ITERATIONS WITH MIXED SUPPORT

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Abstract. In this talk we will consider three properties of iterations with mixed (finite/countable) supports: iterations of arbitrary length preserve ω_1, iterations of length $\leq \omega_2$ over a model of CH have the $\aleph_2$-chain condition and iterations of length $< \omega_2$ over a model of CH do not increase the size of the continuum.

Definition 1. Let $\mathbb{P}_\kappa$ be an iterated forcing construction of length κ, with iterands $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ such that for every $\alpha < \kappa$

$$\Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \text{ is } \sigma\text{-centered" or } \Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \text{ is countably closed"}.$$

Then $\mathbb{P}_\kappa$ is *finite/countable iteration* if and only if for every $p \in \mathbb{P}_\kappa$, $\text{support}(p) = \{\alpha < \kappa : p(\alpha) \neq \dot{\mathbb{1}}_\alpha\}$ is countable and $\text{Fsupport}(p) = \{\alpha : \Vdash \text{"}Q_\alpha \text{ is } \sigma\text{-centered"}, \ p(\alpha) \neq \dot{\mathbb{1}}_\alpha\}$ is finite.

Remark 1. In the context of the above definition, whenever

$$\Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \text{ is } \sigma\text{-centered"}$$

we will say that α is a σ-centered stage and correspondingly, whenever

$$\Vdash \text{"}\dot{\mathbb{Q}}_\alpha \text{ is countably closed"}$$

we will say that α is a countably closed stage.

From now on $\mathbb{P}_\kappa$ is a finite/countable iteration of length κ.

Definition 2. Let $p, q \in \mathbb{P}_\kappa$. We say than $p \leq_D q$ if and only if $p \leq q$ and for every σ-centered stage $\alpha < \kappa$, $p \restriction \alpha \Vdash p(\alpha) = q(\alpha)$. Similarly $p \leq_C q$ if and only if for every countably closed stage α, $p \restriction \alpha \Vdash p(\alpha) = q(\alpha)$.

Claim. Both $\leq_D$ and $\leq_C$ are transitive relations.

Lemma 1. *Let $\langle p_n \rangle_{n\in\omega}$ be a sequence in $\mathbb{P}_\kappa$ such that for every $n \in \omega$, $p_{n+1} \leq_D p_n$. Then there is a condition $p \in \mathbb{P}_\kappa$ such that for every $n \in \omega$, $p \leq_D p_n$.*

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Proof. Define p inductively. It is sufficient to define p for successor stages α. Suppose we have defined $p \restriction \alpha$ so that for every $n \in \omega$, $p \restriction \alpha \leq_D p_n \restriction \alpha$. If α is σ-centered then

$$p \restriction \alpha \Vdash p_0(\alpha) = p_1(\alpha) = \dots$$

and so we can define $p(\alpha) = p_0(\alpha)$. If α is countably closed stage, then

$$p \restriction \alpha \Vdash p_0(\alpha) \geq p_1(\alpha) \geq \dots$$

and since $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is countably closed" there is a $\mathbb{P}_\alpha$-name $p(\alpha)$ such that $p \restriction \alpha \Vdash p(\alpha) \leq p_n(\alpha)$ for every $n \in \omega$. □

Lemma 2. *Let $p, q \in \mathbb{P}_\kappa$ be such that $p \leq q$. Then there is a condition $r \in \mathbb{P}_\kappa$ such that $p \leq_C r \leq_D q$.*

Proof. Again we will define r inductively. Suppose we have defined $r \restriction \alpha$ so that $p \restriction \alpha \leq_C r \restriction \alpha \leq_D q \restriction \alpha$. Then if α is a σ-centered stage, let $r(\alpha) = q(\alpha)$. If α is a countably closed stage, define $r(\alpha)$ to be a $\mathbb{P}_\alpha$-term such that: if $\bar{r} \leq p \restriction \alpha$ then $\bar{r} \Vdash_\alpha r(\alpha) = p(\alpha)$, if $\bar{r}$ is incompatible with $p \restriction \alpha$ then $\bar{r} \Vdash_\alpha r(\alpha) = q(\alpha)$.

To verify $p \leq_C r$ note that if α is a σ-centered stage then $p \restriction \alpha \Vdash p(\alpha) \leq q(\alpha) = r(\alpha)$. If α is a countably closed stage, then $p \restriction \alpha \Vdash p(\alpha) = r(\alpha)$.

