The consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

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Definition

A family $\mathcal{H} \subseteq {}^\omega\omega$ is unbounded, if there is no $g \in {}^\omega\omega$ which dominates all elements of $\mathcal{H}$. The bounding number $\mathfrak{b}$ is the minimal cardinality of an unbounded family.

Definition

A family $S \subseteq [\omega]^\omega$ is splitting, if for every $A \in [\omega]^\omega$ there is $B \in S$ such that both $A \cap B$ and $A \cap B^c$ are infinite. The splitting number $\mathfrak{s}$ is the minimal size of a splitting family.

$\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$

In 1984 S. Shelah showed the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$ using a proper, almost ${}^\omega\omega$ bounding forcing notion of size continuum, which adds a real not split by the ground model reals.

$\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

Given an unbounded $<^*$-directed family $\mathcal{H}$ of size κ we obtain a σ-centered suborder $\mathbb{P}_\mathcal{H}$ of Shelah's poset, which preserves $\mathcal{H}$ unbounded and adds a real not split by $V \cap [\omega]^\omega$.

$\mathbb{M}$ adds a real not split by the ground model reals

If G is $\mathbb{M}$-generic, then $U_G = \cup\{u : \exists A(u, A) \in G\}$ is an infinite set such that $\forall A \in V \cap [\omega]^\omega$, $U_G \subseteq^* A$ or $U_G \subseteq^* A^c$.

$\mathbb{M}$ adds a dominating real

However, if F_G is the enumerating function of U_G, then F_G dominates all ground model reals.

Definition

- Let $s \subseteq \omega$. Then $h \colon [s]^{<\omega} \to \omega$ is called a *logarithmic measure* if $\forall A \in [s]^{<\omega}$, $\forall A_0, A_1$ such that $A = A_0 \cup A_1$, $h(A_i) \geq h(A) - 1$ for $i = 0$, or $i = 1$ unless $h(A) = 0$.
- If s is finite, the pair $x = (s, h)$ is called *a finite logarithmic measure*. The value $h(s) = \|x\|$ is called *the level of* x and $\mathrm{int}(x)$ denotes s.

Definition

Let $P \subseteq [\omega]^{<\omega}$ be upwards closed family, which does not contain singletons. Then P induces a logarithmic measure on $[\omega]^{<\omega}$ defined inductively as follows:

1. $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$
2. $h(e) > 0$ iff $e \in P$
3. for $\ell \geq 1$, $h(e) \geq \ell + 1$ iff whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \max\{k : h(e) \geq k\}$. The elements of P are called positive sets and h is said to be induced by P.

Example

Let $P = \{a \in [\omega]^{<\omega} : |a| \geq 2\}$. Then $h(a) = \min\{j : |a| \leq 2^j\}$ is the logarithmic measure induced by P, called *standard measure*.

Lemma

Let $A \subseteq \omega$ does not contain a set of measure $\geq \ell + 1$ for some $\ell \in \omega$. Then there are A_0, A_1 such that $A = A_0 \cup A_1$ and none of A_0, A_1 contain a set of measure $\geq \ell$.

Lemma

Let $P \subseteq [\omega]^{<\omega}$ be upwards closed family, which does not contain singletons and let h be induced by P. Then if for every $n \in \omega$ and partition $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that A_j contains a positive set, then for every $k \in \omega$, for every $n \in \omega$ and partition $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that A_j contains a set of measure $\geq k$.

Definition

Let Q be the set of all pairs (u, T) where u is a finite subset of ω and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of logarithmic measures such that

1. $\max u < \min s_0$
2. $\max s_i < \min s_{i+1}$ for all $i \in \omega$
3. $\langle h_i(s_i) : i \in \omega \rangle$ is unbounded.

