FRESH FUNCTION SPECTRA

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Abstract. In this paper, we investigate the fresh function spectrum of forcing notions, where a new function on an ordinal is called fresh if all its initial segments are in the ground model. We determine the fresh function spectrum of several forcing notions and discuss the difference between fresh functions and fresh subsets. Furthermore, we consider the question which sets are realizable as the fresh function spectrum of a homogeneous forcing. We show that under GCH all sets with a certain closure property are realizable, while consistently there are sets which are not realizable.

1. Introduction

The distributivity of a forcing is the smallest ordinal δ such that the forcing adds a new function from δ to the ordinals. Clearly, on any ordinal larger than δ, a new function is added as well. The situation becomes non-trivial if we ask on which ordinals a fresh function is added. We give the definition for arbitrary models $V \subseteq W$:

Definition 1.1. Let $V \subseteq W$ be models of ZFC with the same ordinals. For an ordinal δ, a function $f \colon \delta \to Ord$ in W is a *fresh function on δ over V* if

(1) $f \notin V$, and
(2) $f \restriction \gamma \in V$ for any $\gamma < \delta$.

This concept has been considered, e.g., in [Ham01], and also in the context of guessing models (see, e.g., [CK18, Definition 3.3]). The term used in [CK18] is that (V, W) has the *weak δ-approximation property*[1] if (in W) there exists no fresh function on δ over V.

Now we define the central notion of the paper, which to the best of our knowledge has not been considered before. Let $(\mathbb{P}, \leq)$ be any (non-atomic) forcing notion.[2]

Definition 1.2. The *fresh function spectrum of* $\mathbb{P}$ (denoted by $\mathsf{FRESH}(\mathbb{P})$) is the set of regular cardinals λ such that in some generic extension by $\mathbb{P}$, there exists a fresh function on λ (over the ground model V).

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[1] A stronger version is the *δ-approximation property*, which actually implies the weak γ-approximation property for every regular $\gamma \geq \delta$.

[2] From now on, we will only talk about forcing extensions, even though the following and several other concepts (e.g., in Section 5) could be formulated as statements about arbitrary models $V \subseteq W$, as we did in Definition 1.1.

To understand the behaviour of a forcing extension with respect to fresh functions, it is enough to restrict the attention to regular cardinals, because the existence of fresh functions on ordinals is only a matter of their cofinality (see Proposition 5.2).

Clearly, the *distributivity* of $\mathbb{P}$, denoted by $\mathfrak{h}(\mathbb{P})$, is the minimum of the fresh function spectrum $\mathsf{FRESH}(\mathbb{P})$, and $\mathbb{P}$ is *δ-distributive* if $\delta < \mathfrak{h}(\mathbb{P})$; in particular, the distributivity number $\mathfrak{h}$ equals the minimum of $\mathsf{FRESH}(\mathcal{P}(\omega)/fin)$.

Note that, even though the definition of the fresh function spectrum as stated in Definition 1.2 talks about all generic extensions of the ground model by $\mathbb{P}$, it can actually be rephrased as a statement in the ground model: a regular cardinal λ is in $\mathsf{FRESH}(\mathbb{P})$ if and only if there exist $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ such that $p \Vdash$ "$\dot{f}\colon \lambda \to Ord$ is a fresh function".

For the rest of the paper, let us fix the following notation: Let *RegCard* denote the class of infinite regular cardinals, and for $\alpha, \beta \in Ord$, let

$$[\alpha, \beta]_{Reg} := \{\lambda \in RegCard \mid \alpha \leq \lambda \leq \beta\},$$

and define $[\alpha, \beta)_{Reg}$ analogously.

In Section 2, we argue that $\mathbb{P}$ having a certain chain condition or $\mathbb{P} \times \mathbb{P}$ having a certain chain condition implies that certain cardinals do not belong to the fresh function spectrum of $\mathbb{P}$. The same is implied by $\mathbb{P}$ having the Y-c.c.. Furthermore, we consider two-step iterations of forcings with a chain condition and sufficiently closed forcings, connecting to work of Usuba [Usu] and Hamkins [Ham01]. In particular it follows that it is possible to collapse a cardinal without adding a fresh function on this cardinal. In Section 3, we show that cardinals which get collapsed to the distributivity of a forcing belong to its fresh function spectrum and use this to determine the fresh function spectra of collapse forcings, including the Lévy collapse.

In Section 4, we consider the question which sets are realizable as the fresh function spectrum of a homogeneous forcing under GCH. We introduce the notion of Easton closure (see Definition 4.8) and show that any Easton closed set is the fresh function spectrum of an Easton product of Cohen forcings, and that the fresh function spectrum of any such product is Easton closed; in fact, we show the following (see Theorem 4.24):

Theorem. *Assume GCH, and let A be a set of regular cardinals. Then $\mathsf{FRESH}({}^{E}\prod_{\alpha \in A} \mathbb{C}(\alpha))$ is equal to the Easton closure of A.*

In Section 4.7, we show that it is possible that the fresh function spectrum with respect to a single generic extension is not in the ground model. In Section 5, we discuss the relation between fresh functions and fresh subsets. Furthermore, we analyze the general structure of the class of ordinals which have a fresh subset and argue that it is enough to consider indecomposable ordinals.

In Section 6, we show that consistently not all sets are realizable as the fresh function spectrum of a forcing notion. In fact, we show the following (see Theorem 6.3):

Theorem. *Assume that Todorčević's maximality principle holds, and $0^{\#}$ does not exist. If $\mathbb{P}$ is a forcing such that $\omega_1 \in \mathsf{FRESH}(\mathbb{P})$, then $\omega \in \mathsf{FRESH}(\mathbb{P})$ or $\omega_2 \in \mathsf{FRESH}(\mathbb{P})$. In particular, there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$.*

In Section 7, we determine the fresh function spectra of several forcing notions. First, we consider $\mathcal{P}(\omega)/\text{fin}$, $\mathcal{P}(\kappa)/{<}\kappa$, Mathias forcing and Silver forcing, and then we show the following general theorem about Miller-like tree forcings, which include Sacks forcing, Miller forcing, full Miller forcing, and Namba forcing (see Theorem 7.11):

Theorem. *If λ is a cardinal, $\mathbb{P} \subseteq \mathcal{P}(\lambda^{<\omega})$ is a Miller-like forcing and $\delta > \lambda^{<\omega}$ is regular uncountable, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

We also consider Laver forcing and variants of it, using their Y-properness, and remark on minimality of forcing notions. Finally, we consider Namba forcing and Prikry forcing, which are our main examples of forcings which add fresh functions on cardinals for which they do not add fresh subsets.

Finally, in Section 8, we consider refining matrices, which give rise to a combinatorial distributivity spectrum of forcing notions; we compare it with the fresh function spectrum, and we show that each Easton closed set is the combinatorial distributivity spectrum of a homogeneous forcing, by again using an Easton product of Cohen forcings.

2. Chain conditions

In this section, we discuss the connection between certain variants of the chain condition and the fresh function spectrum. In particular two strengthenings of the usual chain condition of a forcing $\mathbb{P}$, the chain condition of $\mathbb{P}\times\mathbb{P}$ and the Y-c.c., are helpful to compute the fresh function spectrum. Furthermore, we consider two-step iterations of forcings with a chain condition and sufficiently closed forcings, connecting to work of Usuba [Usu] and Hamkins [Ham01]. The results of this section are mostly well-known or slight generalizations of known facts.

It is easy to see that no regular cardinal strictly above the size of $\mathbb{P}$ belongs to $\mathsf{FRESH}(\mathbb{P})$: if there were such a fresh function, then all its initial segments are in the ground model, so they can be decided by conditions in $\mathbb{P}$; by cardinality, one condition appears cofinally often, hence forces the entire function to be in the ground model. This also directly follows from the more general fact on the connection of $\mathsf{FRESH}(\mathbb{P})$ and the chain condition of $\mathbb{P}\times\mathbb{P}$ (see Proposition 2.2 and Corollary 2.3).

As an example let us look at Cohen forcings. For a regular cardinal α, let $\mathbb{C}(\alpha)$ denote *α-Cohen forcing*, i.e., the set of partial functions from α to 2 of size strictly smaller than α, ordered by reverse inclusion. If α is inaccessible or GCH holds, $|\mathbb{C}(\alpha)| = \alpha$, therefore its fresh function spectrum is easy to compute (for the general case, see Proposition 6.1):

Proposition 2.1. *Let α be a regular cardinal and assume that $2^{<\alpha} = \alpha$. Then $\mathsf{FRESH}(\mathbb{C}(\alpha)) = \{\alpha\}$.*

Proof. First observe that $\mathbb{C}(\alpha)$ adds an α-Cohen real, which is a fresh function on α. By assumption, $|\mathbb{C}(\alpha)| = |2^{<\alpha}| = \alpha$, so by the above discussion (see also Corollary 2.3) no cardinal above α belongs to the fresh function spectrum. On the other hand, $\mathbb{C}(\alpha)$ is ${<}\alpha$-closed and hence γ-distributive for every $\gamma < \alpha$, so no $\gamma < \alpha$ belongs to the fresh function spectrum. □

The following result generalizes a well-known fact about branches of certain trees, which was essentially proved by Mitchell in [Mit73, Lemma 3.8] (see also [Mit70]). A slightly stronger generalization of Mitchell's result yielding the δ-approximation property was proved in [Ung13]. For the convenience of the reader we give a detailed proof.

Proposition 2.2. *If $\mathbb{P} \times \mathbb{P}$ has the χ-c.c. and $\delta \geq \chi$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

Proof. Assume $\delta \in \mathsf{FRESH}(\mathbb{P})$, i.e., there exists $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ such that p forces $\dot{f} \colon \delta \to Ord$ is not in V and $\dot{f} \restriction \gamma \in V$ for each $\gamma < \delta$. Therefore, we can, by induction on $i < \chi$, construct $\alpha_i < \delta$, $p_i \leq p$, and $q_i \leq p$ such that p_i and q_i decide $\dot{f}$ up to α_i, and α_i is the first point on which p_i and q_i disagree; more precisely, there is $s_i \colon \alpha_i + 1 \to Ord$ and $t_i \colon \alpha_i + 1 \to Ord$ such that

(1) $\alpha_j < \alpha_i$ for each $j < i$,
(2) $p_i \Vdash \dot{f} \restriction (\alpha_i + 1) = s_i$,
(3) $q_i \Vdash \dot{f} \restriction (\alpha_i + 1) = t_i$,
(4) $s_i \neq t_i$, and $s_i \restriction \alpha_i = t_i \restriction \alpha_i$.

Consider $\langle (p_i, q_i) \mid i < \chi \rangle$ and use that $\mathbb{P} \times \mathbb{P}$ has the χ-c.c. to obtain $i_0 < i_1$ such that (p_{i_0}, q_{i_0}) and (p_{i_1}, q_{i_1}) are compatible, and fix $(\bar{p}, \bar{q})$ with $(\bar{p}, \bar{q}) \leq (p_{i_0}, q_{i_0})$ and $(\bar{p}, \bar{q}) \leq (p_{i_1}, q_{i_1})$. It follows that both $\bar{p}$ and $\bar{q}$ force that $\dot{f} \restriction \alpha_{i_1} = s_{i_1} \restriction \alpha_{i_1}$. Moreover, $\bar{p} \Vdash \dot{f} \restriction (\alpha_{i_0} + 1) = s_{i_0}$ and $\bar{q} \Vdash \dot{f} \restriction (\alpha_{i_0} + 1) = t_{i_0}$, but $s_{i_0} \neq t_{i_0}$, which easily yields (using $\alpha_{i_0} < \alpha_{i_1}$) a contradiction. □

Since a forcing of size θ has the θ^+-c.c., this immediately yields the above mentioned fact:

Corollary 2.3. *If $|\mathbb{P}| = \theta$ and $\delta > \theta$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

Recall that if $\mathbb{P}$ is χ-Knaster, then $\mathbb{P} \times \mathbb{P}$ has the χ-c.c.; therefore, we also get the following:

Corollary 2.4. *If $\mathbb{P}$ is χ-Knaster and $\delta \geq \chi$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

As an example, letting $\mathbb{C}_\mu$ be the forcing adding μ-many (ω-)Cohen reals, Corollary 2.4 yields that $\mathsf{FRESH}(\mathbb{C}_\mu) = \{\omega\}$.

Note that if (and only if) there exists a χ-Suslin tree, the χ-c.c. of a forcing itself is not sufficient to obtain the conclusion of Proposition 2.2: Let T be a χ-Suslin tree. It has the χ-c.c., yet $\chi \in \mathsf{FRESH}(T)$ since forcing with the tree adds a new branch which is a fresh function on χ (indeed, $\mathsf{FRESH}(T) = \{\chi\}$, due to the fact that T is $<\chi$-distributive and of size χ). On the other hand, if there exists no χ-Suslin tree and $\mathbb{P}$ has the χ-c.c., then it already follows that $\chi \notin \mathsf{FRESH}(\mathbb{P})$: Let $p \in \mathbb{P}$, and assume $p \Vdash$ "$\dot{f} \colon \chi \to Ord$ is fresh"; then

$$\{g \in Ord^{<\chi} \mid \exists q \leq p \text{ with } q \Vdash g \subseteq \dot{f}\}$$

(the interpretation tree of $\dot{f}$) is a χ-Suslin tree.

However, the χ-c.c. of the forcing $\mathbb{P}$ is always sufficient to obtain a conclusion slightly weaker than the one in Proposition 2.2. This is a generalization of a folklore result about branches of certain trees (see, e.g., [Kun11, Exercise V.4.21]). A further generalization yielding the δ-approximation property has been proved in [Usu].

Proposition 2.5. *If $\mathbb{P}$ has the χ-c.c. and $\delta > \chi$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

The proof of the proposition is based on a lemma about the non-existence of very thin Aronszajn trees, which was first proved by Kurepa (see [Kan09, Proposition 7.9]):

Lemma 2.6. *Let δ be a regular cardinal and $\chi < \delta$. Then each tree of height δ all whose levels are of size less than χ has a cofinal branch.*

In fact, we will use the following stronger lemma, which has a similar proof (see also [Kun11, hint of Exercise III.6.32]). We say that a tree is *well-pruned* if each node can be extended to arbitrarily high levels.

Lemma 2.7. *Let δ be a regular cardinal and $\chi < \delta$. Then for each well-pruned tree T of height δ all whose levels are of size less than χ there exists $\gamma < \delta$ such that T does not split above level γ.*

Proof. Note that we can assume that χ is infinite (because otherwise the statement is trivial). In particular, it follows that δ is uncountable.

First assume χ is regular. For every $\alpha < \delta$ with $\mathrm{cf}(\alpha) = \chi$ consider the αth level T_α of T. For each two nodes $\eta, \xi \in T_\alpha$ for which $\{\tau \in T \mid \tau <_T \eta\} \neq \{\tau \in T \mid \tau <_T \xi\}$ let $\beta_{\eta,\xi} < \alpha$ be such that there exist $\tau_\eta \neq \tau_\xi$ in $T_{\beta_{\eta,\xi}}$ with $\tau_\eta <_T \eta$ and $\tau_\xi <_T \xi$. Let $f(\alpha) := \sup_{\eta,\xi \in T_\alpha} \beta_{\eta,\xi}$. Since $|T_\alpha| < \chi = \mathrm{cf}(\alpha)$ it follows that $f(\alpha) < \alpha$. So $f : \{\alpha \in \delta \mid \mathrm{cf}(\alpha) = \chi\} \to \delta$ is a regressive function on a stationary set. By Fodor's Theorem there exists a stationary subset X on which f is constant with value $\gamma < \delta$. Since X is cofinal in δ and T is well-pruned, it easily follows that T does not split above level γ.

Now assume that χ is singular. Let $\{\nu_i \in \mathit{RegCard} \mid i < \mathrm{cf}(\chi)\}$ be cofinal in χ. It follows that for every level T_α there exists i such that $|T_\alpha| < \nu_i$. So, by the pigeonhole principle, there exists a regular $\nu < \chi$ such that $|T_\alpha| < \nu$ for unboundedly many $\alpha < \delta$. Consider the set $T' := \bigcup\{T_\alpha \mid |T_\alpha| < \nu\}$, which is a well-pruned tree of height δ all whose levels are of size less than ν. Since ν is regular, there exists (by what we proved above) a level above which T' does not split. This implies (again using that T is well-pruned) that the same holds for the tree T. □

We now derive the following lemma from which Proposition 2.5 will easily follow. Moreover, we will use the lemma later (see Theorem 7.11).