To see that $r \leq_D q$ note that if α is a σ-centered stage then by definition $r \restriction \alpha \Vdash r(\alpha) = q(\alpha)$. If α is countably closed stage, it is sufficient to show that $\mathbb{1} \Vdash_\alpha r(\alpha) \leq q(\alpha)$. Let $\bar{r} \in \mathbb{P}_\alpha$. If $\bar{r} \not\perp p \restriction \alpha$ fix a common extension t. Then $t \Vdash p(\alpha) = r(\alpha) \wedge p(\alpha) \leq q(\alpha)$ and so $t \Vdash r(\alpha) \leq q(\alpha)$. If $\bar{r}$ is incompatible with $p \restriction \alpha$ then by definition $\bar{r} \Vdash r(\alpha) = q(\alpha)$. □

Definition 3. Let α be a σ-centered stage and let $\dot{s}_\alpha$ be a $\mathbb{P}_\alpha$-name such that $\Vdash_\alpha (\dot{s}_\alpha \colon \dot{\mathbb{Q}}_\alpha \to \omega) \wedge [\forall p, q \in \dot{\mathbb{Q}}_\alpha (\dot{s}_\alpha(p) = \dot{s}_\alpha(q) \to p \not\perp q)]$. Condition $p \in \mathbb{P}_\kappa$ is *determined* if and only if for every $\alpha \in \mathrm{Fsupport}(p)$ there is $n \in \omega$ such that $p \restriction \alpha \Vdash \dot{s}_\alpha(p(\alpha)) = \check{n}$.

Claim. The set of determined conditions in $\mathbb{P}_\kappa$ is dense.

Proof. Proceed by induction on the length of the iteration κ. It is sufficient to consider successor stages. Let $\alpha = \beta + 1$ and $p \in \mathbb{P}_\alpha$. We can assume that β is a σ-centered stage. By inductive hypothesis there is a determined condition $\bar{r} \leq p \restriction \beta$ such that $\bar{r} \Vdash \dot{s}_\beta(p(\beta)) = \check{n}$ for some $n \in \omega$. If $r \in \mathbb{P}_\alpha$ is such that $r \restriction \beta = \bar{r}$ and $r(\beta) = p(\beta)$, then r is determined and $r \leq p$. □

Lemma 3. *Let q_1, q_2 be (determined) conditions in $\mathbb{P}_\kappa$ such that*

$$\mathit{Fsupport}(q_1) = \mathit{Fsupport}(q_2) = F$$

and for every $\alpha \in F$ *there is* $n \in \omega$ *such that*

$$q_1 \restriction \alpha \Vdash \dot{s}_\alpha(q_1(\alpha)) = \check{n} \text{ and } q_2 \restriction \alpha \Vdash \dot{s}_\alpha(q_2(\alpha)) = \check{n}.$$

Furthermore, let $p \in \mathbb{P}_\kappa$ *such that* $q_1 \leq_C p$ *and* $q_2 \leq p$*. Then* q_1 *and* q_2 *are compatible.*

Proof. The common extension r of q_1 and q_2 will be defined inductively. Suppose we have defined $r \restriction \alpha$ for some $\alpha < \kappa$ such that $r \restriction \alpha \leq q_1 \restriction \alpha$ and $r \restriction \alpha \leq q_2 \restriction \alpha$. If α is a σ-centered stage, then there is $n \in \omega$ such that $r \restriction \alpha \Vdash \dot{s}_\alpha(q_1(\alpha)) = \dot{s}_\alpha(q_2(\alpha)) = \check{n}$ and so there is a $\mathbb{P}_\alpha$-name $r(\alpha)$ for a condition in $\dot{\mathbb{Q}}_\alpha$ such that $r \restriction \alpha \Vdash r(\alpha) \leq q_1(\alpha) \wedge r(\alpha) \leq q_2(\alpha)$. If α is countably closed then $r \restriction \alpha \Vdash q_1(\alpha) = p(\alpha) \wedge q_2(\alpha) \leq p(\alpha)$. Thus we can define $r(\alpha) = q_2(\alpha)$. □

Definition 4. An antichain $\langle q_\xi : \xi < \eta \rangle$ of determined conditions is *concentrated* with *witnesses* $\langle p_\xi : \xi < \eta \rangle$ if and only if $\forall \xi < \eta (q_\xi \leq_C p_\xi)$ and $\forall \zeta < \xi (p_\xi \leq_D p_\zeta)$.

