g_\alpha$, and let M be a countable suitable model containing g as an element. Then $C^\alpha_k \cap \omega_1^M$, $Y^\alpha_k \restriction \omega_1^M$ are elements of M for all $k \in \psi[g_\alpha]$. Fix any $m \in \psi[g_\alpha]$. Then there is $\bar{p} = \langle\langle s, F, s^*\rangle, \langle c_k, y_k\rangle_{k\in\omega}\rangle \in G$ such that $m \in \psi[s]$ and $C^\alpha_m \cap \omega_1^{\mathcal{M}} = c_m$, $Y^\alpha_m \cap \omega_1^{\mathcal{M}} = y_m$. By definition of being a condition we obtain that

$$\mathcal{M} \vDash Y^\alpha_m \cap \omega_1^{\mathcal{M}} \text{ codes a limit ordinal } \bar{\alpha}_m \text{ such that } S_{\bar{\alpha}_m+m} \text{ is not stationary.}$$

Note that for every distinct m_1, m_2 in $\psi[g_\alpha]$ we have that $Y^\alpha_{m_1} \cap \omega_1^{\mathcal{M}} = Y^\alpha_{m_2} \cap \omega_1^{\mathcal{M}}$, and so $\bar{\alpha}_m$ does not depend on m.

To see the other implication, fix g such that for every countable suitable model containing g as an element there exists $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+k}$ is non-stationary in M for all $k \in \psi[g]$. By the Löwenheim-Skolem theorem the same holds for arbitrary suitable models of ZF^- containing g. In particular this holds in $M = L_\Theta[G]$ for some sufficiently large Θ, say $\Theta > \omega_{100}$. Then $\omega_2^M = \omega_2^{L[G]} = \omega_2^L$, $\bar{S}^M = \bar{S}$, and the notions of stationarity of subsets of ω_1 coincide in M and $L[G]$. Thus there is a limit ordinal $\alpha < \omega_2$ such that $S_{\alpha+k}$ is non-stationary for all $k \in \psi[g]$. By the above corollary for every $\beta \notin \{\alpha + k : k \in \psi[g_\alpha]\}$ the set S_β is stationary. Therefore $\psi[g] \subseteq \psi[g_\alpha]$ and so $g = g_\alpha$. $\square$

Thus as the right-hand side of the equivalence stated in Lemma 3.14 is Π^1_2, we obtain:

Theorem 3.15. *There is a generic extension of the constructible universe in which $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there is a maximal cofinitary group with a Π^1_2-definable set of generators.*

4. Remarks

We expect that the techniques of [3] can be modified to produce a generic extension of the constructible universe in which $\mathfrak{b} = \mathfrak{c} = \aleph_3$ and there is a maximal cofinitary group with a Π^1_2-definable set of generators. Of interest remains the following question: Is it consistent that there is a Π^1_2 definable maximal cofinitary group and $\mathfrak{b} = \mathfrak{c} = \aleph_2$?

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