THE CONSISTENCY OF $\mathfrak{t} = \omega_1 < \mathfrak{h} = \omega_2$

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1. Preliminaries

In this section we systemize some well known definitions which will be used throughout the talk.

Definition 1. Suppose E and F are maximal almost disjoint families. We say that E is a refinement of F if and only if for every $x \in E$ there is $y \in F$ such that $x \subseteq^* y$.

In the following consider the partial order $([\omega]^\omega, \subseteq^*)$ consisting of infinite subsets of ω with extension relation almost-inclusion. That is if $A, B \in [\omega]^\omega$ then $A \leq B$ if and only if $A \subseteq^* b$. Note that in this setting $\mathfrak{t}$ is the greatest cardinal κ such that $[\omega]^\omega$ is κ-closed.

Definition 2. The *distributivity cardinal* $\mathfrak{h}$ is defined as the least cardinal κ such that forcing with $[\omega]^\omega$ adds a new real $h\colon \kappa \to V$ (where V denotes the ground model as usual). Equivalently, $\mathfrak{h}$ is the least cardinal such that any collection of less than κ-many maximal almost disjoint families have a common refinement.

The above remark implies $\mathfrak{t} \leq \mathfrak{h}$ and so we have the following inequalities

$$\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s}.$$

Remark 1. Certainly every tower has the strong finite intersection property and has no pseudo-intersection, which establishes the first inequality. To obtain that $\mathfrak{h} \leq \mathfrak{s}$ consider a splitting family $\mathcal{A} = \{a_\alpha : \alpha \in \mathfrak{s}\}$ and let G be a $[\omega]^\omega$-generic filter. Then in $V[G]$ define $f\colon \mathfrak{s} \to 2$ as follows:

$$f(\alpha) = 1 \text{ iff } a_\alpha \in G.$$

Consider any $a \in [\omega]^\omega$ as a condition in the associated partial order. Since the family $\mathcal{A}$ is splitting, there is an $\alpha \in \mathfrak{s}$ such that both

$$a \cap a_\alpha \text{ and } a \cap a_s^c$$

are infinite. But then a does not decide $\dot{f}(\alpha)$ and so f is a new function $\mathfrak{s} \to V$. Here $\dot{f}$ is an $[\omega]^\omega$-name for the function f.

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Recall also that the following:

Definition 3. The Mathias forcing notion $\mathbb{P}$ consists of all pairs

$$(s, A) \in [\omega]^{<\omega} \times [\omega]^{\omega}$$

where $(t, B) \leq (s, A)$ (that is (t, B) is stronger than (s, A)) if and only if t end-extends s, $B \subseteq A$ and $t - s \subseteq B$.

Lemma 1. *There is a two stage iteration $Q * \dot{R}$ of a countably closed forcing notion Q and a σ-centered forcing notion $\dot{R}$ (that is $\mathbb{1} \Vdash_Q$ "$\dot{R}$ is $\sigma -$ centered") such that the Mathias partial order $\mathbb{P}$ is densely embedded into $Q * \dot{R}$.*

Proof. Let $Q = ([\omega]^{\omega}, \subseteq^*)$ and let G be Q-generic filter. Then in $V[G]$ define R to be the partial order consisting of all pairs (s, A) in the Mathias partial order $\mathbb{P}$ for which the pure part A belongs to G with the extension relation inherited from $\mathbb{P}$ and let $\dot{R}$ be a Q-name for R. Then Q is countably closed, $\Vdash_Q$ "$\dot{R}$ is $\sigma -$ centered" and the mapping

$$(s, A) \mapsto (A, (s, A))$$

is a dense embedding of $\mathbb{P}$ into $Q * \dot{R}$. $\square$

We will refer to the above two-stage iteration as *factored Mathias forcing.*

Theorem 1 (CH)**.** Let $\mathbb{P}_{\omega_2}$ be ω_2-stage iteration of Mathias forcing, or factored Mathias forcing. That is for every $\alpha < \omega_2$, we have that $\mathbb{1}_\alpha \Vdash$ "Q_α is Mathias forcing" or respectively for every $\alpha < \omega_2$, α-even $\mathbb{1}_\alpha \Vdash$ "$Q_\alpha * Q_{\alpha+1}$ is factored Mathias forcing". Suppose that $\mathbb{P}_{\omega_2}$ satisfies the following conditions:

(1) $\mathbb{P}_{\omega_2}$ is $\aleph_2$-c.c.
(2) For every $p \in \mathbb{P}_{\omega_2}$ the support of p is bounded
(3) For every $\alpha < \omega_2$ $V^{\mathbb{P}_\alpha} \vDash CH$.
(4) $\mathbb{P}_{\omega_2}$ preserves ω_1.