Lemma 2.8. *Let $\mathbb{P}$ be a forcing and $\dot{f}$ a $\mathbb{P}$-name, δ regular with $\delta > \chi$, and $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{f} \colon \delta \to Ord$ is fresh". Then there exists $\beta < \delta$ such that*

$$|\{g \mid \exists q \leq p \text{ with } q \Vdash \dot{f} \restriction \beta = g\}| \geq \chi. \tag{1}$$

Proof. Assume $|\{g \mid \exists q \leq p \text{ with } q \Vdash \dot{f} \restriction \beta = g\}| < \chi$ for every $\beta < \delta$, i.e., the levels of the interpretation tree $T := \{g \in Ord^{<\delta} \mid \exists q \leq p \text{ with } q \Vdash g \subseteq \dot{f}\}$ are all of size smaller than χ. Since $\dot{f}$ is forced to be fresh (and hence all proper initial segments are in V), T is well-pruned. By Lemma 2.7, there exists some $\gamma < \delta$ such that T does not split above level γ. Let $g \colon \gamma + 1 \to Ord$ and $q \leq p$ be such that $q \Vdash g \subseteq \dot{f}$. Since the interpretation tree does not split above level γ, the whole function $\dot{f}$ is already decided by q. So $q \Vdash$ "$\dot{f}$ is in V", contradicting the fact that $p \Vdash$ "$\dot{f}$ is fresh". □

Proof of Proposition 2.5. Assume $p \in \mathbb{P}$ is a condition, $\dot{f}$ is a name such that p forces that $\dot{f}$ is a fresh function from a regular cardinal δ into the ordinals, and $\chi < \delta$. So, by Lemma 2.8, we can fix $\beta < \delta$ such that (1) holds. Observe that for any two distinct g, g' and q, q' such that $q \Vdash \dot{f} \restriction \beta = g$ and $q' \Vdash \dot{f} \restriction \beta = g'$, the conditions q and q' are incompatible. Therefore, there is an antichain of size χ below p, contradicting the χ-c.c. of $\mathbb{P}$. □

Recall that the ω_1-Cohen real added by $\mathbb{C}(\omega_1)$ is a fresh function over V. However, if this forcing is the second step of a two-step iteration after, e.g., (ω-)Cohen forcing $\mathbb{C}$, the ω_1-Cohen real is not fresh over

V any more (since by a density argument, the ω-Cohen real appears in the ω_1-Cohen real). In fact, the following has been shown in [Ham01, Key Lemma]:

Proposition 2.9. *Assume that $\mathbb{P}$ is non-atomic and $|\mathbb{P}| \leq \beta$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a $\leq\beta$-strategically closed forcing and $\delta > \beta$. Then $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$.*

Using [Usu, Lemma 1.5], the above fact for closed forcings can be generalized to the following:

Proposition 2.10. *Assume that $\mathbb{P}$ is non-atomic, has the χ-c.c., and $\chi \notin \mathsf{FRESH}(\mathbb{P})$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a $<\chi$-closed forcing and $\delta \geq \chi$. Then $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$.*

Proof. The assumption for $\mathbb{P}$ in particular implies that $\mathbb{P}$ has the strong[3] χ-c.c.. Therefore, by [Usu, Lemma 1.5], $\mathbb{P} * \dot{\mathbb{Q}}$ has the χ-approximation property, which easily implies that $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$ for each $\delta \geq \chi$. □

Note that the assumption on $\mathbb{P}$ in Proposition 2.9 is stronger than the assumption in Proposition 2.10, while the assumption on $\mathbb{Q}$ is stronger in Proposition 2.10. It is not so clear how to reconcile the proof strategies of these two propositions in order to only need the weaker assumptions on both $\mathbb{P}$ and $\mathbb{Q}$. For $\chi = \omega_1$, however, we can use a different approach to show that the two weaker assumptions are sufficient:

Proposition 2.11. *Assume that $\mathbb{P}$ is non-atomic, has the c.c.c., and $\omega_1 \notin \mathsf{FRESH}(\mathbb{P})$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a σ-strategically closed forcing and $\delta \geq \omega_1$. Then $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$.*

Proof. A close examination of the proof of [Usu, Lemma 1.5] shows that, in Proposition 2.10, the assumption of $\mathbb{Q}$ being $<\chi$-closed can be weakened to $\mathbb{Q}$ having a dense $<\chi$-closed subset.

By [Jec84, Addenda, 5], a forcing $\mathbb{Q}$ is σ-strategically closed if and only if there exists a forcing $\mathbb{R}$ such that $\mathbb{Q} \times \mathbb{R}$ has a dense σ-closed subset. So we can fix $\dot{\mathbb{R}}$ such that $\mathbb{P}$ forces that $\dot{\mathbb{Q}} * \check{\dot{\mathbb{R}}}$ has a dense σ-closed subset. Now use the above mentioned stronger version of Proposition 2.10 for $\chi = \omega_1$ to conclude that $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}} * \check{\dot{\mathbb{R}}})$ for any $\delta \geq \omega_1$. Since any fresh function in the extension by $\mathbb{P} * \dot{\mathbb{Q}}$ remains fresh in any further extension (see Lemma 4.5), it follows that $\delta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$ for any $\delta \geq \omega_1$, as desired. □

In the hypothesis of the above proposition, the σ-strategic closure cannot be weakened to ω-distributivity, as can be seen by the following counterexample. Assume that there exists a Suslin tree T in V. Note that T is still a Suslin tree in $V[\mathbb{C}]$, i.e., $\Vdash_{\mathbb{C}}$ "$\check{T}$ is a Suslin tree" (this is due to the fact that each set of ground model objects of size ω_1 in $V[\mathbb{C}]$ contains a set of size ω_1 in V). Since $\mathsf{FRESH}(T) = \{\omega_1\}$ and $\mathbb{C} * \check{T}$ is forcing equivalent to $\mathbb{C} \times T$, it follows[4] that $\mathsf{FRESH}(\mathbb{C} * \check{T}) = \{\omega, \omega_1\}$. Since a Suslin tree is always ω-distributive, this shows that σ-strategically closed in Proposition 2.11 cannot be replaced by ω-distributive.

Another strengthening of the c.c.c. is the Y-c.c. as defined in [CZ15]. As the property that $\mathbb{P} \times \mathbb{P}$ has the c.c.c. (see Proposition 2.2), also $\mathbb{P}$ being Y-c.c. is sufficient for ensuring that only ω is in the fresh function spectrum of $\mathbb{P}$. The same holds for the more general notion of Y-properness, also defined in [CZ15].

Proposition 2.12. *If $\mathbb{P}$ is Y-c.c. (or Y-proper) and $\delta \geq \omega_1$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

[3] $\mathbb{P}$ has the strong χ-c.c. if $\mathbb{P}$ has the χ-c.c. and does not add new cofinal branches to χ-Suslin trees.

[4] For details, see the discussion after Example 4.4.

Proof. This follows[5] easily from [CZ15, Theorem 2.8] (or, in case $\mathbb{P}$ is only Y-proper, from [CZ15, Theorem 4.1(1)]). □

Note that "$\mathbb{P}\times\mathbb{P}$ is c.c.c." does not imply "$\mathbb{P}$ is Y-c.c." (with random forcing being a counterexample). On the other hand, consistently, there exists a forcing $\mathbb{P}$ which is Y-c.c., while $\mathbb{P}\times\mathbb{P}$ is not c.c.c. (an example can be constructed[6] from variants of the partition-type forcing discussed in [CZ15, Corollary 3.5]). Note that such a forcing cannot exist in ZFC because Y-c.c. implies c.c.c., and under MA every c.c.c. forcing is Knaster and therefore its square is c.c.c..

3. Collapsing cardinals

In this section, we discuss the connection between collapsing cardinals and fresh functions. Let us start with the following well-known easy fact:

Proposition 3.1. *If a cardinal $\delta > \lambda$ is collapsed to λ (i.e., $|\delta| = \lambda$ in the extension), then there is a new subset of λ.*

The following lemma shows that cardinals which are collapsed to the distributivity of the forcing are in the fresh function spectrum:

Lemma 3.2. *Let λ be a regular cardinal and $\mathbb{P}$ a forcing which collapses λ to $\mathfrak{h}(\mathbb{P})$. Then $\lambda \in \mathsf{FRESH}(\mathbb{P})$.*

Note that it is possible to collapse a cardinal λ (to a cardinal larger than the distributivity) without adding a fresh function on λ, which can be seen, e.g., using Proposition 2.9 with $\mathbb{P} = \mathbb{C}$ and $\dot{\mathbb{Q}} = Coll(\omega_1, \lambda)$, where $Coll(\mu, \lambda)$ denotes the forcing collapsing λ to μ by conditions of size $< \mu$.

Proof of Lemma 3.2. We can assume that $\lambda > \mathfrak{h}(\mathbb{P})$ in the ground model. In the extension by $\mathbb{P}$, we have $|\lambda| = \mathfrak{h}(\mathbb{P})$; in particular, $\mathrm{cf}(\lambda) \leq \mathfrak{h}(\mathbb{P})$, i.e., there exists $\alpha \leq \mathfrak{h}(\mathbb{P})$ and $g\colon \alpha \to \lambda$ cofinal, strictly increasing. Since λ is regular in the ground model, g is a new function in the extension. Now consider the function $k\colon \lambda \to 2$ which maps i to 1 if and only if $i \in \mathrm{rng}(g)$. Since g can be redefined from k, and g is new, k is new as well. Assume there exists $\delta < \lambda$ with $k{\restriction}\delta$ new. Since $\mathrm{rng}(g)$ is cofinal in λ, it follows that $g{\restriction}\beta$ is new for some $\beta < \alpha$. But $\beta < \mathfrak{h}(\mathbb{P})$, so there are no new functions on β, a contradiction. It follows that k is fresh, hence $\lambda \in \mathsf{FRESH}(\mathbb{P})$. □

Recall that $[\mu, \lambda]_{Reg}$ denotes $\{\delta \in RegCard \mid \mu \leq \delta \leq \lambda\}$, and note that from the hypothesis of Lemma 3.2 we actually get that

$$[\mathfrak{h}(\mathbb{P}), \lambda]_{Reg} \subseteq \mathsf{FRESH}(\mathbb{P}). \tag{2}$$

Also note that (2) holds even if λ is singular and $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$.

In the following two propositions we compute the fresh function spectra of $Coll(\mu, \lambda)$ and of the Lévy collapse.

Proposition 3.3. *Assume GCH. Let μ be regular, let λ be a cardinal with $\mathrm{cf}(\lambda) \geq \mu$, and let $\mathbb{P} := Coll(\mu, \lambda)$. Then $\mathsf{FRESH}(\mathbb{P}) = [\mu, \lambda]_{Reg}$.*

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[5]In fact, they show that such forcings have the ω_1-approximation property.

[6]This was personal communication with David Chodounský.

Proof. First note that $\mathfrak{h}(\mathbb{P}) = \mu$, since $\mathbb{P}$ is $<\mu$-closed (and the generic surjection from μ to λ is a fresh function on μ). By Lemma 3.2 (see also (2) above), we have $\mathsf{FRESH}(\mathbb{P}) \supseteq [\mu, \lambda]_{Reg}$. Using GCH and the assumption that $\mathrm{cf}(\lambda) \geq \mu$, it is straightforward to check that $|\mathbb{P}| = \lambda$; so, by Corollary 2.3, no regular cardinal strictly above λ belongs to $\mathsf{FRESH}(\mathbb{P})$, finishing the proof. □

Let $Coll(\mu, <\lambda)$ denote the *Lévy collapse* turning λ into μ^+.

Proposition 3.4. *Let μ be regular, let $\lambda > \mu$ be an inaccessible cardinal, and let $\mathbb{P} := Coll(\mu, <\lambda)$. Then $\mathsf{FRESH}(\mathbb{P}) = [\mu, \lambda)_{Reg}$.*

Proof. Again, $\mathfrak{h}(\mathbb{P}) = \mu$, since $\mathbb{P}$ is $<\mu$-closed. It follows by Lemma 3.2 that $\mathsf{FRESH}(\mathbb{P}) \supseteq [\mu, \lambda)_{Reg}$. Recall that $\mathbb{P}$ can be represented as ${}^{bd}\prod_{i\in\lambda} \mathbb{P}_i$ with $|\mathbb{P}_i| < \lambda$. Since ${}^{bd}\prod_{i\in\lambda} \mathbb{P}_i \times {}^{bd}\prod_{i\in\lambda} \mathbb{P}_i = {}^{bd}\prod_{i\in\lambda}(\mathbb{P}_i \times \mathbb{P}_i)$, it follows from [Jec03, Theorem 15.17(iii)] that $\mathbb{P} \times \mathbb{P}$ has the λ-c.c.; so, by Proposition 2.2, no regular cardinal greater or equal λ belongs to $\mathsf{FRESH}(\mathbb{P})$, finishing the proof. □

4. Realizing fresh function spectra under GCH

If we consider the fresh function spectrum of arbitrary forcing notions, we can ask which sets of regular cardinals are possible fresh function spectra. In this section, we explore the situation under GCH. However, when using lottery sums, any set can be realized as long as the notion of fresh function spectrum from Definition 1.2 is used (see Proposition 4.1). Therefore, to ask a proper question, we either have to modify the definition by focusing on a single generic extension (see Definition 4.2), or restrict ourselves to homogeneous forcings (see Proposition 4.3).

For each regular cardinal α, the fresh function spectrum of the α-Cohen forcing $\mathbb{C}(\alpha)$ is $\{\alpha\}$ under GCH. We will employ the forcings $\mathbb{C}(\alpha)$ as building blocks, using products. In particular, we compute the fresh function spectrum of arbitrary Easton products of such Cohen forcings, which allows us to realize fresh function spectra for a quite large class of sets of regular cardinals (see Definition 4.8 and Theorem 4.24). In particular, using a theorem from pcf theory, we show that successors of singular limits cannot be avoided (see Section 4.4); we also show that regular limits cannot be avoided, unless they are Mahlo (see Section 4.3).

We also argue that some more sets of regular cardinals are realizable as a fresh function spectrum of a forcing, if we allow the forcing to collapse cardinals, and ask open questions (see Section 4.6). Finally, we show that the fresh function spectrum with respect to a single generic extension might be not in the ground model (see Section 4.7).

4.1. **Lottery sums and homogeneity.**

As mentioned above, lottery sums yield arbitrary fresh function spectra under GCH in a trivial way:

Proposition 4.1. *Assume GCH, and let A be a set of regular cardinals. Let $\mathbb{P}$ be the* lottery sum *of the forcings $\mathbb{C}(\alpha)$ with $\alpha \in A$, i.e., the disjoint union of the partial orders $\mathbb{C}(\alpha)$ with $\alpha \in A$ together with an additional weakest element. Then $\mathsf{FRESH}(\mathbb{P}) = A$.*

Proof. Let $\alpha \in A$, and $p \in \mathbb{P}$ which belongs to $\mathbb{C}(\alpha)$. Then, for any generic filter G which contains p, there exists a fresh function on α in $V[G]$. On the other hand, for $\beta \neq \alpha$ there is no fresh function on β in $V[G]$. □

Let's now look at a refined version of the definition of fresh function spectrum:

Definition 4.2. Assume G is a $\mathbb{P}$-generic filter over V. Let $\mathsf{FRESH}(\mathbb{P}, G)$ be the set of regular cardinals λ such that $V[G] \models$ "there exists a fresh function on λ".

Note that $\lambda \in \mathsf{FRESH}(\mathbb{P})$ if and only if $\lambda \in \mathsf{FRESH}(\mathbb{P}, G)$ for some $\mathbb{P}$-generic filter G.

We say that a forcing $\mathbb{P}$ is *homogeneous*, if for each $p, q \in \mathbb{P}$, there is an automorphism $\varphi \colon \mathbb{P} \to \mathbb{P}$ (i.e., a bijection preserving the order) such that $\varphi(p)$ and q are compatible.

Proposition 4.3. *If $\mathbb{P}$ is homogeneous, then $\mathsf{FRESH}(\mathbb{P}) = \mathsf{FRESH}(\mathbb{P}, G)$ for any generic filter G.*

Proof. Let $\lambda \in \mathsf{FRESH}(\mathbb{P})$ then there exists $p \in \mathbb{P}$ such that $p \Vdash$ "there exists a fresh function on λ". We will show that the set of conditions forcing this is actually dense. Let $q \in \mathbb{P}$. Since $\mathbb{P}$ is homogeneous, we can fix an automorphism $\varphi \colon \mathbb{P} \to \mathbb{P}$ such that $\varphi(p)$ is compatible with q. Then $\varphi(p) \Vdash$ "there exists a fresh function on λ". Take $r \leq \varphi(p), q$. Then r forces "there exists a fresh function on λ". Consequently, for any generic filter G, there is a fresh function on λ in $V[G]$, i.e., $\lambda \in \mathsf{FRESH}(\mathbb{P}, G)$. □

Since the forcings we are going to provide are homogeneous anyway, we will always write $\mathsf{FRESH}(\mathbb{P})$ (instead of $\mathsf{FRESH}(\mathbb{P}, G)$) in the following.

4.2. **Basic definitions and facts about Easton products.** Before investigating possible fresh function spectra of products of Cohen forcings in more detail (see in particular Theorem 4.24), let us consider the following special case as an example. It shows that the fresh function spectrum is not always an interval.