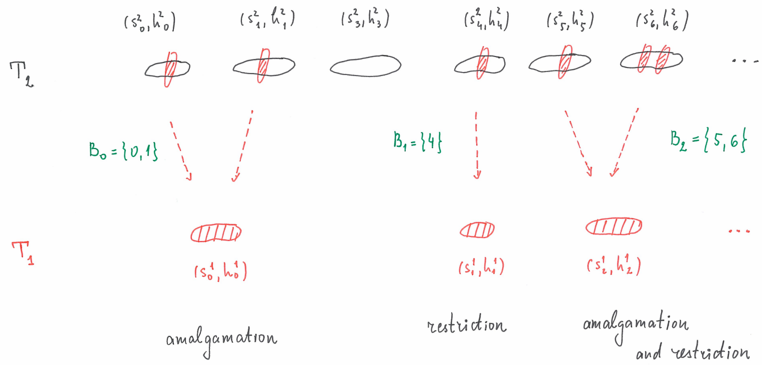
Also $\text{int}(T) = \cup\{s_i : i \in \omega\}$. If $u = \emptyset$, then $(\emptyset, T)$ is *a* pure condition and is denoted by T. Note that if (u, T) is Shelah's condition, then $(u, \text{int}(T))$ is Mathias.

We say $(u_2, T_2) \leq (u_1, T_1)$, where $T_\ell = \langle (s_i^\ell, h_i^\ell) : i \in \omega \rangle$ for $\ell = 1, 2$, if the following conditions hold:

1. u_2 is an end-extension of u_1 and $u_2 \backslash u_1 \subseteq \text{int}(T_1)$
2. $\text{int}(T_2) \subseteq \text{int}(T_1)$ and furthermore there is an infinite sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max u_2 < \min s_1^j$ for $j = \min B_0$, $\max(B_i) < \min(B_{i+1})$ and $s_i^2 \subseteq \bigcup \{s_j^1 : j \in B_i\}$.
3. for every subset e of s_i^2 such that $h_i^2(e) > 0$ there is $j \in B_i$ such that $h_j^1(e \cap s_j^1) > 0$.

If $u_1 = u_2$, then (u_2, T_2) is a pure extension of (u_1, T_1).

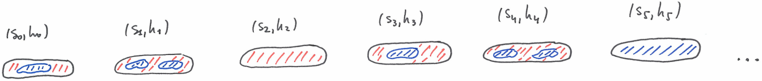
Extensions in $\mathbb{Q}$: $T_1 \leq T_2$



$A \in [\omega]^\omega$

$T \restriction A$ or $T \restriction A^c$ is a pure condition

$T = \langle (s_i, h_i) : i \in \omega \rangle$



$T \restriction A$

$\langle h_i(s_i \cap A) : i \in \omega \rangle$

$T \restriction A^c$

$\langle h_i(s_i \cap A^c) : i \in \omega \rangle$

Definition

Let $\mathcal{F}$ be family of pure conditions. Then $Q(\mathcal{F})$ is the suborder of Q of all $(u, T) \in Q$ such that $\exists R \in \mathcal{F}(R \leq T)$.

- If C is centered, then $Q(C)$ is σ-centered.
- Let $p, q \in Q(C)$. Then $p \not\perp_Q q$ iff $p \not\perp_{Q(C)} q$.
- If $C \subseteq Q(C')$ then C' is said to extend C.
- If $T \not\perp C$ and $\omega = A_0 \cup \cdots \cup A_{n-1}$, then $\exists j \in n$ and $R \leq T(R \not\perp C)$ such that $\text{int}(R) \subseteq A_j$.

$\mathbb{P}_{\mathcal{H}} = Q(C_{\mathcal{H}})$

The forcing notion $\mathbb{P}_{\mathcal{H}}$ is of the form $Q(C_{\mathcal{H}})$. Starting with an arbitrary pure condition T and $C_0 = \{T \backslash v : v \in [\omega]^{<\omega}\}$, we will obtain a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ of centered families such that $\forall \alpha < \beta (C_\alpha \subseteq Q(C_\beta))$ and $C_{\mathcal{H}} = \cup_{\alpha \in \kappa} C_\alpha$.

$C_\alpha \subseteq Q(C_{\alpha+1})$

At successor stages, we will use three distinct countable forcing notions each of which adds a single pure condition with desired combinatorial properties.