Then $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{h} = \omega_2$.

Proof. Let $\langle E_\gamma : \gamma \in \omega_1 \rangle$ be a collection of ω_1 $\mathbb{P}_{\omega_2}$-names for maximal almost disjoint families and let $p \in \mathbb{P}_{\omega_2}$. We can assume that for every $\gamma < \omega_1$

$$p \Vdash |E_\gamma| = \aleph_2$$

and fix sequences of $\mathbb{P}_{\omega_2}$-names for infinite subsets of ω such that for every $\gamma < \omega_1$

$$p \Vdash E_\gamma = \langle x_{\xi,\gamma} : \xi \in \omega_2 \rangle.$$

We will show that there is a $\mathbb{P}_{\omega_2}$-name $\dot{x}$ for an infinite subset of ω such that for all $\gamma < \omega_1$

$$p \Vdash \dot{x} \text{ is almost contained in an element of } E_\gamma.$$

Claim. For every sentence ϕ in the forcing language of $\mathbb{P}_{\omega_2}$ there is an $\alpha < \omega_2$ such that if $q \in \mathbb{P}_{\omega_2}$ and q decides ϕ then $q \restriction \alpha$ decides ϕ.

Proof. Fix a maximal antichain of conditions deciding ϕ. Then since $\mathbb{P}_{\omega_2}$ is $\aleph_2$-c.c. $|A| \leq \aleph_1$. Furthermore the support of every condition is bounded which implies that there is an $\alpha < \omega_2$ such that

$$\bigcup \{\text{support}(a) : a \in A\} \subseteq \alpha.$$

Then certainly, for every q which decides ϕ, $q \restriction \alpha$ decides ϕ. □

Claim. If $p \Vdash \dot{x} \subseteq \omega$ then there is $\alpha = \alpha(\dot{x}) < \omega_2$ such that $p \Vdash \dot{x} \in V[G_{\alpha(\dot{x})}]$.

Proof. For every $n \in \omega$ fix a maximal antichain A_n below p of conditions deciding "$\check{n} \in \dot{x}$" and let $\alpha_n(\dot{x}) < \omega_2$ be such that

$$\bigcup \{\text{support}(a) : a \in A_n\} \subseteq \alpha_n.$$

Let $\alpha = \alpha(\dot{x}) = \sup_{n \in \omega} \alpha_n(\dot{x})$. □

Claim. There is a function $f \colon \omega_2 \to \omega_2$ such that for every $\beta < \omega_2$ and every $\gamma < \omega_1$ we have

$$p \Vdash \langle \dot{x}_{\xi\gamma} : \xi < \beta \rangle \in V[G_{f(\beta)}].$$

Proof. For every $\beta < \omega_2$ let

$$f(\beta) = \sup\{\alpha(x_{\xi\gamma}) : \xi < \beta, \gamma < \omega_1\},$$

where $\alpha(\dot{x}_{\xi\gamma})$ is defined as above. □

Claim. There is a function $g \colon \omega_2 \to \omega_2$ such that for every $\beta < \omega_2$, every $\gamma < \omega_1$ and every $\mathbb{P}_\beta$-name such that $p \Vdash \dot{y} \in [\omega]^\omega$, we have

$$p \Vdash (\exists \xi < g(\beta) |\dot{y} \cap \dot{x}_{\xi\gamma}| = \aleph_0.$$

Proof. Let $\beta < \omega_2$. Fix any $\gamma < \omega_1$. Then $p \Vdash$ "E_γ is mad". Let $\dot{y}$ be a $\mathbb{P}_\beta$-term such that $p \Vdash \dot{y} \in [\omega]^\omega$. Then

$$p \Vdash \exists \xi < \omega_2 (|\dot{y} \cap \dot{x}_{\xi\gamma}| = \aleph_0).$$