Example 4.4. *Assume GCH. Then $\mathsf{FRESH}(\mathbb{C}(\omega) \times \mathbb{C}(\omega_2)) = \{\omega, \omega_2\}$.*

To see that ω and ω_2 are in the fresh function spectrum of the above product, note that a product $\mathbb{P} \times \mathbb{Q}$ is equivalent to the two-step iterations $\mathbb{P} * \check{\mathbb{Q}}$ and $\mathbb{Q} * \check{\mathbb{P}}$. Recall that $\mathsf{FRESH}(\mathbb{C}(\omega)) = \{\omega\}$ and $\mathsf{FRESH}(\mathbb{C}(\omega_2)) = \{\omega_2\}$ (see Proposition 2.1). So both ω and ω_2 are in the fresh function spectrum of the product, since any function which is fresh over a model V stays fresh over V in any further extension:[7]

Lemma 4.5. $\mathsf{FRESH}(\mathbb{P}) \subseteq \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$.

Proof. Assume $\lambda \in \mathsf{FRESH}(\mathbb{P})$, and let f be a witness, i.e., f is a fresh function on λ over V. Clearly, $f \in V[\mathbb{P} * \dot{\mathbb{Q}}]$, and f is still fresh over V, since $f \notin V$, and $f \restriction \delta \in V$ for every $\delta < \lambda$. □

On the other hand, no unexpected cardinals appear in the fresh function spectrum of the iteration. Since $\mathbb{C}(\omega)$ in the extension by $\mathbb{C}(\omega_2)$ is forced to be $\mathbb{C}(\omega)$ of the ground model, the following lemma finishes the computation of $\mathsf{FRESH}(\mathbb{C}(\omega) \times \mathbb{C}(\omega_2))$ (letting $\mathbb{P} = \mathbb{C}(\omega_2)$ and $\dot{\mathbb{Q}} = \check{\mathbb{C}}(\omega)$):

Lemma 4.6. *Assume that $\beta \notin \mathsf{FRESH}(\mathbb{P})$ and $\mathbb{P} \Vdash \beta \notin \mathsf{FRESH}(\dot{\mathbb{Q}})$. Then $\beta \notin \mathsf{FRESH}(\mathbb{P} * \dot{\mathbb{Q}})$.*

Proof. Assume towards a contradiction that in $V[\mathbb{P} * \dot{\mathbb{Q}}]$, there is a fresh function f on β over V. Since $\beta \notin \mathsf{FRESH}(\mathbb{P})$, f is not an element of $V[\mathbb{P}]$, so it must be added by $\mathbb{Q}$ over $V[\mathbb{P}]$. It follows that f is fresh over $V[\mathbb{P}]$. But this is not possible, because $\mathbb{P}$ forces that $\beta \notin \mathsf{FRESH}(\dot{\mathbb{Q}})$, a contradiction. □

[7]In the following lemma and in Lemma 4.6, the analogous statements involving fixed generic extensions as in Definition 4.2 hold true as well.

Let us recall several kinds of products of forcing notions:

Definition 4.7. Let A be a set of regular cardinals, and for every $\alpha \in A$, let $\mathbb{P}_\alpha$ be a forcing.

(1) Let ${}^{full}\prod_{\alpha\in A}\mathbb{P}_\alpha$ denote the *full support product* of the $\mathbb{P}_\alpha$, and let ${}^{bd}\prod_{\alpha\in A}\mathbb{P}_\alpha$ denote the *bounded support product* (i.e., the set of all $p \in {}^{full}\prod_{\alpha\in A}\mathbb{P}_\alpha$ with $\mathrm{supp}(p)$ bounded in $\sup(A)$).
(2) Let ${}^{E}\prod_{\alpha\in A}\mathbb{P}_\alpha$ denote the *Easton product* of the $\mathbb{P}_\alpha$, i.e., the set of all conditions $p \in {}^{full}\prod_{\alpha\in A}\mathbb{P}_\alpha$ with $\mathrm{supp}(p)\cap\lambda$ bounded in λ for each inaccessible λ (i.e., bounded support at regular limits, and full support everywhere else).

We will now introduce two notions which will turn out to be useful in the context of Easton products. These notions are inspired by the notion of "Easton set" as defined in [Dor89, Definition 2.3].

Definition 4.8. A set A of regular cardinals is *Easton closed* if for every limit point λ of A,

- λ regular, not Mahlo $\Rightarrow \lambda \in A$,
- λ singular $\Rightarrow \lambda^+ \in A$.

The *Easton closure* of a set A is the smallest Easton closed set which contains A as a subset.

We will show below (see Theorem 4.24) that, under GCH, for any set A of regular cardinals the fresh function spectrum of ${}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ is equal to the Easton closure of A. The following lemma is similar to [Dor89, Lemma 2.4].

Lemma 4.9. *Assume GCH. Let A be a set of regular cardinals, and for each $\alpha \in A$, let $\mathbb{P}_\alpha$ be a forcing with $|\mathbb{P}_\alpha| \le \alpha$. Let A^* be the Easton closure of A. Then $\left|{}^{E}\prod_{\alpha\in A}\mathbb{P}_\alpha\right| \le \sup(A^*)$.*

We will sometimes abuse notation by indexing an Easton product by cardinals not connected to the size of the forcings; but it should always be clear how to rewrite it in a suitable way.

Proof of Lemma 4.9. In case $\sup(A)$ is a singular cardinal, we have

$$\left|{}^{E}\prod\nolimits_{\alpha\in A}\mathbb{P}_\alpha\right| \le \prod_{\alpha\in A}|\mathbb{P}_\alpha| \le \prod_{\alpha\in A}\alpha \le \sup(A)^{|A|} \le \sup(A)^{\sup(A)} = \sup(A)^+,$$

and $\sup(A)^+ = \sup(A^*)$. In case $\sup(A)$ is a regular limit cardinal (i.e., an inaccessible), the Easton product has bounded support at $\sup(A)$, and we get

$$\left|{}^{E}\prod\nolimits_{\alpha\in A}\mathbb{P}_\alpha\right| \le \sum_{\beta<\sup(A)}\prod_{\alpha\in A\cap\beta}|\mathbb{P}_\alpha| \le \sum_{\beta<\sup(A)}|\beta|^{|\beta|} = \sup(A) = \sup(A^*).$$

Finally, if $\sup(A)$ is a successor cardinal (which is the maximal element of A), then $\left|{}^{E}\prod_{\alpha\in A}\mathbb{P}_\alpha\right| \le \prod_{\alpha\in A}\alpha = \sup(A) = \sup(A^*)$. □

Using the above, the following is easy to see (see for example the proof of [Jec03, Theorem 15.18]):

Proposition 4.10. *Assume GCH. Let $\mathbb{P} := {}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ be an Easton product of Cohen forcings. Then $\mathbb{P}$ does not collapse cardinals (and does not change cofinalities).*

Fact 4.11. *Let λ be a cardinal, $A \subseteq \lambda$ unbounded and $C \subseteq \lambda$ a club. Then there exists $A' \subseteq A$ unbounded in λ such that all limit points of A' are in C.*

Proof. For every $\beta \in C$ let $\varphi(\beta) \in A$ be minimal with $\varphi(\beta) \geq \beta$. Let $A' := \{\varphi(\beta) \mid \beta \in C\}$. Clearly every limit point of A' is a limit point of C, so since C is club, it is an element of C. □

Proposition 4.12. *Let λ be a cardinal which is not Mahlo, and let $A \subseteq \lambda$ be an unbounded subset of regular cardinals. Then there exists an $A' \subseteq A$ unbounded in λ such that ${}^E\prod_{\alpha \in A} \mathbb{C}(\alpha)$ is equivalent to a two-step iteration which starts*

- *with ${}^{bd}\prod_{\alpha \in A'} \mathbb{C}(\alpha)$ in case λ is regular, and*
- *with ${}^{full}\prod_{\alpha \in A'} \mathbb{C}(\alpha)$ in case λ is singular.*

Proof. Let $C \subseteq \lambda$ be a club[8] which does not contain inaccessible cardinals (which exists since λ is not Mahlo). By Fact 4.11, there exists an unbounded subset $A' \subseteq A$ such that all limit points of A' are (in C and hence) not inaccessible.

So if λ is regular, we have ${}^E\prod_{\alpha \in A'} \mathbb{C}(\alpha) = {}^{bd}\prod_{\alpha \in A'} \mathbb{C}(\alpha)$, because the Easton product has bounded support at λ in this case, and no limit points of A' are inaccessible and hence for every $\beta < \lambda$ the product has full support below β.

If λ is singular, we have ${}^E\prod_{\alpha \in A'} \mathbb{C}(\alpha) = {}^{full}\prod_{\alpha \in A'} \mathbb{C}(\alpha)$, because the Easton product has full support at λ in this case, and again no limit points of A' are inaccessible and hence also for every $\beta < \lambda$ the product has full support below β.

The product ${}^E\prod_{\alpha \in A} \mathbb{C}(\alpha)$ is equivalent to a two-step iteration which starts with ${}^E\prod_{\alpha \in A'} \mathbb{C}(\alpha)$, and the conclusion follows. □

4.3. **Fresh functions at regular limits.** Let us recall the following easy and well-known fact about bounded support products:

Proposition 4.13. *Let λ be a regular limit cardinal, $A \subseteq \lambda$ an unbounded set of regular cardinals, and $\mathbb{P}_\alpha$ a non-atomic forcing for each $\alpha \in A$. Then ${}^{bd}\prod_{\alpha \in A} \mathbb{P}_\alpha$ adds a λ-Cohen real.*

Proof. For each $\alpha \in A$, let $p_\alpha \in \mathbb{P}_\alpha$ be such that there exists a condition in $\mathbb{P}_\alpha$ which is incompatible with p_α. Let G be generic for ${}^{bd}\prod_{\alpha \in A} \mathbb{P}_\alpha$. Fix an increasing enumeration $\varphi \colon \lambda \to A$ in the ground model. Now let $r \in 2^\lambda$ be such that $r(\alpha) = 0$ if and only if there is a condition $q \in G$ with $q(\varphi(\alpha)) = p_{\varphi(\alpha)}$. It is easy to see that r is a λ-Cohen real, since the product has bounded support. □

Since the Easton product below the least inaccessible is the bounded support product, we immediately obtain:

Example 4.14. Let λ be the least inaccessible. Then ${}^E\prod_{\alpha \in \lambda \cap RegCard} \mathbb{C}(\alpha)$ adds a λ-Cohen real (in particular, a fresh function on λ).

The following well-known fact can be found in Kanamori's book [Kan09, Proposition 6.2]:

Proposition 4.15. *Let λ be a Mahlo cardinal and $P \subseteq V_\lambda$ be a unary predicate. Then there exists an inaccessible cardinal $\alpha < \lambda$ such that $(V_\alpha, P \cap V_\alpha)$ is an elementary substructure of (V_λ, P).*

[8] Note that, since each Mahlo is regular, C in particular exists in case λ is a singular cardinal. If $\mathrm{cf}(\lambda) = \omega$, we can pick any unbounded set of order-type ω, and the rest of the proof clearly works.

Lemma 4.16. *Let λ be a Mahlo cardinal and let $A \subseteq \lambda$ be a set of regular cardinals. Then $\lambda \notin$ FRESH$({}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha))$.*

Proof. Let $\mathbb{P} := {}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha)$. Assume towards a contradiction that in the extension by $\mathbb{P}$ there exists a fresh function on λ. Since $\mathbb{P}$ has the λ^+-c.c., we can assume without loss of generality that the range of the fresh function is a subset of λ (by using a bijection between a cover of the range and λ). Let $\dot{f}$ be a nice name for such a fresh function from λ to λ. Note that $\mathbb{P}$ and $\dot{f}$ are subsets of V_λ, so we can consider the structure $(V_\lambda, \mathbb{P}, \dot{f})$ with $\mathbb{P}$ and $\dot{f}$ being unary predicates on V_λ. By Proposition 4.15, there exists an inaccessible cardinal $\alpha < \lambda$ such that $(V_\alpha, \mathbb{P}\cap V_\alpha, \dot{f}\cap V_\alpha)$ is elementary in $(V_\lambda, \mathbb{P}, \dot{f})$. In particular $(V_\alpha, \mathbb{P}\cap V_\alpha, \dot{f}\cap V_\alpha) \models \mathbb{P}\cap V_\alpha \Vdash \dot{f}\cap V_\alpha$ is a fresh function on α.

Since α is inaccessible, ${}^{E}\prod_{\beta\in A\cap\alpha}\mathbb{C}(\beta)$ has bounded support in α. Moreover $\mathbb{C}(\beta) \subseteq V_\alpha$, therefore $\mathbb{P}\cap V_\alpha = {}^{E}\prod_{\beta\in A\cap\alpha}\mathbb{C}(\beta) =: \mathbb{P}_{<\alpha}$. Since $\mathbb{P}_{<\alpha}$ is a complete subforcing of $\mathbb{P}$, we have $(\dot{f}\cap V_\alpha)^{G\cap\mathbb{P}_{<\alpha}} = \dot{f}^G\restriction\alpha$ for any generic filter G for $\mathbb{P}$. By the above, $(\dot{f}\cap V_\alpha)^{G\cap\mathbb{P}_{<\alpha}}$ is new, contradicting the fact that $\dot{f}^G$ is fresh. □

Example 4.17. Let λ be a Mahlo cardinal. Then ${}^{E}\prod_{\alpha\in\lambda\cap RegCard}\mathbb{C}(\alpha)$ does not add a fresh function on λ (in particular, no λ-Cohen real).

4.4. **Pcf theory: fresh functions at successors of singular limits.** We will use a theorem of Shelah to show that the full support product of Cohen forcings at a singular cardinal λ adds a λ^+-Cohen real. We start with a few well-known facts of pcf theory (see, for example, [Jec03]):

Proposition 4.18. *Let A be a set of regular cardinals. Then the following holds.*

(1) $\mathrm{pcf}(A) \subseteq RegCard$.
(2) For each $\mu \in \mathrm{pcf}(A)$, *we have* $\mu \leq |\prod A|$.

Recall that for a linear order $(L, <)$, the true cofinality $\mathrm{tcf}(L)$ is the minimal size of a cofinal subset of L, which is always a regular cardinal.

Proposition 4.19. *Assume GCH. Let $A \subseteq RegCard$ with $\sup(A) =: \lambda$ being a singular cardinal and let $\mathcal{D}$ be an ultrafilter on A which does not contain any bounded subset of A. Then $\mathrm{tcf}(\prod A/\mathcal{D}) = \lambda^+$.*

Proof. $|\prod A| \leq \lambda^+$, so $\mathrm{tcf}(\prod A/\mathcal{D}) \leq \lambda^+$.

Let $\delta < \lambda$ and $\langle f_\alpha \mid \alpha < \delta\rangle$ be a sequence in $\prod A$. Define $g \in \prod A$ as follows: for $\beta \in A$ let $g(\beta) = 0$ if $\beta \leq \delta$ and $g(\beta) = \sup_{\alpha<\delta}(f_\alpha(\beta) + 1)$ if $\beta > \delta$. Since $\delta < \beta$ and β regular, $\sup_{\alpha<\delta}(f_\alpha(\beta) + 1) < \beta$, so g is well-defined. Every co-bounded subset of A is in $\mathcal{D}$, so $g >_{\mathcal{D}} f_\alpha$ for every $\alpha < \delta$. Therefore $\langle f_\alpha \mid \alpha < \delta\rangle$ is not cofinal in $\prod A/\mathcal{D}$ and hence $\mathrm{tcf}(\prod A/\mathcal{D}) \geq \lambda$. Since λ is singular, $\mathrm{tcf}(\prod A/\mathcal{D}) > \lambda$ and together with $\mathrm{tcf}(\prod A/\mathcal{D}) \leq \lambda^+$ it follows that $\mathrm{tcf}(\prod A/\mathcal{D}) = \lambda^+$. □

Theorem 4.20. *Let A be a set of regular cardinals, $\lambda = \sup(A) \notin A$, $2^{<\nu} = 2^\lambda$, $\nu > \lambda$, $\nu \in \mathrm{pcf}(A)$, and moreover there is an ultrafilter $\mathcal{D}$ on A not containing any bounded subset of A such that $\nu = \mathrm{tcf}(\prod A/\mathcal{D})$. Then the forcing ${}^{full}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ adds a ν-Cohen real.*

Proof. See [She00]. □

Corollary 4.21. *Assume GCH. Let λ be a singular cardinal and let $A \subseteq \lambda$ be an unbounded subset of regular cardinals. Then ${}^{full}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ adds a λ^+-Cohen real.*

Proof. It follows by Proposition 4.19 and Theorem 4.20 (letting $\nu = \lambda^+$) that ${}^{full}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ adds a λ^+-Cohen real. ◻

Example 4.22. Assume GCH. Then ${}^{E}\prod_{n\in\omega}\mathbb{C}(\aleph_{2n})$ adds an $\aleph_{\omega+1}$-Cohen real (in particular, a fresh function on $\aleph_{\omega+1}$).