Definition

Let Q_{fin} be the poset of all $\bar{r} = \langle r_0, \dots, r_n \rangle$ of finite measure such that $\forall i \in n$, $\max \operatorname{int}(r_i) < \min \operatorname{int}(r_{i+1})$ and $\|r_i\| < \|r_{i+1}\|$ with extension relation end-extension.

Definition

Let $\bar{r} \in Q_{fin}$ and T a pure condition. Then $\bar{r} \leq T$ if there is a pure condition $R \leq T$ such that $\bar{r} \subseteq R$.

Definition

Let T be a pure condition. Then $\mathbb{P}(T)$ is the suborder of Q_{fin} of all finite sequences $\bar{r}$ extending T.

Lemma

Let $T \not\leq X$, $n \in \omega$. Then

$$D_T(X, n) = \{\bar{r} \in \mathbb{P}(T) : \exists r_j \in \bar{r}(r_j \leq X \text{ and } \|r_j\| \geq n)\}$$

is dense in $\mathbb{P}(T)$.

Corollary

Let $C \not\perp T$, let G be $\mathbb{P}(T)$-generic filter. Then in $V[G]$

- $R_G = \cup G = \langle r_i : i \in \omega \rangle \leq T$.
- $\exists C'$ *such that* $C \cup \{R_G\} \subseteq Q(C')$, $|C| = |C'|$.

Proof.

Since $G \cap D_T(X, n) \neq \emptyset$ for all $X \in C$, $n \in \omega$, the set $I_X = \langle i : r_i \leq X \rangle$ is infinite and so $R_G \wedge X = \langle r_i : i \in I_X \rangle$ is a common extension of R_G and X. If $X \leq Y$ then $I_X \subseteq I_Y$ and so $R_G \wedge X \leq R_G \wedge Y$. Then $C' = \{R_G \wedge X\}_{X \in C}$ is centered. □

Lemma

Let $cov(\mathcal{M}) = \kappa$, C centered $|C| < \kappa$, $\dot{f}$ a good $Q(C)$-name for a real. Then there is $T = \langle r_i : i \in \omega \rangle$ of logarithmic measures of strictly increasing levels, such that

- *$\forall X \in C$ the set $J_X = \{i : r_i \leq X\}$ is infinite and*
- *$\forall i \forall v \subseteq i \forall s \subseteq int(r_i)$ which is r_i-positive $\exists w \subseteq s \exists p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T) \leq p$.*

The proof uses two countable forcing notions, the first of which produces a pure condition which is preprocessed for $\dot{f}$.

Under $\text{cov}(\mathcal{M}) = \kappa$ certain subfamilies of $[\omega]^{<\omega}$ induce logarithmic measures which take arbitrarily high values. The second forcing notions amalgamates such measures into the pure condition T.

Theorem

Let $cov(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ unbounded, $<^*$-directed, $|\mathcal{H}| = \kappa$, C centered, $|C| < \kappa$, $\dot{f}$ good $Q(C)$-name for a real. Then $\exists C' \exists h \in \mathcal{H}$, such that C' extends C, $|C'| = |C|$ and $\forall C''$ extending C', $\Vdash_{Q(C'')}$ "$\check{h} \not<^* \dot{f}$".

Proof

Let $T = \langle r_i : i \in \omega \rangle$ satisfy the preceding Lemma for C and $\dot{f}$. Then $\forall i \in \omega$ let $g(i)$ be the maximal k such that there are $v \subseteq i$, $w \subseteq \text{int}(r_i)$, $p \in \mathcal{A}_i(\dot{f})$ with $p \Vdash \check{k} = \dot{f}(i)$ and $(v \cup w, T) \leq p$.

- $\forall X \in C$ $J_X = \{i : r_i \leq X\}$ is infinite. Then $\forall n \in \omega$ let $F_X(n) = g(J_X(i+1))$ iff $n \in (J_X(i), J_X(i+1)]$ where $J_X(n)$ is the n-th element of J_X. Then $\forall X \in C \exists h_X \in \mathcal{H}(h_X \not\leq^* F_X)$.
- Let $h \in \mathcal{H}$ dominate all h_X's. Then $J = \{i : g(i) < h(i)\}$ and $I_X = J_X \cap J$ are infinite. Let $R = \langle r_i \rangle_{i \in J}$, $R \wedge X = \langle r_i \rangle_{i \in I_X}$ and $C' = \{R \wedge X\}_{X \in C}$.