Fix a maximal antichain $A_\gamma(\dot{y})$ below p such that for every $q \in A_\gamma(\dot{y})$ there is $\xi_q \in \omega_2$ such that

$$q \Vdash |\dot{y} \cap \dot{x}_{\xi\gamma}| = \aleph_0.$$

Then $|A_\gamma(\dot{y})| \leq \aleph_1$ and so there is $\alpha_\gamma(\dot{y}) < \omega_2$ such that

$$\bigcup\{\text{support}(a) : a \in A_\gamma(\dot{y})\} \subseteq \alpha_\gamma(\dot{y}).$$

Then $\alpha(\dot{y}) = \sup_{\gamma\in\omega_1} \alpha_\gamma(\dot{y})$ is also smaller than ω_2. However $V^{\mathbb{P}_\beta} \vDash CH$ and so we can define

$$g(\beta) = \sup\{\alpha_\gamma(\dot{y}) : \dot{y} \text{ is } \mathbb{P}_\beta - \text{name s.t. } p \Vdash_\beta \dot{y} \in [\omega]^\omega\}.$$

□

Let $\alpha < \omega_2$ be such that $\text{cof}(\alpha) = \omega_1$ and $\forall \beta < \alpha$, $f(\beta) < \alpha$ and $g(\beta) < \alpha$. Then the definition of f implies that for every $\gamma < \omega_1$

$$p \Vdash \langle x_{\xi\gamma} : \xi < \alpha\rangle \in V[G_\alpha]$$

and furthermore the definition of g implies that

$$V[G_\alpha] \vDash \forall\gamma < \omega_1(\langle \dot{x}_{\xi\gamma} : \xi < \alpha\rangle \text{ is mad})$$

since every real in $V[G_\alpha]$ appears in some $V[G_\beta]$ for $\beta < \alpha$. Really, suppose $\dot{x}$ is a $\mathbb{P}_\alpha$-name for an infinite subset of ω, which does not appear in $V[G_\beta]$ for any $\beta < \alpha$. Then in $V[G_\alpha]$ we can define a cofinal function $f\colon \omega \to \alpha$ as follows:

$$f(n) = \gamma \text{ iff } \exists q \in G \restriction \gamma(q \text{ decides } "\check{n} \in \dot{x}),$$

which is a contradiction since $V[G_\alpha]$ preserves ω_1.

However, the Mathias generic real is almost contained in a member of every maximal almost disjoint family from the ground model and so if g_α is the α-th Mathias real, then

$$V[G] \vDash \forall\gamma < \omega_1 \exists \xi_\gamma < \alpha(\text{range}(g_\alpha) \subseteq \dot{x}_{\xi\gamma}).$$

□

The following theorem is due to Baumgartner.

Theorem 2. Let $\mathbb{P}$ be the Mathias partial order and let $\langle x_\alpha : \alpha < \kappa\rangle$ be a tower in $[\omega]^\omega$. Then $\langle x_\alpha : \alpha < \kappa\rangle$ remains a tower in $V^{\mathbb{P}}$.

Proof. Suppose not. Then there is a $\mathbb{P}$-generic extension $V[G]$ such that

$$V[G] \vDash \exists x \in [\omega]^\omega \forall \alpha < \kappa(x \subseteq^* x_\alpha).$$

Then there is a $\mathbb{P}$-name for an infinite subset of ω and a condition $p = (s_0, A_0) \in G$ such that for every $\alpha < \kappa$

$$(s_0, A_0) \Vdash \dot{x} \subseteq^* x_\alpha.$$

We can assume that the condition (s_0, A_0) is pre-processed for $\dot{x}$. That is for every $k \in \omega$ and $t \leq (s_0, A_0)$ (that is t end-extends s_0 and $t - s_0 \subseteq A_0$) if there is $C \subseteq A_0$ such that $(t, C) \Vdash \check{k} \in \dot{x}$ then there is some $m \in \omega$ such that $(t, A_0 - m) \Vdash \check{k} \in \dot{x}$. Then we can define for every $s \leq (s_0, A_0)$ the set

$$F_s = \{k : \exists C \subseteq A_0((s, C) \Vdash \check{k} \in \dot{x})\} = \{k : (\exists m)(s, A_0 - m) \Vdash \check{k} \in \dot{x}\}.$$

Claim. There is $(s, A) \leq (s_0, A_0)$ such that for every $t \leq (s, A)$ the set F_s is finite.