Remark 4.23. The bounded support product ${}^{bd}\prod_{\alpha\in A}\mathbb{C}(\alpha)$ has size $\sup(A) =: \lambda$ (under GCH), so nothing above λ belongs to its fresh function spectrum (see Corollary 2.3), in particular, λ^+ does not. However, for singular λ, there are two problems with this "solution" to avoid λ^+: first, the forcing collapses cardinals (namely λ to $\mathrm{cf}(\lambda)$), and second, more importantly, this results in $[\mathrm{cf}(\lambda), \lambda)_{Reg}$ being contained in the fresh function spectrum (see[9] Lemma 3.2 and the discussion thereafter), so only co-bounded sets of regular cardinals can be realized in such a way. For more on that and related open questions, see Section 4.6.

4.5. **Realizing any Easton closed set.** We now prove our main result about Easton products of Cohen forcings.

Theorem 4.24. *Assume GCH, and let A be a set of regular cardinals. Then* $\mathsf{FRESH}({}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha))$ *is equal to the Easton closure of A.*

Proof. Let $\mathbb{P}$ denote ${}^{E}\prod_{\alpha\in A}\mathbb{C}(\alpha)$. By Proposition 2.1, $\mathsf{FRESH}(\mathbb{C}(\alpha)) = \{\alpha\}$. So, using Lemma 4.5, it is easy to see that any $\alpha \in A$ is in $\mathsf{FRESH}(\mathbb{P})$, because $\mathbb{P}$ can be seen as an iteration starting with $\mathbb{C}(\alpha)$.

Assume now that λ is a regular cardinal which is a limit point of A, yet not Mahlo. By Proposition 4.12, there exists a set $A' \subseteq A \cap \lambda$ unbounded in λ, such that ${}^{E}\prod_{\alpha\in A\cap\lambda}\mathbb{C}(\alpha)$ is equivalent to a two-step iteration which starts with ${}^{bd}\prod_{\alpha\in A'}\mathbb{C}(\alpha)$. By Proposition 4.13, ${}^{bd}\prod_{\alpha\in A'}\mathbb{C}(\alpha)$ adds a λ-Cohen real. Therefore, $\mathbb{P}$ adds a λ-Cohen real, so in particular, $\lambda \in \mathsf{FRESH}(\mathbb{P})$.

Now assume that λ is a singular cardinal which is a limit point of A (in particular λ is not Mahlo). Again, by Proposition 4.12, there exists a set $A' \subseteq A \cap \lambda$ unbounded in λ, such that ${}^{E}\prod_{\alpha\in A\cap\lambda}\mathbb{C}(\alpha)$ is equivalent to a two-step iteration which starts with ${}^{full}\prod_{\alpha\in A'}\mathbb{C}(\alpha)$. By Corollary 4.21, ${}^{full}\prod_{\alpha\in A'}\mathbb{C}(\alpha)$ adds a λ^+-Cohen real. Therefore, $\mathbb{P}$ adds a λ^+-Cohen real, so in particular, $\lambda^+ \in \mathsf{FRESH}(\mathbb{P})$.

We finish the proof by showing that no regular cardinal outside of the Easton closure of A belongs to $\mathsf{FRESH}(\mathbb{P})$. Let A^* denote the Easton closure of A. By Lemma 4.9 $|\mathbb{P}| \leq \sup(A^*)$, so by Corollary 2.3, no cardinal above $\sup(A^*)$ is in $\mathsf{FRESH}(\mathbb{P})$.

Now let $\beta \leq \sup(A^*)$ be a regular cardinal with $\beta \notin A^*$. Let $\mathbb{P}_{<\beta}$ denote ${}^{E}\prod_{\alpha\in A\cap\beta}\mathbb{C}(\alpha)$ and let $\mathbb{P}_{>\beta}$ denote ${}^{E}\prod_{\alpha\in A\setminus\beta}\mathbb{C}(\alpha)$. Since $\beta \notin A$ we have $\mathbb{P} = \mathbb{P}_{>\beta} \times \mathbb{P}_{<\beta}$. Note that $\mathbb{P}_{>\beta}$ is $\leq\beta$-closed, so $\beta \notin \mathsf{FRESH}(\mathbb{P}_{>\beta})$, and $(\mathbb{P}_{<\beta})^V = (\mathbb{P}_{<\beta})^{V[\mathbb{P}_{>\beta}]}$. The latter implies that $\mathbb{P} = \mathbb{P}_{>\beta} * \mathbb{P}_{<\beta}$. To finish the proof that $\beta \notin \mathsf{FRESH}(\mathbb{P})$, by Lemma 4.6 it suffices to show that $\beta \notin \mathsf{FRESH}(\mathbb{P}_{<\beta})$ (in the extension by $\mathbb{P}_{>\beta}$). In case β is not Mahlo, it is straightforward to check that $\sup(A^* \cap \beta) < \beta$, so, since the Easton closure of $A \cap \beta$ is $A^* \cap \beta$, we can apply Lemma 4.9 to obtain that $|\mathbb{P}_{<\beta}| < \beta$. By Corollary 2.3, $\beta \notin \mathsf{FRESH}(\mathbb{P}_{<\beta})$. In case β is Mahlo, we apply Lemma 4.16 to obtain that $\beta \notin \mathsf{FRESH}(\mathbb{P}_{<\beta})$. ◻

By definition, an Easton closed set coincides with its Easton closure, so it follows that we can realize every Easton closed set as a fresh function spectrum (see also Proposition 4.10):

[9]Since the lemma has to be applied to a forcing with distributivity $\mathrm{cf}(\lambda)$, one has to first split the product into a lower and an upper part, if necessary.

Corollary 4.25. *Assume GCH. For every set A which is Easton closed, there is a homogeneous forcing $\mathbb{P}$ (which does not collapse cardinals or change cofinalities) such that $\mathsf{FRESH}(\mathbb{P}) = A$.*

In particular, this yields examples of forcings $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P})$ being any given finite set of regular cardinals (since finite sets are always Easton closed).

4.6. **Questions.** In view of Corollary 4.25, it is natural to ask whether any set which is not Easton closed can ever be realized as the fresh function spectrum of a forcing (under GCH).

Recall that the Easton product of Cohen forcings as discussed above does not collapse cardinals (or change cofinalities). It is not clear to us how to realize a set which is not Easton closed without collapsing cardinals:

Conjecture 4.26. *Assume GCH, and assume that $\mathbb{P}$ does not collapse cardinals. Then*[10] *$\mathsf{FRESH}(\mathbb{P}, G)$ is Easton closed for any generic filter G.*

However, when allowing to collapse cardinals, it is indeed possible to realize some more sets. If λ is a singular cardinal and μ is a regular cardinal with $\mu \leq \mathrm{cf}(\lambda)$, then – under GCH – $\mathsf{FRESH}(Coll(\mu, \lambda)) = [\mu, \lambda]_{Reg} = [\mu, \lambda)_{Reg}$ (see Proposition 3.3). Note that $\mathsf{FRESH}(Coll(\mu, \lambda))$ is not Easton closed, because it does not contain λ^+. If λ is an inaccessible cardinal, and $\mu < \lambda$ is a regular cardinal, then the fresh function spectrum of the Lévy collapse $Coll(\mu, {<}\lambda)$ equals $[\mu, \lambda)_{Reg}$ (see Proposition 3.4). Since $\lambda \notin \mathsf{FRESH}(Coll(\mu, {<}\lambda))$, this fresh function spectrum is not Easton closed, provided that λ is not Mahlo. In particular, any initial segment of the class of regular cardinals can be realized as the fresh function spectrum of a forcing (at least, when we allow to collapse cardinals).

It is possible to combine the above with the method from Theorem 4.24: by taking the product of a collapse as above with an Easton product of Cohen focings on an Easton closed set below μ, it is possible to extend the above fresh function spectra. In this way, certain co-bounded sets can be realized. We do not know, however, if it is possible to have a set as the fresh function spectrum of a forcing which is unbounded and co-unbounded in λ, where λ is a singular cardinal, or an inaccessible which is not Mahlo.

In particular, we do not know whether non-trivial fresh function spectra below $\aleph_\omega$ are possible:

Question 4.27. Let $A \subsetneq \{\aleph_n \mid n \in \omega\}$ be infinite. Is there a forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}, G) = A$ for some generic filter G?

As discussed above, the forcing collapsing $\aleph_\omega$ to ω has $\{\aleph_n \mid n \in \omega\}$ as its fresh function spectrum. We do not know whether this set can be realized by a forcing which does not collapse cardinals.

Question 4.28. Let λ be the least inaccessible, and let $A \subseteq \lambda$ be an unbounded and co-unbounded set of regular cardinals. Is there a forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}, G) = A$ for some generic filter G?

4.7. **Fresh function spectra which are not in the ground model.** We now turn our attention to the notion $\mathsf{FRESH}(\mathbb{P}, G)$ for non-homogeneous forcings. We show that it is possible that $\mathsf{FRESH}(\mathbb{P}, G)$ is not in the ground model, by providing the following example. Note that the forcing $\mathbb{P}$ in the example is not homogeneous, which is necessary by Proposition 4.3.

[10] Note that here we have to use $\mathsf{FRESH}(\mathbb{P}, G)$ instead of $\mathsf{FRESH}(\mathbb{P})$ because of Proposition 4.1.

Example 4.29. *Assume GCH. Let λ be the* least inaccessible*. Let $\mathbb{P}_\alpha$ (for $\alpha < \lambda$) be the* lottery sum *of $\mathbb{C}(\aleph_{\omega\cdot\alpha+2})$ and $\mathbb{C}(\aleph_{\omega\cdot\alpha+3})$, and let $\mathbb{P} := {}^{E}\prod_{\alpha<\lambda}\mathbb{P}_\alpha$. Then for each generic filter G, $\mathsf{FRESH}(\mathbb{P}, G) \notin V$.*

Proof. Fix a generic filter G for $\mathbb{P}$. In $V[G]$, let $f\colon \lambda \to 2$ be the function which records which parts of the lottery sums are chosen by the generic filter, i.e., f is defined by $f(\alpha) = i$ if and only if there exists $p \in G$ with $p(\alpha) \in \mathbb{C}(\aleph_{\omega\cdot\alpha+2+i})$. Clearly this function is well-defined, and it is a λ-Cohen real. We will finish the proof by showing that for $\alpha < \lambda$ and $i \in 2$, we have $\aleph_{\omega\cdot\alpha+2+i} \in \mathsf{FRESH}(\mathbb{P}, G)$ if and only if $f(\alpha) = i$. Indeed this shows that $\mathsf{FRESH}(\mathbb{P}, G)$ is not in the ground model,[11] because f can be derived from it.

By Proposition 2.1, $\mathsf{FRESH}(\mathbb{C}(\alpha)) = \{\alpha\}$ for any regular cardinal α. Let $\alpha \in \lambda$. Note that $G(\alpha) = \{p(\alpha) \mid p \in G\}$ is a generic filter for $\mathbb{P}_\alpha$, which yields an $\aleph_{\omega\cdot\alpha+2+f(\alpha)}$-Cohen real, i.e., a fresh function on $\aleph_{\omega\cdot\alpha+2+f(\alpha)}$. Therefore, since $\mathbb{P}$ can be seen as an iteration starting with $\mathbb{P}_\alpha$, it is easy to see that $\aleph_{\omega\cdot\alpha+2+f(\alpha)} \in \mathsf{FRESH}(\mathbb{P}, G)$.

Fix $\alpha \in \lambda$ and $i \neq f(\alpha)$. We show that $\aleph_{\omega\cdot\alpha+2+i} \notin \mathsf{FRESH}(\mathbb{P}, G)$. Let $\mathbb{P}_{<\alpha}$ denote ${}^{E}\prod_{\beta<\alpha}\mathbb{P}_\beta$, and $\mathbb{P}_{>\alpha}$ denote ${}^{E}\prod_{\beta>\alpha}\mathbb{P}_\beta$. Thus we have $\mathbb{P} = \mathbb{P}_{>\alpha} \times \mathbb{P}_\alpha \times \mathbb{P}_{<\alpha}$. Note that $\mathbb{P}_{>\alpha}$ is $<\aleph_{\omega\cdot(\alpha+1)}$-closed, so $\aleph_{\omega\cdot\alpha+2+i} \notin \mathsf{FRESH}(\mathbb{P}_{>\alpha})$, and $(\mathbb{P}_\alpha)^V = (\mathbb{P}_\alpha)^{V[\mathbb{P}_{>\alpha}]}$. So $\mathbb{P}_{>\alpha} \times \mathbb{P}_\alpha = \mathbb{P}_{>\alpha} * \mathbb{P}_\alpha$. Moreover $\mathbb{P}_{>\alpha} * \mathbb{P}_\alpha$ is $<\aleph_{\omega\cdot\alpha}$-closed, so $(\mathbb{P}_{<\alpha})^V = (\mathbb{P}_{<\alpha})^{V[\mathbb{P}_{>\alpha} * \mathbb{P}_\alpha]}$. Consequently, $\mathbb{P} = \mathbb{P}_{>\alpha} * \mathbb{P}_\alpha * \mathbb{P}_{<\alpha}$.

Assume towards a contradiction that there is a fresh function on $\aleph_{\omega\cdot\alpha+2+i}$ in $V[G]$. Since $\mathbb{P}_{>\alpha}$ is $<\aleph_{\omega\cdot(\alpha+1)}$-closed, it has not been added by $\mathbb{P}_{>\alpha}$. Since $G(\alpha)$ is basically a generic filter for $\mathbb{C}(\aleph_{\omega\cdot\alpha+2+f(\alpha)})$ and $\mathsf{FRESH}(\mathbb{C}(\aleph_{\omega\cdot\alpha+2+f(\alpha)})) = \{\aleph_{\omega\cdot\alpha+2+f(\alpha)}\}$, the fresh function on $\aleph_{\omega\cdot\alpha+2+i}$ has not been added by $\mathbb{P}_\alpha$. Finally $|\mathbb{P}_{<\alpha}| \leq \aleph_{\omega\cdot\alpha+1}$ (see Lemma 4.9 and the subsequent comment), so the fresh function has not been added by $\mathbb{P}_{<\alpha}$ either (see Corollary 2.3). □

5. Fresh functions vs. fresh subsets

Let $(\mathbb{P}, \leq)$ be any (non-atomic) forcing notion.[12] Let δ be an ordinal. A set $A \subseteq \delta$ in the extension by $\mathbb{P}$ over V is a *fresh subset of* δ if $A \notin V$ and $A \cap \alpha \in V$ for each $\alpha < \delta$. If we identify A with its characteristic function, a fresh subset of δ is a fresh function from δ to 2.

Proposition 5.1. *If there is a new function $f\colon \delta \to \delta$, then there is a new subset of δ. If δ is regular and $f\colon \delta \to \delta$ is fresh, then there is a fresh subset of δ.*

Proof. The graph of f can be seen as a subset of $\delta \times \delta$. By using a bijection φ (in V) between $\delta \times \delta$ and δ, this gives a new subset A of δ. Now assume that δ is a regular cardinal. If $\delta = \omega$, then this subset is fresh. If δ is uncountable, note that there exists a club $C \subseteq \delta$ such that $\varphi^{-1}[\alpha] = \alpha \times \alpha$ for each $\alpha \in C$. Assume f is fresh and $\gamma < \delta$. Fix $\alpha \in C$ with $\gamma \leq \alpha$. Hence $A \cap \gamma$ can be derived from $f \restriction \alpha$, which is in V. Therefore A is fresh. □

Before further discussing fresh subsets, let us give a proof of the fact that the existence of fresh functions is only a matter of cofinality:

[11] In fact, $\mathsf{FRESH}(\mathbb{P}, G) = \{\aleph_{\omega\cdot\alpha+2+f(\alpha)} \mid \alpha \in \lambda\} \cup \{\lambda\} \cup \{\aleph_{\omega^2\cdot\beta+1} \mid 0 < \beta < \lambda\}$. Since $\mathbb{P}$ clearly has size λ, no regular cardinal strictly above λ belongs to $\mathsf{FRESH}(\mathbb{P}, G)$ (see Corollary 2.3). On the other hand, $\mathbb{P}$ is σ-closed, so $\omega \notin \mathsf{FRESH}(\mathbb{P}, G)$.

[12] Instead of speaking about forcing extensions, we could (as in Definition 1.1) formulate most of this section in terms of arbitrary models $V \subseteq W$.

Proposition 5.2. *Let δ be an ordinal and $\lambda = \mathrm{cf}(\delta)$. Let $V[G]$ be a forcing extension of V. Then, in $V[G]$, there exists a fresh function on λ if and only if there exists a fresh function on δ.*

Proof. In V, let $\{\beta_i \mid i < \lambda\}$ be an increasing cofinal sequence in δ. Let $f\colon \lambda \to Ord$ be a fresh function over V. Let $g\colon \delta \to Ord$ be defined by $g(\beta_i) = f(i)$ for each $i < \lambda$, and $g(\alpha) = 0$ for $\alpha \notin \{\beta_i \mid i < \lambda\}$. Clearly, g is a fresh function over V.