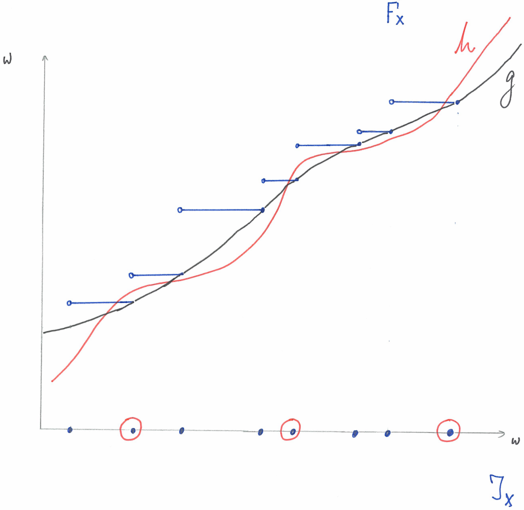
$J_X = \{ i : \tau_i \le X \}$

$J = \{ i : g(i) < h(i) \}$

$\exists^\infty i \in J_X \, (F_X(i) < h(i))$

otherwise $h \le^* F_X$

Therefore $I_X = J_X \cap J$ is infinite



$\forall C''$ extending $C' \Vdash_{Q(C'')} \check{h} \not<^* \dot{f}$

- Let C'' be centered, $C' \subseteq Q(C'')$, $a \in [\omega]^{<\omega}$, $k_0 \in \omega$ and let $(b, R') \in Q(C'')$ be an extension of (a, R). There is $i \in J$, $i > k_0$ such that $b \subseteq i$ and $s = \text{int}(R') \cap \text{int}(r_i)$ is r_i-positive. Then $\exists w \subseteq s \exists p \in \mathcal{A}_i(\dot{f})$ such that $(b \cup w, T) \leq p$.
- Therefore $(b \cup w, R')$ extends (b, R') and p. Let $k \in \omega$ be such that $p \Vdash \dot{f}(i) = \check{k}$. Then by definition of g, $k \leq g(i)$ and since $i \in J$, $g(i) < h(i)$. Thus $(b \cup w, R') \Vdash_{Q(C'')}$ "$\dot{f}(i) = \check{k} \leq \check{g}(i) < \check{h}(i)$".

Lemma

Let $cov(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ be an unbounded, directed family of cardinality κ and let $\forall\lambda < \kappa(2^\lambda \leq \kappa)$. Then there is a centered family C, $|C| = \kappa$ such that $Q(C)$ preserves $\mathcal{H}$ unbounded and adds a real not split by $V \cap [\omega]^\omega$.

Let $\mathcal{N} = \{\dot{f}_\alpha\}_{\alpha<\kappa}$ enumerate all $Q(C')$ names for functions in ${}^\omega\omega$ where $|C'| < \kappa$. Let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha<\kappa}$ enumerate $V \cap [\omega]^\omega$. By induction of length κ obtain a sequence $\langle C_\alpha : \alpha < \kappa\rangle$ such that $\forall \alpha < \beta C_\alpha \subseteq Q(C_\beta)$, $|C_\alpha| < \kappa$ as follows:

- Begin with any T and $C_0 = \{T \backslash v : v \in [\omega]^{<\omega}\}$
- If α is a limit, let $C_\alpha = \cup_{\beta<\alpha} C_\beta$

If $\alpha = \beta + 1$, let $\dot{g}_\alpha$ be the name with least index in $\mathcal{N} \backslash \{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name.