Lemma 4. *There are no uncountable concentrated antichains.*

Proof. Suppose to the contrary that $\langle q_\xi : \xi < \omega_1 \rangle$ is a concentrated antichain with witnesses $\langle p_\xi : \xi < \omega_1 \rangle$. For every $\xi < \omega_1$ let $F_\xi = \mathrm{Fsupport}(q_\xi)$. Since a subset of a concentrated antichain is a concentrated antichain, we can assume that $\langle F_\xi : \xi < \omega_1 \rangle$ form a Δ-system with root F such that for some $\alpha < \kappa$, $F \subseteq \alpha < \min F_\xi \backslash F$ for every $\xi < \omega_1$.

Claim. $\langle q_\xi \restriction \alpha : \xi < \omega_1 \rangle$ is a concentrated antichain in $\mathbb{P}_\alpha$.

Proof. Suppose to the contrary that there are $\zeta < \xi$ such that for some $\bar{r} \in \mathbb{P}_\alpha$, $\bar{r} \leq q_\zeta \restriction \alpha$ and $\bar{r} \leq q_\xi \restriction \alpha$. Then for every $\gamma \geq \alpha$, define $r(\gamma)$ as follows: if $\gamma \in F_\zeta$ let $r(\gamma) = q_\zeta(\gamma)$, otherwise let $r(\gamma) = q_\xi(\gamma)$. Inductively we will show that r is a common extension of q_ζ and q_ξ. It is sufficient to show that for all countably closed stages γ if $r \restriction \gamma \leq q_\xi \restriction \gamma$ and $r \restriction \gamma \leq q_\zeta \restriction \gamma$, then $r \restriction \gamma \Vdash r(\gamma) \leq q_\zeta(\gamma) \wedge r(\gamma) \leq q_\xi(\gamma)$. Note that

$$r \restriction \gamma \Vdash (q_\zeta(\gamma) = p_\zeta(\gamma)) \wedge (p_\xi(\gamma) \leq p_\zeta(\gamma)) \wedge (q_\xi(\gamma) = p_\xi(\gamma))$$

and so $r \restriction \gamma \Vdash r(\gamma) = q_\xi(\gamma) \leq q_\zeta(\gamma)$. □

For every $\xi < \omega_1$ let $f_\xi : F \to \omega$ be such that $f_\xi(\gamma) = n$ if and only if $q_\xi \restriction \gamma \Vdash \dot{s}_\gamma(q_\xi(\gamma)) = \check{n}$. Since there are only countably many such functions, there are $\zeta < \xi$ such that $f_\zeta = f_\xi$. Then $q_\zeta \restriction \alpha$, $q_\xi \restriction \alpha$ and $p_\zeta \restriction \alpha$ satisfy the hypothesis of Lemma 3 and so $q_\zeta \restriction \alpha$ and $q_\xi \restriction \alpha$ are compatible, which is a contradiction. □

Lemma 5. *Let $p \in \mathbb{P}_\kappa$ and let $\dot{x}$ be a $\mathbb{P}_\kappa$-name such that $p \Vdash \dot{x} \in V$. Then there is $q \leq_D p$ and a ground model countable set X such that $q \Vdash \dot{x} \in \check{X}$.*

Proof. Inductively construct concentrated antichain $\langle q_\xi : \xi < \eta < \omega_1 \rangle$ with witnesses $\langle p_\xi : \xi < \eta < \omega_1 \rangle$ such that for all $\xi < \eta$, $q_\xi \leq p$, $p_\xi \leq_D p$ and $\exists x_\xi \in V$, $q_\xi \Vdash \dot{x} = \check{x}_\xi$. Furthermore we will have that if $\xi \neq \zeta$, then $x_\xi \neq x_\zeta$. Suppose $\langle q_\xi : \xi < \eta \rangle$ and $\langle p_\xi : \xi < \eta \rangle$ have been defined. Since η is countable, by Lemma 1 there is condition p' such that $\forall \xi < \eta (p' \leq_D p_\xi)$. *Case 1.* If $p' \Vdash \dot{x} \in \{x_\xi : \xi < \eta\}$, then let $q = p'$ and $X = \{x_\xi : \xi < \eta\}$. *Case 2.* Otherwise, there is $q_\eta \leq p'$, $x_\eta \in V$ such that $x_\eta \notin \{x_\xi : \xi < \eta\}$ and $q_\eta \Vdash \dot{x} = \check{x}_\eta$. By Lemma 2 there is p_η such that $q_\eta \leq_C p_\eta \leq_D p'$. This extends the concentrated antichain and so completes the inductive step. Since there are no uncountable concentrated antichains at some countable stage of the construction Case 1 must occur. □