Conversely, let $g\colon \delta \to Ord$ be a fresh function over V. Fix an ordinal γ such that $g\colon \delta \to \gamma$, and let $\iota\colon \gamma^{<\delta} \to |\gamma^{<\delta}|$ be a bijection in V. Define $f\colon \lambda \to Ord$ by $f(i) = \iota(g \restriction \beta_i)$. It is straightforward to check that f is a fresh function over V. □

For fresh subsets, the situation is different: their existence depends not only on the cofinality of the ordinal. In fact, for the question whether $\mathbb{P}$ adds a fresh subset of an ordinal, we can restrict ourselves to indecomposable ordinals, for the following reason. Assume η is a decomposable ordinal, i.e., there exists $\alpha \in Ord$ and $0 < \beta < \eta$ which is indecomposable with $\eta = \alpha + \beta$. If there is a fresh subset A of β, then $\{\alpha + \gamma \mid \gamma \in A\}$ is a fresh subset of η. On the other hand, if A is a fresh subset of η, then $\{\gamma \mid \alpha + \gamma \in A\}$ is a fresh subset of β.

Proposition 5.3.

(1) If $\mathbb{P}$ adds a fresh subset of δ, then in particular $\mathrm{cf}(\delta) \in \mathsf{FRESH}(\mathbb{P})$.
(2) If $\mathbb{P}$ adds a fresh subset of δ, and $\eta > \delta$ is indecomposable with $\mathrm{cf}(\delta) = \mathrm{cf}(\eta)$, then $\mathbb{P}$ adds a fresh subset of η.
(3) If $\delta \in \mathsf{FRESH}(\mathbb{P})$, then there is an (indecomposable) ordinal $\eta \geq \delta$ with $\mathrm{cf}(\eta) = \delta$ such that $\mathbb{P}$ adds a fresh subset of η.

Proof. (1) The characteristic function of the fresh subset of δ is a fresh function on δ, hence, by Proposition 5.2, $\mathrm{cf}(\delta) \in \mathsf{FRESH}(\mathbb{P})$.
(2) In V, let $\langle \xi_i \mid i < \mathrm{cf}(\delta)\rangle$ be increasing, cofinal in δ. Since $\eta > \delta$ is indecomposable, $\mathrm{ordertype}(\eta \setminus \alpha) = \eta$ for each $\alpha < \eta$. Therefore we can fix $\langle \zeta_i \mid i < \mathrm{cf}(\delta)\rangle$ increasing, cofinal in η such that $\mathrm{ordertype}(\zeta_{i+1} \setminus \zeta_i) \geq \mathrm{ordertype}(\xi_{i+1} \setminus \xi_i)$ for all $i < \mathrm{cf}(\delta)$. Let $A \subseteq \delta$ be fresh, and let $A_i := A \cap (\xi_{i+1} \setminus \xi_i)$. Then we can copy each A_i into $\zeta_{i+1} \setminus \zeta_i$, yielding a fresh subset of η.
(3) Let $f : \delta \to \mu$ be a fresh function, and let $\eta := \mu \cdot \delta$. Note that η is an indecomposable ordinal of cofinality δ. Now $\{(\mu \cdot \alpha) + f(\alpha) \mid \alpha \in \delta\}$ is a fresh subset of η. □

Corollary 5.4. *There exists a sequence $\langle \xi_\lambda \mid \lambda \in \mathsf{FRESH}(\mathbb{P})\rangle$ such that for every indecomposable ordinal η the following holds: $\mathbb{P}$ adds a fresh subset of η if and only if $\mathrm{cf}(\eta) \in \mathsf{FRESH}(\mathbb{P})$ and $\eta \geq \xi_{\mathrm{cf}(\eta)}$.*

For many forcings the following holds: for each ordinal δ, there exists a fresh subset of δ if and only if there exists a fresh function on δ. On the other hand, Prikry forcing and Namba forcing are examples of forcings for which this is not the case (see Theorem 7.21 and Theorem 7.23). For Prikry forcing, the ξ_λ's in the above corollary can be chosen to be $\xi_\omega = \xi_\kappa = \kappa$. For Namba forcing they can be chosen to be $\xi_\omega = \xi_{\omega_1} = \xi_{\omega_2} = \omega_1$. However, it is not always possible to choose all the ξ_λ's to be the same as we show in Example 7.22.

For the next result, we need the following well-known theorem:

Theorem 5.5 (Jensen's Covering Theorem)**.** *If $0^\#$ does not exist, then for every uncountable set Y of ordinals there exists a constructible set $X \supseteq Y$ such that $|X| = |Y|$.*

Using this theorem, we can turn new functions with arbitrary range into new subsets:

Proposition 5.6. *Assume that $0^\#$ does not exist, and let λ be a regular uncountable cardinal. Then the following holds. If $\mathbb{P}$ adds a new function from λ to the ordinals, then $\mathbb{P}$ adds a new subset of λ. In particular, if λ is the distributivity of $\mathbb{P}$, then $\mathbb{P}$ adds a fresh subset of λ.*

Proof. Let $f : \lambda \to Ord$ be a new function. Since $0^\#$ does not exist, we can apply Jensen's Covering Theorem, so we can cover $\mathrm{ran}(f)$ (which is a set of size at most λ in the extension) by a set $X \in V$ such that $|X| = \lambda$ in the extension. Let δ be such that V satisfies $|X| = \delta$. If $\delta > \lambda$, then $\mathbb{P}$ collapses δ to λ, hence it adds a new subset of λ (see Proposition 3.1). If $\delta = \lambda$, we can take a bijection $\iota\colon X \to \lambda$ which is in V, to obtain $\iota \circ f : \lambda \to \lambda$ which is new; therefore, by Proposition 5.1, there is a new subset of λ. □

Question 5.7. Assume that $0^\#$ does not exist, and let λ be a regular uncountable cardinal. If $\mathbb{P}$ adds a fresh function from λ to the ordinals, does $\mathbb{P}$ necessarily add a fresh subset of λ?

The above proposition does not hold without assumptions about the non-existence of large cardinals. For example, Magidor forcing adds a fresh function on ω_1 and no new subset of ω_1, provided that a measurable cardinal of Mitchell order ω_1 exists. Note that Jensen's Covering Theorem does not hold for countable sets, so the above proof does not work for $\lambda = \omega$. In fact, the proposition fails for $\lambda = \omega$ (even if $0^\#$ does not exist): Assuming CH, Namba forcing adds a new function from ω to ω_2, but does not add a new subset of ω (see Theorem 7.23). If there exists a measurable cardinal μ (so $0^\#$ exists), also Prikry forcing is an example: it adds a new cofinal function from ω to μ, without adding any new bounded subset of μ (see Theorem 7.21). However, for proper forcings the following holds.

Proposition 5.8. *If $\mathbb{P}$ is proper and $\omega \in \mathsf{FRESH}(\mathbb{P})$, then $\mathbb{P}$ adds a new real.*

Proof. It is easy to transform the witnessing new function from ω to the ordinals added by $\mathbb{P}$ into a new real, by covering its range by a countable set of the ground model (which is possible by properness) and using Proposition 5.1. □

6. Omitting fresh function spectra

In this section, we want to discuss the question whether it is consistent that for some regular cardinal λ there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\lambda\}$.

In Proposition 2.1, we have seen that in some cases Cohen forcing is such a forcing. In general, Cohen forcing is not good for having a single specific value in the fresh function spectrum:

Proposition 6.1. *Let α be a regular cardinal. Then $\mathsf{FRESH}(\mathbb{C}(\alpha)) = [\alpha, 2^{<\alpha}]_{\mathrm{Reg}}$.*

Proof. Since $\mathfrak{h}(\mathbb{C}(\alpha)) = \alpha$ and $|\mathbb{C}(\alpha)| = 2^{<\alpha}$, it follows from Corollary 2.3 that $\mathsf{FRESH}(\mathbb{C}(\alpha)) \subseteq [\alpha, 2^{<\alpha}]_{\mathrm{Reg}}$. Now note that $\mathbb{C}(\alpha)$ collapses 2^μ to α for each $\mu < \alpha$. If $2^\mu = 2^{<\alpha}$ for some $\mu < \alpha$, the statement follows directly from Lemma 3.2 (and (2) afterwards). The statement is also clear if $\alpha = 2^{<\alpha}$. If $\alpha < 2^{<\alpha}$ and $2^\mu < 2^{<\alpha}$ for every $\mu < \alpha$, then $2^{<\alpha}$ has cofinality α, so it is singular. Then in $[\alpha, 2^{<\alpha})_{\mathrm{Reg}}$, there is a cofinal sequence of cardinals which get collapsed to α. So $\mathsf{FRESH}(\mathbb{C}(\alpha)) = [\alpha, 2^{<\alpha})_{\mathrm{Reg}} = [\alpha, 2^{<\alpha}]_{\mathrm{Reg}}$, again by Lemma 3.2 (and (2) afterwards). □

In particular, for strongly inaccessible λ (or $\lambda = \omega$), we have $\mathsf{FRESH}(\mathbb{C}(\lambda)) = \{\lambda\}$. For successor cardinals, this is the case if GCH at the predecessor holds. Another possibility to get fresh function spectrum $\{\lambda\}$ is a Suslin tree of height λ (as discussed in Section 2).

Let us now concentrate on the case $\lambda = \omega_1$. We will show that it is consistent that there is no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$ (in such a model, CH necessarily fails, and there is no Suslin tree). For that, we will make use of the following principle which is known to be consistent:

Definition 6.2. *Todorčević's maximality principle* is the following assertion: If $\mathbb{P}$ is a forcing which adds a fresh subset of ω_1, then $\mathbb{P}$ collapses ω_1 or ω_2.

For a tree $(T, \leq)$ of height ω_1, we say that T is *special* if there exists a function $f\colon T \to \omega$ such that if $x \leq y, z$ and $f(x) = f(y) = f(z)$, then $y \leq z$ or $z \leq y$. If T has no uncountable branch, then this definition of special is equivalent to the usual one (i.e., to the existence of a function $f\colon T \to \omega$ such that if $x \leq y$, then $f(x) \neq f(y)$). Consider the assertions

(3) $2^{\aleph_0} = \aleph_2$, and every tree of height and size ω_1 is special,

(4) Todorčević's maximality principle,

(5) there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$.

Todorčević showed in [Tod82] that (3) implies (4). In Theorem 6.3 below, we show that if $0^\#$ does not exist then (4) implies (5). We do not know whether the assumption that $0^\#$ does not exist can be omitted. Moreover, we do not know whether (5) implies (4) or (3). Recall that both $2^{\aleph_0} \geq \aleph_2$ as well as "there are no Suslin trees" are necessary for (5), and the same applies to (3) and (4). According to [Tod82, Introduction], the consistency strength of (3) is exactly the existence of an inaccessible. We do not know, however, whether the consistency of (4) or (5) needs an inaccessible.

Theorem 6.3. *Assume that Todorčević's maximality principle holds and $0^\#$ does not exist. If $\mathbb{P}$ is a forcing such that $\omega_1 \in \mathsf{FRESH}(\mathbb{P})$, then $\omega \in \mathsf{FRESH}(\mathbb{P})$ or $\omega_2 \in \mathsf{FRESH}(\mathbb{P})$. In particular, there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$.*

Proof. Let $\mathbb{P}$ be a forcing such that $\omega_1 \in \mathsf{FRESH}(\mathbb{P})$. In case $\omega \in \mathsf{FRESH}(\mathbb{P})$, we are finished, so assume from now on that $\omega \notin \mathsf{FRESH}(\mathbb{P})$ (i.e., ω_1 is the distributivity of $\mathbb{P}$). By Proposition 5.6 (using that $0^\#$ does not exist), $\mathbb{P}$ adds a fresh subset of ω_1. So, by Todorčević's maximality principle, $\mathbb{P}$ collapses ω_1 or ω_2. If $\mathbb{P}$ collapses ω_1, a new subset of ω is added (see Proposition 3.1), which contradicts our case assumption that $\omega \notin \mathsf{FRESH}(\mathbb{P})$. If $\mathbb{P}$ collapses ω_2 (to ω_1), then (since ω_1 is the distributivity of $\mathbb{P}$) we can apply Lemma 3.2 to conclude that $\omega_2 \in \mathsf{FRESH}(\mathbb{P})$. □

Corollary 6.4. *It is consistent, relative to the existence of an inaccessible, that there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$.*

Proof. Assume the existence of an inaccessible. By passing to L, we can assume that $0^\#$ does not exist (and there is still an inaccessible). Now there is a forcing extension in which Todorčević's maximality principle holds (and $0^\#$ does still not exist). By Theorem 6.3, there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$ in this model. □

As mentioned above, we do not know whether the inaccessible is really necessary:

Question 6.5. Does it follow from the consistency of ZFC that consistently there is no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\omega_1\}$?

We do not know whether Theorem 6.3 can be generalized[13] to higher cardinals:

Question 6.6. Is it consistent that there exists a successor cardinal $\lambda > \omega_1$ such that there exists no forcing $\mathbb{P}$ with $\mathsf{FRESH}(\mathbb{P}) = \{\lambda\}$?

7. Fresh function spectra of several forcing notions

In this section, we are going to determine the fresh function spectra of several forcing notions. In Section 7.1 and Section 7.2, we compute the fresh function spectra of $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\kappa)/{<}\kappa$, respectively. Using that, we analyze the fresh function spectra of Mathias forcing and Silver forcing in Section 7.3. In Section 7.4, we prove a general theorem about Miller-like tree forcings, which we then use to determine the fresh function spectra of Sacks forcing, Miller forcing, and full Miller forcing in Section 7.5, where we also consider Laver forcing and its Y-properness. The rest of Section 7.5 is devoted to a discussion of minimality of forcing notions, and its relation to minimality for reals. Finally, in Section 7.6, we consider Namba forcing and Prikry forcing, which are our main examples of forcings for which there are fresh functions on ordinals which do not have fresh subsets.

7.1. $\mathcal{P}(\omega)/$fin.

To compute the fresh function spectrum of $\mathcal{P}(\omega)/fin$, we are going to use Lemma 3.2, together with the fact that $\mathcal{P}(\omega)/\text{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$; this follows from the base matrix theorem.

We say that a forcing $\mathbb{P}$ has a *base matrix* if there exists a (refining) system $\{A_\xi \mid \xi < \mathfrak{h}(\mathbb{P})\}$ of maximal antichains of $\mathbb{P}$ such that $\bigcup_{\xi<\mathfrak{h}(\mathbb{P})} A_\xi$ is dense in $\mathbb{P}$ (see also Definition 8.1). Recall the base matrix theorem which guarantees the existence of base matrices for certain forcings. Most importantly, it holds true for $\mathcal{P}(\omega)/\text{fin}$, which has been proved in [BPS80]; let us state the more general version from [BDH15, Theorem 2.1]:

Theorem 7.1. *Assume that $\mathbb{P}$ is homogeneous with respect to distributivity, i.e., $\mathfrak{h}(\{q \in \mathbb{P} \mid q \leq p\}) = \mathfrak{h}(\mathbb{P})$ for each $p \in \mathbb{P}$. Moreover, assume that there exists a dense subset $D \subseteq \mathbb{P}$ such that $|D| = \mathfrak{c}$ and D is σ-closed. Then $\mathbb{P}$ has a base matrix (of height $\mathfrak{h}(\mathbb{P})$).*

$\mathcal{P}(\omega)/\text{fin}$ satisfies the assumptions of the following proposition[14] for $\lambda = \mathfrak{c}$. Its proof goes back to [BPS80].

Proposition 7.2. *If $\mathbb{P}$ has a base matrix (of height $\mathfrak{h}(\mathbb{P})$), and for each $p \in \mathbb{P}$, there is an antichain of size λ below p, then $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$.*

[13] Todorčević's maximality principle has been generalized to larger cardinals in [GS21]. For the consistency proof, they use a supercompact cardinal. It is unclear to us whether the generalized principle is consistent together with something like "$0^\#$ does not exist".

[14] Note that also a Suslin tree has a base matrix (namely the levels of the tree itself), but clearly does not satisfy the assumption because of its chain condition.

Proof. Let $\mathcal{A} = \{A_\xi \mid \xi < \mathfrak{h}(\mathbb{P})\}$ be a base matrix[15] of height $\mathfrak{h}(\mathbb{P})$. For each $\xi < \mathfrak{h}(\mathbb{P})$ and each $p \in A_\xi$, let $\{q^i_p \mid i < \lambda\}$ be (an injective enumeration of) an antichain of size λ below p (such an antichain exists by assumption).