Proof. Suppose the claim is not true. That is for every $(s, A) \leq (s_0, A_0)$ there is $t \leq (s, A)$ such that F_t is infinite. However there are only countably many F_t's and so there is some $\alpha < \kappa$ such that for every $t \leq (s, A)$ such that F_t is infinite, $F_t \not\subseteq^* x_\alpha$. Otherwise, for every $\beta < \kappa$ there is an infinite F_t such that $F_t \subseteq x_\beta$. However if F is an infinite subset of ω such that $F \subseteq^* F_t$ for every infinite F_t, then F is a pseudo-intersection of $\langle x_\alpha : \alpha < \kappa \rangle$ which belongs to the ground model which is a contradiction to $\langle x_\alpha : \alpha < \kappa \rangle$ being a tower. Since $(s_0, A_0) \Vdash \dot{x} \subseteq^* x_\alpha$, there is an extension $(s, A) \in G$ and $j \in \omega$ such that

$$(s, A) \Vdash \dot{x} - j \subseteq x_\alpha.$$

By assumption there is $t \leq (s, A)$ such that F_t is infinite. But then there is $k \in F_t - x_\alpha - j$ and so by definition of F_t there is some $m \in \omega$ such that $(t, A_0 - m) \Vdash \check{k} \in \dot{x}$. However $(t, A - m)$ extends both (s, A) and $(t, A_0 - m)$ and so

$$(t, A - m) \Vdash (\check{k} \in \dot{x} - j) \wedge (\dot{x} - j \subseteq x_\alpha)$$

which is a contradiction since $k \notin x_\alpha$. □

Furthermore we have the following property.

Claim. Suppose (s_0, A_0) is a condition in $\mathbb{P}$ such that for every $s \leq (s_0, A_0)$ F_s is finite. Then there is $B \subseteq A_0$ such that for every $t \leq (s_0, B)$

$$(s_0, B) \Vdash \check{F}_t \subseteq \dot{x}.$$

Proof. We will construct the set B inductively. Suppose we have defined $b_0 < b_1 < \cdots < b_{n-1}$ and a set $B_n \subseteq A_0$ such that $b_0 > \max s_0$, $b_{n-1} < \min B_n$ and such that for evert t which end-extends s_0 and such that $t \backslash s_0 \subseteq \{b_0, \ldots, b_{n-1}\}$, $(t, B_n) \Vdash F_t \subseteq \dot{x}$. Let $b_n = \min B_n$. Consider any t which end-extends s_0 such that $t \backslash s_0 \subseteq \{b_i\}_{i \leq n-1}$. Then $F_{t^\frown b_n}$ is finite and for every $k \in T_{t^\frown b_n}$ there is $n_t^k \in \omega$ such that

$$(t^\frown b_n, A_0 - n_t^k) \Vdash \check{k} \in \dot{x}$$

and since $B_n \subseteq A$ this implies that

$$(t^\frown b_n, B_n - n_t^k) \Vdash \check{k} \in \dot{x}.$$

Let $n_t = \max\{n_t^k : k \in F_{t^\frown b_n}\}$. Then if m is the maximum of all such n_t's the set $B_n - m$ has the property that for every t which end-extends s_0 and such that $t\backslash s_0 \subseteq \{b_i\}_{i\leq n}$

$$(t^\frown b_n, B_n - m) \Vdash \check{F}_{t^\frown b_n} \subseteq \dot{x}.$$

Let $B_{n+1} = B_n - m$. With this the inductive construction is complete. The set $B = \cap\{\{b_0, \ldots, b_{n-1}\} \cup B_n\} = \{b_i\}_{i\in\omega}$ has the desired properties. □

Thus we can assume that the chosen condition (s_0, A_0) has the properties that for every $s \leq (s_0, A_0)$, F_s is finite and there is $m \in \omega$ such that $(s, A_0 - m) \Vdash F_s \subseteq \dot{x}$. Inductively, we will obtain an infinite subset A of A_0 such that for every $s \leq (s_0, A)$ one of the following two conditions holds:

(1) $\forall a \in A - (\|s\| + 1) F_{s^\frown a} = F_s$.
(2) $(\exists \alpha < \kappa)(\forall j \in \omega)(\exists m_j \in \omega)(\forall a \in A - m_j) F_{s^\frown a} - x_\alpha - j \neq \emptyset$.