Corollary 1. *Let p be a condition in $\mathbb{P}_\kappa$ and let $\dot{f}$ be a $\mathbb{P}_\kappa$-name such that $p \Vdash \dot{f} \colon \omega \to V$. Then there is $q \leq_D p$ and $X \in V$, X countable such that $q \Vdash \dot{f}"\omega \subseteq \check{X}$. Therefore $\mathbb{P}_\kappa$ preserves ω_1.*

Proof. Inductively define a sequence of conditions $\langle p_n \rangle_{n\in\omega}$ in $\mathbb{P}_\kappa$, where $p_{-1} = p$ and a sequence of countable sets $\{X_n\}_{n\in\omega} \subseteq V$ such that for every n, $p_{n+1} \leq_D p_n$ and $p_n \Vdash \dot{f}(n) \in \check{X}_n$. Let $q \in \mathbb{P}_\kappa$ be such that $q \leq_D p_n$ for every n and let $X = \cup_{n\in\omega} X_n$. Then X is a countable ground model set, $q \leq_D p$ and $q \Vdash \dot{f}"\omega \subseteq \check{X}$. □

Theorem 1 (CH)**.** Let $\mathbb{P}_{\omega_2}$ be a finite/countable support iteration with iterands $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ such that $\forall \alpha < \omega_2$, $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| = \aleph_1$. Then $\mathbb{P}_{\omega_2}$ is $\aleph_2$-c.c.

Proof. It is sufficient to show that for every $\alpha < \omega_2$ there is a dense subset D_α in $\mathbb{P}_\alpha$ of cardinality $\aleph_1$. Suppose $\langle q_\xi : \xi < \omega_2 \rangle$ is an antichain in $\mathbb{P}_{\omega_2}$ of size $\aleph_2$. We can assume that $\langle F_\xi : \xi < \omega_2 \rangle$ where $F_\xi =$ Fsupport(q_ξ) form a Δ-system with root F such that for some $\alpha < \omega_2$,

$$F \subseteq \alpha < \min F_\xi \backslash F$$

for every $\xi < \omega_2$. Then $\langle q_\xi \restriction \alpha : \xi < \omega_2 \rangle$ is an antichain in $\mathbb{P}_\alpha$ of size $\aleph_2$ which is not possible. As an additional requirement we will have that for every $p \in \mathbb{P}_\alpha$ there is $d \in D_\alpha$ such that $d \leq_D p$.

Proceed by induction. Suppose $\alpha = \beta + 1$ and we have defined D_β $\leq_D$-dense in $\mathbb{P}_\beta$ of size $\aleph_1$. Let $\{\dot{d}_\xi : \xi < \omega_1\}$ be a set of $\mathbb{P}_\beta$-terms such that $\Vdash_\beta$ "$\dot{\mathbb{Q}}_\beta = \{\dot{d}_\xi : \xi < \omega_1\}$". For every countable antichain $A \subseteq D_\beta$ and function $f \colon A \to \omega_1$ let $\dot{q}(f)$ be a $\mathbb{P}_\beta$-term such that for every $p \in \mathbb{P}_\beta$ the following holds: if there is $a \in A$ such that $p \leq a$ then $p \Vdash \dot{q}(f) = \dot{d}_{f(a)}$; if p is incompatible with every element of A then $p \Vdash \dot{q}(f) = \dot{\mathbb{1}}_\beta$. The collection T of all such names is of size $\aleph_1$ and so

$$D_\alpha = \{p \in \mathbb{P}_\alpha : p \restriction \beta \in D_\beta \text{ and } p(\beta) \in T\}$$

is of size $\aleph_1$ as well. We will show that D_α is $\leq_D$-dense in $\mathbb{P}_\alpha$. Consider arbitrary $p \in \mathbb{P}_\alpha$ and let A be a maximal antichain of conditions in D_β such that for every $a \in A$ there is $\xi < \omega_1$ such that $a \Vdash p(\beta) = \dot{d}_\xi$.

Claim. There is $q \leq_D p \restriction \beta$ s. t. $A' = \{a \in A : a \not\perp q\}$ is countable.