- If $\dot{g}_\alpha$ is good, let C_α extend C_β, $|C_\alpha| = |C_\beta|$ such that
  1. $\exists h_\alpha \in \mathcal{H} \forall C''$ extending $C_\alpha \Vdash_{Q(C'')}$ "$\check{h}_\alpha \not<^* \dot{g}_\alpha$"
  2. $\exists T_\alpha \in Q(C_\alpha)(\text{int}(T_\alpha) \subseteq A_\alpha$ or $\text{int}(T_\alpha) \subseteq A^c_\alpha)$.
- If $\dot{g}_\alpha$ is not good, let C_α extend C_β, $|C_\alpha| = |C_\beta|$ such that
  1. $\dot{g}_\alpha$ is not a $Q(C_\alpha)$-name,
  2. $\exists T_\alpha \in Q(C_\alpha)(\text{int}(T_\alpha) \subseteq A_\alpha$ or $\text{int}(T_\alpha) \subseteq A^c_\alpha)$.

Then let $C = \cup_{\alpha<\kappa} C_\alpha$.

$\mathcal{H}$ is unbounded

If $\dot{f}$ is a $Q(C)$-name, then $\exists\beta \in \kappa$ such that $\dot{f}$ is a good $Q(C_\beta)$-name and is the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name. Then $(\mathcal{H} \text{ is unbounded})^{V^{Q(C)}}$.

$\exists$ a real not split by the ground model reals

Let G be $Q(C)$-generic. Then for every $A \in V \cap [\omega]^\omega$ there is (u, T) in G such that $\text{int}(T) \subseteq A$ or $\text{int}(T) \subseteq A^c$. Note also that if $U_G = \cup\{u : \exists T(u, T) \in G\}$, then $U_G \subseteq^* \text{int}(T)$ for all T such that $\exists u(u, T) \in G$.

Theorem

Let $\mathcal{H} \subseteq {}^\omega\omega$ be unbounded family such that every countable subfamily of $\mathcal{H}$ is dominated by an element of $\mathcal{H}$ and let $\langle \mathbb{P}_\gamma : \gamma \leq \alpha \rangle$ be a finite support iteration of ccc forcing notions of length α, $cf(\alpha) = \omega$ such that $\forall \gamma < \alpha$ $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}_\gamma}}$. Then $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}_\alpha}}$.

Theorem

Let $\mathcal{H} \subseteq {}^\omega\omega$ be unbounded, directed family, $|\mathcal{H}| = \kappa$. Then for every partial order $\mathbb{P}$ of size less than κ, $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}}}$.

Theorem (*GCH*)

Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

Add κ Hechler reals to obtain a model V of $\mathfrak{b} = \mathfrak{c} = \kappa$. Let $\mathcal{H} = V \cap {}^{\omega}\omega$. Define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+ \rangle$ such that $\forall \alpha < \kappa^+$

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \text{ is } \textit{ccc} \text{ and } |\dot{\mathbb{Q}}_\alpha| \leq \mathfrak{c}\text{”}$$

as follows. If α is a limit, let $\mathbb{P}_\alpha$ be the finite support iteration of $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$. If $\alpha = \beta + 1$ is a successor, then

- Let $\dot{\mathbb{Q}}_\beta$ be $\mathbb{P}_\beta$-name for $\mathbb{C}(\kappa)$ and $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. $\exists C$ such that $Q(C)$ preserves $\mathcal{H}$ unbounded and destroys $V^{\mathbb{P}_\alpha} \cap [\omega]^\omega$ as a splitting family.
- Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$ name for $Q(C)$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.
- Let $\mathcal{A} \subseteq V^{\mathbb{P}_{\alpha+1}} \cap {}^\omega\omega$ be unbounded of size less than κ. Then let $\dot{\mathbb{Q}}_{\alpha+1}$ be $\mathbb{P}_{\alpha+1}$-name for $\mathbb{H}(\mathcal{A})$; $\mathbb{P}_{\alpha+2} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{Q}}_{\alpha+1}$.

Then in $V^{\mathbb{P}_{\kappa^+}}$ $\mathcal{H}$ is unbounded and there are no splitting families of size less than κ^+. Using a suitable bookkeeping device one can guarantee that there are no unbounded families of size less than κ.