Again, suppose we have defined $\{a_0, \ldots, a_{n-1}\}$ and $A_n \subseteq A_0$ such that for every s which end-extends s_0 and such that $s - s_0 \subseteq \{a_i\}_{i\leq n}$ the corresponding two conditions above hold (A substituted by A_n). Let $a_n = \min A_n$. Then successively consider all end-extensions s of s_0 such that $s - s_0 \subseteq \{a_i\}_{i\leq n}$ and define a set $A_{s,n}$ which is contained in $A_{s',n}$ for every s' considered prior to s and A_n as follows.

If $B^* = \bigcup\{F_{s^\frown a} : a \in A_n\}$ is finite, then for every $k \in B^*$ either the set $B_k = \{a \in A_n : k \in A_n\}$ is finite or it is infinite. If B_k is finite than we can remove the corresponding a's from A_n (note also that in this case k does not belong to F_s). If B_k is infinite, then for every $b \in B_k$ (by inductive hypothesis) we have $(s^\frown b, B_k) \Vdash \check{k} \in \dot{x}$ and so $(s, B_k) \Vdash \check{k} \in \dot{x}$ which implies that $k \in F_s$.

If $B^* = \bigcup\{F_{s^\frown a} : a \in A_n\}$ is infinite, then let $\alpha < \kappa$ be such that $B^* \not\subseteq^* x_\alpha$. Define $A_{s,n}$ so that if a is the j-th element of $A_{s,n}$ then there is $k \geq j$ such that $k \in F_{s^\frown a} - x_\alpha - j$.

Then define A_{n+1} to be the intersection of all such $A_{s,n}$'s. Finally, let $A = \{a_n\}_{n\in\omega}$. Then $A \subseteq A_0$ and for every $s \leq (s_0, A)$ one to the two conditions above hold. Again since there are only countably many $s \leq (s_0, A)$ we can choose an $\alpha < \kappa$ such that α is greater than all β's associated to finite sequences $s \leq (s_0, A)$ by part (ii) of the above two conditions. Than since (s_0, A) extends (s, A)

$$(s_0, A) \Vdash \dot{x} \subseteq x_\alpha$$

and so there is some $(s, B) \leq (s_0, A)$ and $j \in \omega$ such that

$$(s, B) \Vdash \dot{x} - j \subseteq x_\alpha.$$

If for every $b \in B$, $F_{s^\frown b} = F_s$, then $(s, B) \Vdash \dot{x} \subseteq \check{F}_s$ which is a contradiction, since F_s is finite. Otherwise, we can find $b \in B$ such that there is $k \in F_{s^\frown b} - x_\alpha - j$. Then there is some $m \in \omega$ such that

$$(s^\frown b, B - m) \Vdash \check{k} \in \dot{x}$$

which is a contradiction since $(s^\frown b, B - m)$ is an extension of (s, B) and so we would obtain $(s^\frown b, B - m) \Vdash \check{k} \in x_\alpha$, which is not possible. $\square$

2. Mixed-support iteration of factored Mathias forcing

We will begin with a well known definition of iterated forcing:

Definition 4. A partial order $\mathbb{P}_\kappa$ is a κ-stage iteration if and only if $\mathbb{P}_\kappa$ is a set of κ-sequences and there is a sequence $\langle Q_\alpha : \alpha < \kappa \rangle$ such that of $\mathbb{P}_\alpha = \{p \restriction \alpha : p \in \mathbb{P}_\kappa\}$ for all $\alpha < \kappa$, the following holds:

(1) $(\forall \alpha < \kappa)$ $\mathbb{P}_\alpha$ is an α-stage iteration, with stages $\langle Q_\beta : \beta < \alpha \rangle$. Let $\Vdash_\alpha$ denote forcing with $\mathbb{P}_\alpha$.
(2) $(\forall \alpha < \kappa) \Vdash_\alpha$ "Q_α is a partial order".
(3) $(\forall p \in \mathbb{P}_\kappa)$ $(\forall \alpha < \kappa) \Vdash_\alpha p_\alpha \in Q_\alpha$ and there is $r \in \mathbb{P}_{\alpha+1}$ where $r \restriction \alpha = p \restriction \alpha$ and $r(\alpha) = \dot{q}$.
(4) $\forall p, q \in \mathbb{P}_\kappa$ $(p \leq q)$ if and only is $(\forall \alpha < \kappa)$ $p \restriction \alpha \Vdash_\alpha p(\alpha) \leq q(\alpha)$.
(5) $(\forall \beta < \alpha \leq \kappa)$ $(\forall p \in \mathbb{P}_\alpha, q \in \mathbb{P}_\beta)$ if $q \leq p \restriction \beta$ then $q \wedge p \in \mathbb{P}_\alpha$, where $q \wedge p(\gamma) = q(\gamma)$ for all $\gamma < \beta$ and $q \wedge p(\gamma) = p(\gamma)$ for $\gamma \geq \beta$.
(6) The trivial condition $\mathbb{1} \in \mathbb{P}_\kappa$, where for every $\alpha < \kappa$, $\mathbb{1}(\alpha)$ is forced to be the trivial condition in Q_α.