Let $\dot{f} := \{((\xi, i), q^i_p) \mid \xi < \mathfrak{h}(\mathbb{P}) \ \wedge \ p \in A_\xi \ \wedge \ i < \lambda\}$; note that $\dot{f}$ is a name for a partial function from $\mathfrak{h}(\mathbb{P})$ to λ (due to the fact that – given $\xi < \mathfrak{h}(\mathbb{P})$ – the set of q^i_p with $i < \lambda$ and $p \in A_\xi$ form an antichain). For each $i < \lambda$, the set $\{q^i_p \mid p \in \bigcup_{\xi<\mathfrak{h}(\mathbb{P})} A_\xi\}$ is dense: indeed, given $p \in \mathbb{P}$, there is – due to the fact that $\mathcal{A}$ is a base matrix – $\xi < \mathfrak{h}(\mathbb{P})$ and $p' \in A_\xi$ such that $p' \leq p$, and hence $q^i_{p'} \leq p$; by definition of $\dot{f}$, we have that $q^i_{p'} \Vdash \dot{f}(\xi) = i$. It follows that every $i < \lambda$ is forced to be in the range of $\dot{f}$, as desired. □

From the above proposition, Lemma 3.2 (see also (2) afterwards), and Corollary 2.3 we immediately get the following:

Corollary 7.3. *Assume that $\mathbb{P}$ has a base matrix (of height $\mathfrak{h}(\mathbb{P})$), and for each $p \in \mathbb{P}$, there is an antichain of size $|\mathbb{P}|$ below p. Then* $\mathsf{FRESH}(\mathbb{P}) = [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]_{Reg}$.

As discussed above, $\mathcal{P}(\omega)/\text{fin}$ fulfills the assumptions of the above corollary, so we finally obtain the fresh function spectrum of $\mathcal{P}(\omega)/\text{fin}$:

Corollary 7.4. $\mathsf{FRESH}(\mathcal{P}(\omega)/\mathit{fin}) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

7.2. $\mathcal{P}(\kappa)/{<}\kappa$. As for $\mathcal{P}(\omega)/\text{fin}$, the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ is an interval. Under the assumption that there are antichains of size θ in $\mathcal{P}(\kappa)/{<}\kappa$, Shelah [She07] has shown that the forcing $\mathcal{P}(\kappa)/{<}\kappa$ collapses θ to ω (based on work of Balcar-Simon [BS88] which shows that it collapses the generalized bounding number $\mathfrak{b}_\kappa$ to ω). We will use Lemma 3.2 and Proposition 2.2 to compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$. As a preparation, we prove the following:

Lemma 7.5. *If $\mathcal{P}(\kappa)/{<}\kappa$ has the χ-c.c., then $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$ has the χ-c.c..*

Proof. Let A be an antichain in $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$. We will show that there is an antichain in $\mathcal{P}(\kappa)/{<}\kappa$ of the same size. Fix a bijection $\iota\colon \kappa \times \kappa \to \kappa$. Define a mapping $\varphi\colon [\kappa]^\kappa \times [\kappa]^\kappa \to [\kappa]^\kappa$ as follows. For $(a, b) \in [\kappa]^\kappa \times [\kappa]^\kappa$, let $a =: \{\alpha_i \mid i < \kappa\}$, and $b =: \{\beta_i \mid i < \kappa\}$ be the increasing enumerations of a and b. Let $\varphi(a, b) := \{\iota(\alpha_i, \beta_i) \mid i < \kappa\}$. It is straightforward to check that φ preserves incompatibility, i.e., if (a_0, b_0) and (a_1, b_1) are incompatible, then $\varphi(a_0, b_0)$ and $\varphi(a_1, b_1)$ are incompatible. Therefore, $\{\varphi(a, b) \mid (a, b) \in A\}$ is an antichain in $\mathcal{P}(\kappa)/{<}\kappa$ of the same size as A. □

We can now compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$. It turns out that it only depends on the size of the antichains:

Proposition 7.6. *If χ is minimal such that $\mathcal{P}(\kappa)/{<}\kappa$ has the χ-c.c., then* $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \chi)_{Reg}$.

Proof. By Shelah [She07], $\mathcal{P}(\kappa)/{<}\kappa$ collapses θ to ω, if there exists an antichain of size θ. Hence, every cardinal smaller than χ is collapsed to ω. Since $\mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa) = \omega$, it follows by Lemma 3.2 that every regular cardinal smaller than χ belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$.

On the other hand, by Lemma 7.5, $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$ has the χ-c.c., hence by Proposition 2.2 no regular cardinal $\geq \chi$ belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$. □

[15] The base matrix is not required to be refining in this proof.

Note that there are always antichains of size κ^+ (actually even of size[16] $\mathfrak{b}_\kappa$) in $\mathcal{P}(\kappa)/{<}\kappa$, hence

$$[\omega, \kappa^+]_{Reg} \subseteq \mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa).$$

If $2^{<\kappa} = \kappa$, then there are antichains of size 2^κ (by the same argument as for ω, i.e., by identifying $2^{<\kappa}$ with κ and taking the set of branches through the tree $2^{<\kappa}$), so

$$\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]_{Reg} \text{ whenever } 2^{<\kappa} = \kappa.$$

To get a model of $2^{<\kappa} = \kappa$ where 2^κ is large, start with a model of GCH and add many κ-Cohen reals.

On the other hand, it is also consistent that 2^κ is large and there are no large antichains in $\mathcal{P}(\kappa)/{<}\kappa$. In fact, the following was shown in [Bau76]. We present a proof for the convenience of the reader.

Proposition 7.7. *Let V be a model of $2^\kappa = \theta$ and $\mu > \theta$ with $\mathrm{cf}(\mu) > \kappa$. Then there exists a cofinality preserving extension of V such that in $\mathcal{P}(\kappa)/{<}\kappa$, there are no antichains of size θ^+, and $2^\kappa = \mu$.*

Note that each antichain in V remains an antichain of the same size in the extension (since cardinalities are preserved). Also note that, starting with a model of GCH, the proposition easily yields a model satisfying $\kappa^+ < 2^\kappa$ and $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \kappa^+]_{Reg}$.

Proof of Proposition 7.7. We add μ many ω-Cohen reals,[17] i.e., force with $\mathbb{C}_\mu$. Then in the extension, clearly $2^\kappa = \mu$ holds true, and there are no antichains of size θ^+ in $\mathcal{P}(\kappa)/{<}\kappa$, which can be seen as follows.

Assume towards a contradiction that $A = \{a_i \mid i < \theta^+\}$ is an antichain of size θ^+ in $V[\mathbb{C}_\mu]$. Now work in V and fix names $\dot{a}_i$ for the sets a_i. For $i, j < \theta^+$, let $\dot{\zeta}_{i,j}$ be such that it is forced that $\dot{a}_i \cap \dot{a}_j \subseteq \dot{\zeta}_{i,j} < \kappa$. Since $\mathbb{C}_\mu$ has the c.c.c., there exist countable sets $Z_{i,j} \subseteq \kappa$ in the ground model such that it is forced that $\dot{\zeta}_{i,j} \in Z_{i,j}$. Now let $\gamma_{i,j} := \sup(Z_{i,j}) < \kappa$. This defines a mapping from $\theta^+ \times \theta^+$ to κ. Since $\theta^+ = (2^\kappa)^+$ in V, we can apply the Erdős-Rado Theorem to get a set $Y \subseteq \theta^+$ of size κ^+ and $\gamma < \kappa$ such that $\gamma_{i,j} = \gamma$ for all $i, j \in Y$. Therefore, $\{a_i \setminus \gamma \mid i \in Y\}$ is a family of κ^+ many (non-empty) disjoint subsets of κ in $V[\mathbb{C}_\mu]$, a contradiction. □

It also follows from the above that the largest size of antichains in $\mathcal{P}(\kappa)/{<}\kappa$ can be strictly between κ^+ and 2^κ. We can proceed as follows. Start with a ground model V satisfying GCH, and let $\kappa^+ < \theta < \mu$ with $\mathrm{cf}(\theta) > \kappa$ and $\mathrm{cf}(\mu) > \kappa$. First add θ many κ-Cohen reals and then add μ many ω-Cohen reals. In the resulting model $2^\kappa = \mu$, and $\mathcal{P}(\kappa)/{<}\kappa$ has the θ^+-c.c. and there exists an antichain of size θ, hence

$$\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \theta]_{Reg}$$

holds in this model.

[16] It is easy to construct a $<^*$-increasing family of functions in κ^κ of size $\mathfrak{b}_\kappa$; the graphs of these functions form an antichain on $\kappa \times \kappa$.

[17] In fact, the proof shows that we only have to demand that the forcing to blow up 2^κ has the κ-c.c. and preserves cofinalities (for example, adding many ν-Cohen reals with $\nu < \kappa$ would work).

7.3. **Mathias forcing and Silver forcing.** Recall that *Mathias forcing* Ma is the poset of pairs (s, A) with $s \in 2^{<\omega}$ and $A \in [\omega]^{\omega}$, where the extension relation is defined as follows: $(t, B) \leq (s, A)$ if $t \trianglerighteq s$, $B \subseteq A$, and for each $n \geq |s|$, if $t(n) = 1$, then $n \in A$.

Proposition 7.8. *FRESH*$(\mathsf{Ma}) = \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

In particular, $\mathsf{FRESH}(\mathsf{Ma})$ is an interval (namely $[\omega, \mathfrak{c}]_{Reg}$) if and only if $\mathfrak{h} = \omega_1$.

Proof of Proposition 7.8. Since Mathias forcing adds new reals, it is clear that $\omega \in \mathsf{FRESH}(\mathsf{Ma})$. It is well-known and easy to see that Mathias forcing can be written as a two-step iteration as follows:

$$\mathsf{Ma} \cong \mathcal{P}(\omega)/\text{fin} * \mathsf{Ma}(G),$$

where $\mathsf{Ma}(G)$ is the Mathias forcing with respect to the generic ultrafilter G added by $\mathcal{P}(\omega)/\text{fin}$, i.e., the subposet of Mathias forcing consisting of those pairs $(s, A) \in \mathsf{Ma}$ for which $A \in G$. Since $\mathsf{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$ (see Corollary 7.4), it follows by Lemma 4.5 that $[\mathfrak{h}, \mathfrak{c}]_{Reg} \subseteq \mathsf{FRESH}(\mathsf{Ma})$.

Fix $\beta \notin \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}]_{Reg}$. On the one hand, $\beta \notin \mathsf{FRESH}(\mathcal{P}(\omega)/\text{fin})$. On the other hand, $\mathcal{P}(\omega)/\text{fin}$ forces that $\mathsf{Ma}(G)$ is σ-centered and hence $\mathsf{Ma}(G) \times \mathsf{Ma}(G)$ has the c.c.c., so Proposition 2.2 yields that $\mathcal{P}(\omega)/\text{fin}$ forces that $\beta \notin \mathsf{FRESH}(\mathsf{Ma}(G))$. Now we can apply Lemma 4.6 to conclude that $\beta \notin \mathsf{FRESH}(\mathsf{Ma})$. □

Recall that *Silver forcing* Si is the poset of partial functions f from ω to 2 with co-infinite domain, and a condition g is stronger than f if g extends f.

Proposition 7.9. *FRESH*$(\mathsf{Si}) \supseteq \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

Proof. Since Silver forcing adds new reals, it is clear that $\omega \in \mathsf{FRESH}(\mathsf{Si})$. It is well-known and easy to see that Silver forcing can be written as a two-step iteration as follows:

$$\mathsf{Si} \cong \mathcal{P}(\omega)/\text{fin} * C(G),$$

where $C(G)$ is Gregorieff forcing with respect to the generic ultrafilter G added by $\mathcal{P}(\omega)/\text{fin}$, i.e., the subposet of Silver forcing consisting of those functions f for which $\omega \setminus \text{dom}(f) \in G$. Since $\mathsf{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$ (see Corollary 7.4), it follows by Lemma 4.5 that $[\mathfrak{h}, \mathfrak{c}]_{Reg} \subseteq \mathsf{FRESH}(\mathsf{Si})$.
□

Since $C(G)$ has no small chain condition, and is not Y-proper,[18] it is not clear to us how to prove that regular cardinals strictly between ω and $\mathfrak{h}$ do not belong to $\mathsf{FRESH}(\mathsf{Si})$.

7.4. **A general fact about tree forcings.** Now we want to discuss forcings whose conditions are trees of height ω of sequences of ordinals, i.e., forcings $\mathbb{P}$ such that $\mathbb{P} \subseteq \mathcal{P}(\lambda^{<\omega})$ for some cardinal λ. We will use the following notation: Let $p \in \mathbb{P}$ be a condition. For a node $s \in p$, let $\text{succ}_p(s) := \{\alpha \in Ord \mid s^\frown\alpha \in p\}$. A node s is a *splitting node* if $|\text{succ}_p(s)| > 1$, i.e., if it has more than one immediate successor in p. Let $\text{split}(p)$ denote the set of splitting nodes of p. The nth *split level* of a condition p, denoted by $\text{split}_n(p)$, is the set of splitting nodes of p which have exactly n proper initial segments which are splitting. In particular, $\text{split}_0(p) = \{\text{stem}(p)\}$. For $s \in p$, let $p^{[s]} := \{t \in p \mid s \trianglelefteq t \text{ or } t \trianglelefteq s\}$.

[18]This is due to the fact that Si and hence $C(G)$ does not add unbounded reals (compare with [CZ15, Theorem 4.1(4)] which says that all Y-proper forcings do).

Definition 7.10. We say that a forcing $\mathbb{P}$ is *Miller-like* if it consists of trees of height ω of sequences of ordinals and has the following properties:

(1) For every $p \in \mathbb{P}$ the following hold:
   (a) For all $t \in p$ there exists $s \in p$ with $t \trianglelefteq s$ and s is a splitting node.
   (b) $p^{[t]} \in \mathbb{P}$ for all $t \in p$.
(2) For every $p \in \mathbb{P}$ with stem s and $p_\alpha \leq p^{[s^\frown \alpha]}$ for every $\alpha \in \mathrm{succ}_p(s)$, the union $\bigcup_{\alpha \in \mathrm{succ}_p(s)} p_\alpha \in \mathbb{P}$.
(3) $\mathbb{P}$ has *fusion*, i.e., for any sequence $p_0 \geq p_1 \geq \dots$ such that $\mathrm{split}_n(p_n) = \mathrm{split}_n(p_{n+1})$ for each n, $\bigcap_{n\in\omega} p_n \in \mathbb{P}$.

We will now provide a general theorem which we will use to compute the fresh function spectra of tree forcings on ω and of Namba forcing (see Section 7.5 and Section 7.6).

Theorem 7.11. *If λ is a cardinal, $\mathbb{P} \subseteq \mathcal{P}(\lambda^{<\omega})$ is a Miller-like forcing and $\delta > \lambda$ is regular uncountable, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

Proof. Assume towards a contradiction that $p \in \mathbb{P}$ and $\dot f$ are such that $p \Vdash \dot f : \delta \to Ord$ is fresh.

Claim 7.12. *There exists $q \leq p$ and $\{\beta_s \mid s \in \mathrm{split}(q)\} \subseteq \delta$ such that for every $s \in \mathrm{split}(q)$, letting $\mathrm{succ}_q(s) =: \{\alpha_i \mid i \in \chi\}$ (with $\chi = |\mathrm{succ}_q(s)|$), there exist functions g_i ($i \in \chi$) with $q^{[s^\frown \alpha_i]} \Vdash \dot f \restriction \beta_s = g_i$ for each i, and $g_i \neq g_j$ for $i \neq j$.*

We will prove this claim below. Let us first finish the proof of the theorem, using the claim. Let $\gamma := \sup\{\beta_s \mid s \in \mathrm{split}(q)\}$ which is $< \delta$ since $\delta > \lambda$ is regular uncountable and $|\{\beta_s \mid s \in \mathrm{split}(q)\}| \leq |\lambda^{<\omega}| = \max(\omega, \lambda)$, and let $q' \leq q$ and $g \in V$ such that $q' \Vdash \dot f \restriction \gamma = g$. Note that if $s^\frown \alpha_i \in q'$ for $s \in \mathrm{split}(q)$ then $q' \nVdash \dot f \restriction \beta_s \neq g_i$, thus, since q' decides $\dot f \restriction \beta_s$, it follows that $q' \Vdash \dot f \restriction \beta_s = g_i$. Hence for every $s \in \mathrm{split}(q)$ for every $i \neq j$ either $s^\frown \alpha_i \notin q'$ or $s^\frown \alpha_j \notin q'$. So q' is not a condition, a contradiction. □

To prove Claim 7.12, let us first prove the following claim.