Proof. Fix an enumeration $\langle a_\xi : \xi < \omega_1\rangle$ of A and let $\dot{x} = \{\langle \check{\xi}, a_\xi\rangle : \xi < \omega_1\}$. Then $\dot{x}$ is a $\mathbb{P}_\beta$-name for an ordinal and so repeating the proof of Lemma 4 we can obtain $X \in V \cap [\omega_1]^\omega$ and $q \leq_D p$ such that $q \Vdash \dot{x} \in \check{X}$. Then $q \Vdash \dot{x} \leq \sup \check{X}$ and so $\forall \xi > \sup X (q \perp a_\xi)$. □

Let $f\colon A' \to \omega_1$ be such that $f(a) = \xi$ if and only if $a \Vdash p(\beta) = \dot{d}_\xi$. Then $q \Vdash \dot{q}(f) = p(\beta)$. By inductive hypothesis, we can assume that $q \in D_\beta$ and so if $r \in \mathbb{P}_\alpha$ is such that $r \restriction \beta = q$ and $r(\beta) = \dot{q}(f)$ then $r \leq_D p$ and $r \in D_\alpha$.

Suppose α is a limit and for every $\beta < \alpha$ we have defined a $\leq_D$-dense subset D_β of $\mathbb{P}_\beta$ of size $\aleph_1$. Let $\bar{D}_\beta$ be the image of D_β under the canonical embedding of $\mathbb{P}_\beta$ into $\mathbb{P}_\alpha$. Then $\bar{D} = \cup_{\beta<\alpha}\bar{D}_\beta$ is of size $\aleph_1$ and furthermore there is a set $D \subseteq \mathbb{P}_\alpha$ of size $\aleph_1$ which contains $\bar{D}$ and such that for every sequence $\langle p_n\rangle_{n\in\omega} \subseteq \bar{D}$ for which $\forall n(p_{n+1} \leq_D p_n)$ there is $p' \in D$ such that $\forall n \in \omega(p' \leq_D p_n)$. We will show that D is $\leq_D$-dense in $\mathbb{P}_\alpha$.

Let $p \in \mathbb{P}_\alpha$. If $\sup(\text{support}(p)) = \beta < \alpha$ then by inductive hypothesis there is $d \in \bar{D}_\beta$ such that $d \leq_D p$. Otherwise fix an increasing and cofinal sequence $\langle \alpha_n\rangle_{n\in\omega}$ in α such that $\text{Fsupport}(p) \subseteq \alpha_0$. Inductively define a sequence $\langle d_n\rangle_{n\in\omega}$ such that for all n, $d_n \in \mathbb{P}_{\alpha_n}$ and $d_{n+1} \leq_D d_n \wedge p \restriction \alpha_{n+1}$. If $d \in D$ is such that $\forall n(d \leq_D d_n)$, then $\forall n(d \leq_D p \restriction \alpha_n)$ and so $d \leq_D p$. □

Lemma 6 (CH)**.** *A forcing notion which preserves ω_1 and has a dense subset of size $\aleph_1$ does not increase the size of the continuum.*

Proof. Suppose $\mathbb{P}$ is a forcing notion which preserves ω_1 and has a dense subset D of size $\aleph_1$. Let T be the collection of all pairs $\langle p, \dot{y}\rangle$ where $p \in D$, $\dot{y}$ is a $\mathbb{P}$-name for a subset of ω and for every $n \in \omega$, there is a countable antichain of conditions in D, deciding "$\check{n} \in \dot{y}$ which is maximal below p. Then $|T| \leq 2^{\aleph_0} = \aleph_1$. We will show that $V^{\mathbb{P}} \models 2^{\aleph_0} \leq |T|$.

Consider any $p \in P$ and $\dot{y}$ a $\mathbb{P}$-name for a subset of ω. Then for every $n \in \omega$ let $\langle r_{n,\xi} : \xi < \omega_1 \rangle$ be a maximal antichian of conditions in D deciding "$\check{n} \in \dot{y}$". Let $\dot{f}$ be a $\mathbb{P}$-term such that $\dot{f}(n) = \xi$ iff $r_{n,\xi} \in \dot{G}$. Then $p \Vdash \dot{f} : \omega \to \omega_1$ and since $\mathbb{P}$ preserves ω_1 and D is dense, there is $q \in D$ such that $q \leq p$ and $q \Vdash \dot{f}"\omega \subseteq \check{\beta}$ for some $\beta < \omega_1$. Then $\langle q, \dot{y} \rangle$ is a pair in T. □

Corollary 2 (CH)**.** *Let $\mathbb{P}_{\omega_2}$ be a finite/countable iteration of length ω_2 with iterands $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ such that $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| \leq 2^{\aleph_0}$. Then for every $\alpha < \omega_2$, $V^{\mathbb{P}_\alpha} \models CH$.*

Proof. Proceed by induction on α, repeating the proof of Theorem 1 and using Lemma 6. □

References

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