For limit $\alpha \leq \kappa$ we have to specify how $\mathbb{P}_\alpha$ is constructed from

$$\{p : \mathrm{dom}(p) = \alpha \text{ and } (\forall \beta < \alpha) p \restriction \beta \in \mathbb{P}_\beta\}.$$

Usually we require $\mathbb{P}_\alpha$ to consists of all conditions for which

$$\mathrm{support}(p) = \{\beta \in \mathrm{dom}(p) : p(\beta) \neq \dot{\mathbb{1}}\}$$

is finite or countable. Then we refer to the iteration $\mathbb{P}_\kappa$ as *finite* respectively *countable support iteration.*

In particular, we will be interested in mixed support iteration:

Definition 5. For κ any ordinal, let $\mathbb{P}_\kappa$ be an iterated forcing construction such that for every $\alpha < \kappa$ either $\Vdash_\alpha$ "Q_α is $\sigma -$ centered" or $\Vdash_\alpha$ "Q_α is countably closed". We thus speak about σ-centered stages and countably closed stages. For $p \in \mathbb{P}_\kappa$ let

$$\mathrm{Fsupport}(p) = \{\alpha < \kappa : \alpha \text{ is a } \sigma \text{ centered stage}\}.$$

Then $\mathbb{P}_\kappa$ is *the finite/countable support iteration* of the Q_α is for every $p \in \mathbb{P}_\kappa$, support(p) is countable, Fsupport(p) is finite and $(\forall \alpha < \kappa) \Vdash_\kappa p(\alpha) \in Q_\alpha$.

Definition 6. We say that p is a direct extension of q, denoted $p \leq_D q$ if $p \leq q$ and for all σ-centered stages $\alpha < \kappa$, $p \restriction \alpha \Vdash_\alpha p(\alpha) = q(\alpha)$. Similarly, we say p is a C-extension of q, denoted $p \leq_C q$ if $p \leq q$ and for all countably closed stages $\alpha < \kappa$,$p \restriction \alpha \Vdash_\alpha p(\alpha) = q(\alpha)$.

Remark 2. Both of the relations $\leq_D$ and $\leq_C$ are transitive.

Lemma 2. *Let $\{p_n\}_{n\in\omega}$ be a sequence in $\mathbb{P}_\kappa$ such that for every n $p_{n+1} \leq_D p_n$. Then there is a condition $p \in \mathbb{P}_\kappa$ such that $p \leq_D p_n$ for all n.*

Proof. Construct p inductively. If α is a limit and $p \restriction \beta$ is defined for every $\beta < \alpha$, then $p \restriction \alpha$ is clear. At successor stage $\alpha + 1$ there are two cases. If α is a countably closed stage and

$$p \restriction \alpha \Vdash p_0(\alpha) \geq p_1(\alpha) \geq \cdots \geq p_n(\alpha) \ldots$$

then since Q_α is countably closed we can choose $p(\alpha)$ to be a $\mathbb{P}_\alpha$-name for an element of Q_α such that $p \restriction \alpha \Vdash p(\alpha) \leq p_n(\alpha)$ for every n. If α is a σ-centered stage and

$$p \restriction \alpha \Vdash p_0(\alpha) = p_1(\alpha) = \cdots = p_n(\alpha) = \ldots$$

then we can simply define $p(\alpha) = p_0(\alpha)$. □

Lemma 3. *Let $p \leq q$ in $\mathbb{P}_\kappa$. Then there is $r \in \mathbb{P}_\kappa$ such that $p \leq_C r \leq_D q$.*

Proof. The condition r is defined by induction on α. If α is a limit and $r \restriction \beta$ is defined for every $\beta < \alpha$ then $r \restriction \alpha$ is clear. So, consider successor stages $\alpha + 1$. If α is a σ-centered stage, then define $r(\alpha) = q(\alpha)$. If α is a countably closed stage we define $r(\alpha)$ to be a $\mathbb{P}_\alpha$-term as follows:

(1) if $\bar{r} \leq p \restriction \alpha$, then $\bar{r} \Vdash r(\alpha) = p(\alpha)$
(2) if $\bar{r} \perp p \restriction \alpha$, then $\bar{r} \Vdash r(\alpha) = q(\alpha)$.