Claim 7.13. *For $\tilde p \leq p$ and $s = \mathrm{stem}(\tilde p)$ there exists $\beta < \delta$ and $\tilde q \leq \tilde p$ with $\mathrm{stem}(\tilde q) = s$ and $\mathrm{succ}_{\tilde q}(s) = \mathrm{succ}_{\tilde p}(s) =: \{\alpha_i \mid i \in \chi\}$ (with $\chi = |\mathrm{succ}_{\tilde p}(s)|$) such that there exist pairwise distinct functions g_i ($i \in \chi$) with $\tilde q^{[s^\frown \alpha_i]} \Vdash \dot f \restriction \beta = g_i$ for each i.*

Proof. For every α_i, using Lemma 2.8, let $\beta_i < \delta$ be such that $|\{g \mid \exists q \leq \tilde p^{[s^\frown \alpha_i]}$ with $q \Vdash \dot f \restriction \beta_i = g\}| \geq \chi$; this is possible since $\chi \leq \lambda < \delta$. Let $\beta := \sup_{i \in \chi} \beta_i < \delta$. By induction on $i < \chi$, we are going to construct $\tilde q$. First, let $q_0 \leq \tilde p^{[s^\frown \alpha_0]}$ and g_0 be such that $q_0 \Vdash \dot f \restriction \beta = g_0$. Inductively proceed as follows. Assume that for every $j < i$, we have defined $q_j \leq \tilde p^{[s^\frown \alpha_j]}$ and g_j such that $q_j \Vdash \dot f \restriction \beta = g_j$, and $g_j \neq g_k$ for $j \neq k$. Let $q_i \leq \tilde p^{[s^\frown \alpha_i]}$ and g_i be such that $q_i \Vdash \dot f \restriction \beta = g_i$ and $g_i \neq g_j$ for every $j < i$; this is possible since $i < \chi$, and β is such that there are at least χ many possibilities to decide $\dot f \restriction \beta$ by extensions of $\tilde p^{[s^\frown \alpha_i]}$. Now $\tilde q := \bigcup_{i<\chi} q_i \leq \tilde p$, which is a condition by Definition 7.10(2), and is as desired. □

Proof of Claim 7.12. Proceed by induction, using Claim 7.13, as follows. First apply Claim 7.13 to p yielding $p_0 \leq p$ and $\beta_{\mathrm{stem}(p)}$. Recall that $\mathrm{split}_n(p)$ denotes the nth split level of p. Now assume that $p_n \leq p$ and $\{\beta_s \mid s \in \mathrm{split}_m(p)$ for some $m \leq n\}$ is defined and satisfies the conclusion of the claim for all such s. For every $s \in \mathrm{split}_{n+1}(p_n)$, apply Claim 7.13 to $p_n^{[s]}$, yielding $q_s \leq p_n^{[s]}$ and β_s; let $p_{n+1} := \bigcup_{s \in \mathrm{split}_{n+1}(p_n)} q_s$.

By repeatedly using Definition 7.10(2), one can show that $p_{n+1} \in \mathbb{P}$. Finally, let $q := \bigcap_{n\in\omega} p_n$. It is straightforward to check (using Definition 7.10(3), i.e., that $\mathbb{P}$ has fusion) that q is a condition, and that it has the desired property. □

7.5. **Tree forcings on ω and minimality.** We now discuss tree forcings on ω; their fresh function spectra always contain ω since new reals are added. Mathias forcing and Silver forcing, which have been considered in Section 7.3, can also be viewed as tree forcings on ω even though the standard representation given there does not consist of trees. The tree forcings we are going to consider here have fresh function spectrum $\{\omega\}$, whereas the fresh function spectra of Mathias forcing and Silver forcing also contain uncountable cardinals. Note that also the classical c.c.c. forcings Cohen forcing, random forcing, and Hechler forcing can be viewed as tree forcings on ω. Since they are Knaster, they all have fresh function spectrum $\{\omega\}$ (see Corollary 2.4). Let us now turn to classical non-c.c.c. tree forcings on ω. Recall that *Sacks forcing* Sa is the collection of perfect trees in $2^{<\omega}$ (as usual for tree forcings, ordered by inclusion). *Miller forcing* Mi is the collection of superperfect trees in $\omega^{<\omega}$ (where a tree is superperfect if each node can be extended to a node with infinitely many immediate successors). *Full Miller forcing* FuMi is the collection of full Miller trees, where a Miller tree T is full if each node can be extended to a node t such that $t^\frown n \in T$ for each $n \in \omega$.

It is well-known and easy to check that the above mentioned tree forcings Sa, Mi, and FuMi are Miller-like tree forcings (see Definition 7.10) in $\omega^{<\omega}$ (or even in $2^{<\omega}$ in case of Sacks forcing). Therefore Theorem 7.11 immediately yields the following:

Corollary 7.14.

(1) $\mathit{FRESH}(\mathsf{Sa}) = \{\omega\}$.
(2) $\mathit{FRESH}(\mathsf{Mi}) = \{\omega\}$.
(3) $\mathit{FRESH}(\mathsf{FuMi}) = \{\omega\}$.

For an ideal[19] $\mathcal{I}$ on ω, let $\mathsf{La}(\mathcal{I})$ be the poset of all trees $T \subseteq \omega^{<\omega}$ such that for every node $s \in T$ extending the stem, the set $\{n \in \omega : s^\frown n \in T\}$ does not belong to $\mathcal{I}$. Let $\mathsf{Mi}(\mathcal{I})$ be the poset of all trees T such that for every node there exists a node t extending it with the property that $\{n \in \omega : t^\frown n \in T\}$ does not belong to $\mathcal{I}$. In particular, if $\mathcal{I}$ is the ideal of finite sets, $\mathsf{La}(\mathcal{I})$ is *Laver forcing* La and $\mathsf{Mi}(\mathcal{I})$ is Miller forcing Mi.

Note that $\mathsf{Mi}(\mathcal{I})$ is Miller-like for any ideal $\mathcal{I}$ on ω, hence Theorem 7.11 implies that

$$\mathsf{FRESH}(\mathsf{Mi}(\mathcal{I})) = \{\omega\}$$

for any ideal $\mathcal{I}$. On the other hand, $\mathsf{La}(\mathcal{I})$ (and hence La) is never Miller-like (in fact, Definition 7.10(2) fails). So we cannot apply Theorem 7.11 to compute their fresh function spectra. Instead, we make use of another property of these forcings (discussed in Section 2):

Theorem 7.15. *Let $\mathcal{I}$ be an ideal on ω which is the intersection of F_σ ideals. Then*

(1) $\mathsf{La}(\mathcal{I})$ *is Y-proper,*
(2) $\mathsf{Mi}(\mathcal{I})$ *is Y-proper.*

[19]We always assume our ideals to contain all finite sets.

Proof. The statement about $\mathsf{La}(\mathcal{I})$ is shown in [CZ15, Theorem 4.8]. Even though it is not explicitly mentioned in [CZ15], it is easy to adapt their proof to obtain the analogous result for $\mathsf{Mi}(\mathcal{I})$. □

Note that Sacks forcing Sa is not Y-proper since it does not add an unbounded real, but all Y-proper forcings do (see [CZ15, Theorem 4.1(4)]). We do not know, however, the status of full Miller forcing:

Question 7.16. Is FuMi Y-proper?

Using Proposition 2.12, we can determine the fresh function spectra of Laver forcings:

Corollary 7.17. *$\mathit{FRESH}(\mathsf{La}) = \{\omega\}$. More generally, $\mathit{FRESH}(\mathsf{La}(\mathcal{I})) = \{\omega\}$ whenever $\mathcal{I}$ is an ideal on ω which is the intersection of F_σ ideals.*

If $\mathcal{I}$ is not of the form needed in the above theorem it is not clear in general whether $\mathsf{La}(\mathcal{I})$ is Y-proper. However, if $\mathcal{I}$ is an analytic P-ideal, then $\mathsf{La}(\mathcal{I})$ is Y-proper if and only if it is of the above form (see [CZ15, Theorem 4.9]). The asymptotic density zero ideal $\mathcal{Z} = \{a \subseteq \omega : \lim_{n\to\infty} \frac{|a\cap n|}{n} = 0\}$ is an example of an analytic P-ideal which is not of the above form. So it is natural to ask the following:

Question 7.18. Is $\mathsf{FRESH}(\mathsf{La}(\mathcal{Z})) = \{\omega\}$?

In the rest of the section we comment on the relation between minimality of a forcing and its fresh function spectrum. Recall that Sa, Mi, and La are *minimal* (i.e., they cannot be written as the iteration of two non-atomic forcings) and their fresh function spectra are singletons in ZFC. The minimality of tree forcings has been studied extensively in [Gro87]. Minimality, however, is not sufficient for the fresh function spectrum being a singleton:

Example 7.19. *There is a minimal Prikry-type forcing $\mathbb{P}$, which singularizes a measurable cardinal κ, with $\mathit{FRESH}(\mathbb{P}) = \{\omega, \kappa\}$.*

Proof. The forcing defined in [KRS13] is a Prikry-type forcing which is minimal. As in Theorem 7.21, it follows that the fresh function spectrum is $\{\omega, \kappa\}$. □

A forcing $\mathbb{P}$ is *minimal for reals* if whenever $\mathbb{P}$ is written as a two-step iteration $\mathbb{P}_0 * \dot{\mathbb{P}}_1$ and a real x belongs to the extension by $\mathbb{P}_0$ but not to the ground model, then the second iterand $\dot{\mathbb{P}}_1$ has to be the trivial forcing.

Proposition 7.20. *Assume $\mathbb{P}$ is a proper forcing such that $\mathit{FRESH}(\mathbb{P}) = \{\omega\}$. Then the following are equivalent:*

(1) $\mathbb{P}$ is minimal for reals.
(2) $\mathbb{P}$ is minimal.

Proof. The implication from (2) to (1) is clear. For the other direction, assume $\mathbb{P}$ is not minimal. Therefore, $\mathbb{P}$ can be written as $\mathbb{P}_0 * \dot{\mathbb{P}}_1$, where both $\mathbb{P}_0$ and $\dot{\mathbb{P}}_1$ are (forced to be) non-atomic. So, by Lemma 4.5, $\omega \in \mathsf{FRESH}(\mathbb{P}_0)$. Since $\mathbb{P}_0$ (as a complete subforcing of $\mathbb{P}$) is proper, $\mathbb{P}_0$ adds a new real by Proposition 5.8. Therefore, $\mathbb{P}$ is not minimal for reals. □

In particular, for forcings such as Sacks, Miller, Laver, etc., it is enough to prove minimality for reals in order to prove minimality.

7.6. **Prikry forcing and Namba forcing.** In this section, we want to discuss Prikry and Namba forcing, and compute their fresh function and fresh subset spectra. These two forcings are examples of forcings which add a new function on ω, but no real[20] (i.e., no new subset of ω). A good reference for properties of these forcings is [Jec03].

For a normal measure $\mathcal{U}$ on a measurable cardinal κ, let $\mathbb{P}_{\mathcal{U}}$ denote *Prikry forcing with respect to* $\mathcal{U}$, i.e., the poset of pairs (s, A) with $s \in [\kappa]^{<\omega}$ and $A \in \mathcal{U}$ with $\max(s) < \min(A)$, where the extension relation is defined as follows: $(t, B) \leq (s, A)$ if $t \supseteq s$, $B \subseteq A$, and $t \setminus s \subseteq A$.

Theorem 7.21. *Let $\mathcal{U}$ be a normal measure on a measurable cardinal κ. Then the following holds:*

(1) $\mathsf{FRESH}(\mathbb{P}_{\mathcal{U}}) = \{\omega, \kappa\}$.
(2) Let δ be an indecomposable ordinal. Then $\mathbb{P}_{\mathcal{U}}$ adds a fresh subset of δ if and only if $\delta \geq \kappa$ and $\mathrm{cf}(\delta) \in \{\omega, \kappa\}$.

Proof. Clearly, $\omega \in \mathsf{FRESH}(\mathbb{P}_{\mathcal{U}})$, since the generic Prikry function is a new function from ω to κ. Moreover, the range of the generic Prikry function (i.e., the Prikry sequence viewed as a subset of κ) is clearly a fresh subset of κ, since all its proper initial segments are finite; in particular, $\kappa \in \mathsf{FRESH}(\mathbb{P}_{\mathcal{U}})$. Further recall the well-known fact that Prikry forcing does not add new bounded subsets of κ. It is easy to check that $\mathbb{P}_{\mathcal{U}}$ is κ-centered (i.e., a union of κ-many centered sets). Consequently, $\mathbb{P}_{\mathcal{U}} \times \mathbb{P}_{\mathcal{U}}$ is κ^+-c.c., so, by Proposition 2.2, $\lambda \notin \mathsf{FRESH}(\mathbb{P}_{\mathcal{U}})$ for any regular $\lambda > \kappa$.

To show (1), it remains to prove that no regular cardinal strictly between ω and κ belongs to the fresh function spectrum. Fix λ regular with $\omega < \lambda < \kappa$ and assume towards a contradiction that $\mathbb{P}_{\mathcal{U}}$ adds a fresh function f from λ to the ordinals. Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-c.c., we can cover the range of f by a set of size κ in the ground model, and therefore, by taking a ground model bijection between this cover and κ, we can assume without loss of generality that f is in fact a fresh function from λ to κ. Note that the range of this function has to be unbounded in κ: if not, then there would exist a new bounded subset of κ, which is not the case. Further note that for each $\alpha < \lambda$, the range of $f \restriction \alpha$ is bounded in κ, because otherwise κ would be singular in the ground model, witnessed by the ground model function $f \restriction \alpha$. Therefore, in the extension, we can construct a cofinal subset X of λ such that $f \restriction X$ is a strictly increasing function whose range is cofinal in κ. Note that X is in the ground model (because no new bounded subsets of κ are added) and hence it has order-type λ. Consequently, $\mathrm{cf}(\kappa) = \lambda$ holds true in the extension, contradicting the fact that $\mathrm{cf}(\kappa) = \omega$ holds true in the extension (as witnessed by the generic Prikry function).

Let us now show (2). Recall Proposition 5.3 and the fact that no new bounded subsets of κ are added. As mentioned above, Prikry forcing adds a fresh subset of κ, so, using Proposition 5.3, the characterization is established for ordinals of cofinality κ. It remains to show that Prikry forcing adds fresh subsets of indecomposable ordinals $\delta \geq \kappa$ with $\mathrm{cf}(\delta) = \omega$. The minimal such ordinal is $\kappa \cdot \omega$. By Proposition 5.3 it is enough to show that Prikry forcing adds a fresh subset of $\kappa \cdot \omega$. Let $g \colon \omega \to \kappa$ be the Prikry function. Then $\{(\kappa \cdot n) + g(n) \mid n \in \omega\}$ is a fresh subset of $\kappa \cdot \omega$. □

Using the above theorem, we give an example of a forcing for which the ξ_λ's from Corollary 5.4 cannot be chosen to be all the same; in fact, it is necessary to choose them in such a way that $\xi_{\omega_1} \leq \omega_1$ and $\xi_\omega \geq \kappa$. For simplicity of the argument, we use CH, although the same forcing is always such an example.

[20] For Namba forcing, this is only true under CH (see the discussion before Theorem 7.23).

Example 7.22. *Assume CH.*

(1) $\mathsf{FRESH}(\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1, \kappa\}$.
(2) Let δ be an indecomposable ordinal. Then $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1)$ adds a fresh subset of δ if and only if $\mathrm{cf}(\delta) = \omega_1$*, or $\delta \geq \kappa$ and* $\mathrm{cf}(\delta) \in \{\omega, \kappa\}$.

Proof. First note that $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1) = \mathbb{C}(\omega_1) * \check{\mathbb{P}}_{\mathcal{U}}$, therefore by Lemma 4.5, $\omega_1 \in \mathsf{FRESH}(\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1))$. Since $\mathbb{P}_{\mathcal{U}}$ does not add new bounded subsets of κ, CH holds in the extension by $\mathbb{P}_{\mathcal{U}}$, and $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1) = \mathbb{P}_{\mathcal{U}} * \check{\mathbb{C}}(\omega_1) = \mathbb{P}_{\mathcal{U}} * \mathbb{C}(\omega_1)$. So $\mathsf{FRESH}(\mathbb{C}(\omega_1)) = \{\omega_1\}$ in the extension by $\mathbb{P}_{\mathcal{U}}$. Therefore, using Lemma 4.5 and Lemma 4.6, we know that $\mathsf{FRESH}(\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1, \kappa\}$.

To prove (2), first recall Proposition 5.3 and note that (under CH) $\mathbb{C}(\omega_1)$ adds a fresh subset of an indecomposable ordinal δ if and only if $\mathrm{cf}(\delta) = \omega_1$. Using Theorem 7.21(2) and the above representations of $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}(\omega_1)$, one can easily finish the proof by applying the straightforward analogues of Lemma 4.5 and Lemma 4.6 for fresh subsets instead of fresh functions. □

Namba forcing Nb is the collection of trees in $\omega_2^{<\omega}$ (ordered by inclusion) which have the property that each node can be extended to a node with ω_2 many immediate successors.

It is well-known that Namba forcing does not add reals if CH holds (see for example the proof of [Jec03, Theorem 28.10]). However, it does add reals if CH does not hold, which can be seen as follows. Fix an injection $\varphi\colon \omega_2 \to 2^\omega$ in the ground model, and let $f\colon \omega \to \omega_2$ be the Namba generic function. Then $\varphi \circ f\colon \omega \to 2^\omega$ is new, and this can easily be translated into a new real.

Theorem 7.23. *Assume CH. Then the following holds:*

(1) $\mathsf{FRESH}(\mathsf{Nb}) = \{\omega, \omega_1, \omega_2\}$.
(2) Let δ be an indecomposable ordinal. Then Nb *adds a fresh subset of δ if and only if $\delta \geq \omega_1$ and* $\mathrm{cf}(\delta) \in \{\omega, \omega_1, \omega_2\}$.

In case GCH (at ω_2) holds, the size of Nb is ω_3, hence it is easy to argue that no cardinal larger than ω_3 belongs to $\mathsf{FRESH}(\mathsf{Nb})$ (see Corollary 2.3). However, even under GCH, the more difficult argument given here is necessary to show that ω_3 does not belong to $\mathsf{FRESH}(\mathsf{Nb})$.

Proof of Theorem 7.23. First recall that (similarly as in the case of Prikry forcing above) Namba forcing Nb adds a new cofinal sequence of length ω to ω_2, hence $\omega, \omega_2 \in \mathsf{FRESH}(\mathsf{Nb})$ and there is a fresh subset of ω_2. Furthermore, as discussed above, Namba forcing does not add new subset of ω (or any countable ordinal) under CH. Note that ω_2 is collapsed to ω_1, so, by Proposition 3.1, Nb adds a new subset of ω_1, which is fresh by the above; in particular, $\omega_1 \in \mathsf{FRESH}(\mathsf{Nb})$.

It is easy to check that Namba forcing is a Miller-like tree forcing (see Definition 7.10) in $\omega_2^{<\omega}$. Therefore, by Theorem 7.11, no regular cardinal above ω_2 belongs to $\mathsf{FRESH}(\mathsf{Nb})$, finishing the proof of (1).

For (2), using Proposition 5.3 and the fact that Nb adds fresh subsets of ω_1 and ω_2, we only have to deal with ordinals of countable cofinality. Namba forcing does not add fresh subsets of countable ordinals, so we consider indecomposable uncountable ordinals of countable cofinality. The smallest such ordinal is $\omega_1 \cdot \omega$. Let $f\colon \omega \to \omega_2$ be the Namba generic function (which is fresh). Let $\varphi\colon \omega_2 \to \mathcal{P}(\omega_1)$ be an injection in the ground model. Then $\varphi \circ f\colon \omega \to \mathcal{P}(\omega_1)$ is again fresh and therefore $\{(\omega_1 \cdot n) + \beta \mid n \in \omega \wedge \beta \in \varphi(f(n))\}$ is a fresh subset of $\omega_1 \cdot \omega$. □

Let us remark that the above theorem in particular shows that Namba forcing is another example of a forcing (compare with the remark after Lemma 3.2 involving Proposition 2.9) which collapses a cardinal λ without adding a fresh function on λ. Indeed, since Nb turns ω_2 into an ordinal of size ω_1 and cofinality ω, it follows from [Jec03, Corollary 23.20] that Nb collapses ω_3, but $\omega_3 \notin \mathsf{FRESH}(\mathsf{Nb})$ by the theorem.

8. Refining matrices and the combinatorial distributivity spectrum

In this last section, we focus on a more combinatorial version of a spectrum related to distributivity. We define refining matrices and the connected notion of combinatorial distributivity spectrum. These notions have been introduced by the authors of this paper in [FKWb], where the existence of refining matrices for $\mathcal{P}(\omega)/\text{fin}$ of height larger than $\mathfrak{h}$ has been shown to be consistent. A special sort of refining matrices are base matrices (see also Section 7.1). Base matrices for $\mathcal{P}(\omega)/\text{fin}$ (of height $\mathfrak{h}$) have been introduced in the seminal paper [BPS80], and recently base matrices of various heights have been constructed by Brendle in [Bre]. A structural analysis of such matrices has been done in [FKWa]. In Section 8.1, we compare the combinatorial distributivity spectrum of arbitrary forcing notions with their fresh function spectrum, and in Section 8.2, we consider Easton products of Cohen forcings once again.

Recall from the introduction that $\mathfrak{h}(\mathbb{P})$ denotes the distributivity of a forcing $\mathbb{P}$. For (maximal) antichains A and B in $\mathbb{P}$, we say that B *refines* A if for each $q \in B$ there is a $p \in A$ such that $q \leq p$. It is well-known and easy to see that $\mathfrak{h}(\mathbb{P})$ is the least λ such that there is a system of λ many maximal antichains without common refinement. To get a sensible definition of spectrum, we consider systems which are in addition required to be refining:

Definition 8.1. Let $(\mathbb{P}, \leq)$ be any (non-atomic) separative[21] forcing notion. We say that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *refining matrix of height λ for* $\mathbb{P}$ if

(1) A_ξ is a maximal antichain in $\mathbb{P}$, for each $\xi < \lambda$,
(2) A_η refines A_ξ whenever $\eta \geq \xi$, and
(3) there is no common refinement, i.e., there is no maximal antichain B which refines every A_ξ.

The *combinatorial distributivity spectrum of* $\mathbb{P}$ (denoted by $\mathsf{COM}(\mathbb{P})$) is the set of regular cardinals λ such that there exists a refining matrix of height λ for $\mathbb{P}$.

We say that q *intersects* a refining matrix $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ if for each $\xi < \lambda$ there is an $a \in A_\xi$ with $q \leq a$. Note that Definition 8.1(3) is equivalent to

(3′) $\{q \in \mathbb{P} \mid q \text{ intersects } \mathcal{A}\}$ is not dense in $\mathbb{P}$.

It is easy to see that the existence of refining matrices of some height is only a matter of its cofinality: if δ is singular with $\text{cf}(\delta) = \lambda$, then there exists a refining matrix of height δ for $\mathbb{P}$ if and only if there exists one of height λ (i.e., $\lambda \in \mathsf{COM}(\mathbb{P})$). Therefore, as in case of the fresh function spectrum $\mathsf{FRESH}(\mathbb{P})$ (see Proposition 5.2), the restriction in the definition of $\mathsf{COM}(\mathbb{P})$ to regular cardinals makes sense.

It is straightforward to check that the least element of $\mathsf{COM}(\mathbb{P})$ is just the distributivity of $\mathbb{P}$; in particular, the minima of the combinatorial distributivity spectrum and the fresh function spectrum coincide:

$$\mathfrak{h}(\mathbb{P}) = \min(\mathsf{COM}(\mathbb{P})) = \min(\mathsf{FRESH}(\mathbb{P})).$$

[21]We assume our forcings to be separative, because otherwise the given definition does not properly reflect its intention. To solve this problem, $\leq$ can be replaced by $\leq^*$, where $q \leq^* p$ if there is no $r \leq q$ which is incompatible to p.

8.1. **COM**($\mathbb{P}$) **vs. FRESH**($\mathbb{P}$)**.** We now establish a relation between the combinatorial distributivity spectrum and the fresh function spectrum by showing how to transform refining matrices into fresh functions:

Proposition 8.2. *COM*($\mathbb{P}$) $\subseteq$ *FRESH*($\mathbb{P}$).

Proof. Assume that $\delta \in \mathsf{COM}(\mathbb{P})$, and fix a witnessing refining matrix $\mathcal{A} = \{A_\alpha \mid \alpha < \delta\}$ of height δ. For each $\alpha < \delta$, fix a bijection $\varphi_\alpha : A_\alpha \to |A_\alpha|$. Now let $\dot{f}$ be a name for the function f from δ to the ordinals which is defined as follows: for each $\alpha < \delta$, let $a \in A_\alpha$ be the (unique) condition which belongs to the generic filter, and let $f(\alpha) = \varphi_\alpha(a)$.

We claim that, in some generic extension, $\dot{f}$ is evaluated to a fresh function on δ, witnessing that $\delta \in \mathsf{FRESH}(\mathbb{P})$. In fact, we will prove that there exists $p \in \mathbb{P}$ such that p forces

(1) $\dot{f} : \delta \to Ord$,
(2) $\dot{f} \restriction \alpha \in V$ for each $\alpha < \delta$, and
(3) $\dot{f} \notin V$.

Clearly, (1) is forced (by $\mathbb{1}_{\mathbb{P}}$), because each A_α is a maximal antichain. To show that (2) is forced (by $\mathbb{1}_{\mathbb{P}}$), we show that, for any $\alpha < \delta$, the set of conditions which decide $\dot{f} \restriction \alpha$ (and therefore force $\dot{f} \restriction \alpha \in V$) is dense. Let $\alpha < \delta$ and $q \in \mathbb{P}$. There exists $a_\alpha \in A_\alpha$ such that q and a_α are compatible. Let $r \leq q, a_\alpha$. Since $\mathcal{A}$ is refining, for every $\beta \leq \alpha$ there exists $a_\beta \in A_\beta$ with $r \leq a_\beta$, thus $r \Vdash \dot{f}(\beta) = \varphi_\beta(a_\beta)$ for every $\beta \leq \alpha$. Consequently, r decides $\dot{f} \restriction \alpha$, as desired.

Finally, by definition of a refining matrix (see property (3′)), the set of conditions intersecting the matrix is not dense in $\mathbb{P}$, i.e., we can fix $p \in \mathbb{P}$ such that there exists no condition stronger than p which intersects $\mathcal{A}$. We show that p forces (3). Assume towards a contradiction that $p \nVdash \dot{f} \notin V$. So we can fix $q \leq p$ and $g \in V$ such that $q \Vdash \dot{f} = g$. This implies that for every $\alpha < \delta$ there exists exactly one $a_\alpha \in A_\alpha$ which is compatible with q. It easily follows that[22] $q \leq a_\alpha$ for each α, so q intersects $\mathcal{A}$, a contradiction. □

In case $\mathbb{P}$ is a complete Boolean algebra, it is easy to show that the combinatorial distributivity spectrum and the fresh function spectrum coincide:

Proposition 8.3. *Let $\mathbb{P}$ be a complete Boolean algebra. Then* *FRESH*($\mathbb{P}$) = *COM*($\mathbb{P}$).

Proof. By Proposition 8.2, $\mathsf{COM}(\mathbb{P}) \subseteq \mathsf{FRESH}(\mathbb{P})$. For the other direction, let $\delta \in \mathsf{FRESH}(\mathbb{P})$. We will construct a refining matrix witnessing $\delta \in \mathsf{COM}(\mathbb{P})$. Fix a name $\dot{f}$ and $p \in \mathbb{P}$ such that p forces the following: $\dot{f} : \delta \to Ord$ with $\dot{f} \notin V$, but $\dot{f} \restriction \gamma \in V$ for any $\gamma < \delta$.

For each $\alpha < \delta$, let

$$A_\alpha := \{[\![\dot{f} \restriction \alpha = g]\!] \mid g \in Ord^\alpha \wedge [\![\dot{f} \restriction \alpha = g]\!] \neq 0\},$$

i.e., A_α is the maximal antichain in the complete Boolean algebra $\mathbb{P}$ according to what $\dot{f} \restriction \alpha$ is forced to be. The set of conditions which are intersecting the matrix $\{A_\alpha \mid \alpha < \delta\}$ is not dense: if it were, there has to be a $q \leq p$ such that for each $\alpha < \delta$, there is an $a \in A_\alpha$ satisfying $q \leq a$; but then $q \Vdash \dot{f} \in V$, contradicting $p \Vdash \dot{f} \notin V$.

[22]If $\mathbb{P}$ is not separative, one has to replace $\leq$ by $\leq^*$ (see footnote 21).

To finish the proof, we have to show that the matrix $\{A_\alpha \mid \alpha < \delta\}$ is refining. Fix $\beta < \gamma < \delta$, and fix $b \in A_\gamma$. We can fix $g \in Ord^\gamma$ satisfying $b = [\![\dot{f} \restriction \gamma = g]\!] \neq 0$. Let $c := [\![\dot{f} \restriction \beta = g \restriction \beta]\!]$. Clearly, $b \leq c$, and $c \in A_\beta$ (since $0 \neq b \leq c$). □

The above proof makes essential use of the assumption that $\mathbb{P}$ is a *complete* Boolean algebra. Therefore, we strongly conjecture that equality does not always hold for arbitrary forcing notions $\mathbb{P}$.

Question 8.4. Is $\mathsf{COM}(\mathbb{P}) = \mathsf{FRESH}(\mathbb{P})$ for every forcing notion $\mathbb{P}$?

We can also ask whether the combinatorial distributivity spectrum is invariant under equivalence of forcing notions. In fact, this question is equivalent to the above question, which can be seen as follows. Let $r.o.(\mathbb{P})$ denote the canonical complete Boolean algebra associated to $\mathbb{P}$, which is forcing equivalent to $\mathbb{P}$. Clearly, the fresh function spectrum is invariant under equivalence of forcing notions, so

$$\mathsf{FRESH}(r.o.(\mathbb{P})) = \mathsf{FRESH}(\mathbb{P}),$$

and by Proposition 8.3,

$$\mathsf{FRESH}(r.o.(\mathbb{P})) = \mathsf{COM}(r.o.(\mathbb{P})).$$

Therefore, if $\mathsf{COM}(r.o.(\mathbb{P})) = \mathsf{COM}(\mathbb{P})$, then $\mathsf{COM}(\mathbb{P}) = \mathsf{FRESH}(\mathbb{P})$, and vice versa.

8.2. **Combinatorial distributivity spectrum of Easton products.** In Section 4, we discussed which sets of regular cardinals can be realized as the fresh function spectrum of a homogeneous forcing. Here, we show that each Easton closed set (see Definition 4.8) is the combinatorial distributivity spectrum of a homogeneous forcing, namely an Easton product of Cohen forcings.

Proposition 8.5. *Assume GCH. If X is Easton closed, then $\mathsf{COM}({}^E\prod_{\beta\in X} \mathbb{C}(\beta)) = X$.*

Proof. First note that $\mathsf{COM}({}^E\prod_{\beta\in X} \mathbb{C}(\beta)) \subseteq \mathsf{FRESH}({}^E\prod_{\beta\in X} \mathbb{C}(\beta)) = X$, using Proposition 8.2 and Theorem 4.24.

To show that $X \subseteq \mathsf{COM}({}^E\prod_{\beta\in X} \mathbb{C}(\beta))$, fix $\alpha \in X$. We construct a refining matrix of height α for ${}^E\prod_{\beta\in X} \mathbb{C}(\beta)$. Since $\min(\mathsf{COM}(\mathbb{C}(\alpha))) = \min(\mathsf{FRESH}(\mathbb{C}(\alpha))) = \alpha$, it follows that $\alpha \in \mathsf{COM}(\mathbb{C}(\alpha))$. Let $\mathcal{A} = \{A_\xi \mid \xi < \alpha\}$ be a refining matrix of height α for $\mathbb{C}(\alpha)$.

For $\xi < \alpha$, let $A'_\xi := \{p \in {}^E\prod_{\beta\in X} \mathbb{C}(\beta) \mid \exists a \in A_\xi \text{ with } p(\alpha) = a \wedge p(\beta) = \mathbb{1}_{\mathbb{C}(\beta)} \text{ for all } \beta \neq \alpha\}$. Now we will show that $\mathcal{A}' := \{A'_\xi \mid \xi < \alpha\}$ is a refining matrix of height α for ${}^E\prod_{\beta\in X} \mathbb{C}(\beta)$: If $p \neq p' \in A'_\xi$, it follows that $a := p(\alpha) \neq p'(\alpha) =: a'$, so a and a' are two distinct elements of A_ξ, which is an antichain, so a is incompatible with a' and therefore p is incompatible with p', so A'_ξ is an antichain. Similarly, it can be shown that A'_ξ is maximal. It is easy to see that $\mathcal{A}'$ is refining, since $\mathcal{A}$ is refining.

To finish the proof we show that property (3′) holds for $\mathcal{A}'$. Assume towards a contradiction that the set of conditions intersecting $\mathcal{A}'$ is dense in ${}^E\prod_{\beta\in X} \mathbb{C}(\beta)$. We show that this implies that the set of conditions intersecting $\mathcal{A}$ is dense in $\mathbb{C}(\alpha)$, which contradicts the fact that $\mathcal{A}$ is a refining matrix. Fix $a \in \mathbb{C}(\alpha)$. Let $p_a \in {}^E\prod_{\beta\in X} \mathbb{C}(\beta)$ be such that $p_a(\alpha) = a$ and $p_a(\beta) = \mathbb{1}_{\mathbb{C}(\beta)}$ for all $\beta \neq \alpha$. By assumption, there exists $q \in {}^E\prod_{\beta\in X} \mathbb{C}(\beta)$ intersecting $\mathcal{A}'$ with $q \leq p_a$. It is easy to see that $q(\alpha) \leq a$ and $q(\alpha)$ intersects $\mathcal{A}$. □

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