With this the inductive construction is defined. It remains to verify that $p \leq_C r$ and $r \leq_D q$.

By induction on α verify that $p \restriction \alpha \leq r \restriction \alpha$ and for countably closed stages $p \restriction \alpha \Vdash p(\alpha) = r(\alpha)$ (holds by definition of $r(\alpha)$), and for σ-centered stages $p \restriction \alpha \Vdash p(\alpha) \leq q(\alpha) = r(\alpha)$ (again by definition of $r(\alpha)$).

Similarly, by induction on α verify that $r \restriction \alpha \leq q \restriction \alpha$ and for countably closed stages $r \restriction \alpha \Vdash r(\alpha) \leq q(\alpha)$, and for σ-centered stages

$r \restriction \alpha \Vdash r(\alpha) = q(\alpha)$. The latter holds by definition of $r(\alpha)$, so it remains to verify the former. Consider any $\bar{r} \leq r \restriction \alpha$. If $\bar{r} \leq p \restriction \alpha$, then

$$\bar{r} \Vdash r(\alpha) = p(\alpha) \wedge p(\alpha) \leq q(\alpha).$$

If $\bar{r} \perp p \restriction \alpha$, then again by definition of $r(alpha)$ $\bar{r} \Vdash r(\alpha) = q(\alpha)$. Therefore every extension of $r \restriction \alpha$ forces that "$r(\alpha) \leq q(\alpha)$ and so $r \restriction \alpha \Vdash r(\alpha) \leq q(\alpha)$. □

Definition 7. Suppose α is a σ-centered stage. Then in $V^{\mathbb{P}_\alpha}$ define a function $s : Q_\alpha \to \omega$ so that

$$\Vdash_\alpha (\forall p, q \in Q_\alpha)(s(p) = s(q) \implies p \not\perp q).$$

Remark 3. Abusing notation we will identify s with its $\mathbb{P}_\alpha$-name $\dot{s}$.

Definition 8. Condition $p \in \mathbb{P}_\kappa$ is said to be determined if for all $\alpha \in$ Fsupport(p) there is $n \in \omega$ such that

$$p \restriction \alpha \Vdash s(p(\alpha)) = \check{n}.$$

Lemma 4. *The set of determined conditions in $\mathbb{P}_\kappa$ is dense. Suppose determined conditions q_1 and q_2 are given with*

$$Fsupport(q_1) = Fsupport(q_2)$$

and for all α in this finite support there is $n \in \omega$ such that $q_1 \restriction \alpha \Vdash s(q_1(\alpha)) = \check{n}$ and $q_2 \restriction \alpha \Vdash s(q_2(\alpha)) = \check{n}$. Suppose also that for some $p \in \mathbb{P}_\kappa$, $q_1 \leq p$ and $q_2 \leq_C p$. Then q_1 and q_2 are compatible.

Proof. Proceed by induction on κ. Let $p \in \mathbb{P}_\kappa$. If κ is a limit, then there is $\alpha < \kappa$ such that Fsupport$(p) \subseteq \alpha$. Then by inductive hypothesis there is a determined $\bar{r} \leq p \restriction \alpha$ and so $\bar{r} \wedge p^\alpha$ is a determined condition extending p. At successor σ-centered stages $\alpha + 1$, we can find determined $\bar{r} \leq p \restriction \alpha$ such that for some $n \in \omega$ $\bar{r} \Vdash s(p(\alpha)) = \check{n}$. Then $\bar{r} \wedge r^\alpha$ is a determined extension of p.

To obtain the second claim of the Lemma, we will define a common extension r of q_1 and q_2 inductively. Suppose α is a limit and for every $\beta < \alpha$ we have defined $r \restriction \beta$. Then let $r \restriction \alpha = \cup_{\beta<\alpha} r \restriction \beta$. At countably closed stages let $r(\alpha) = q_1(\alpha)$. Then

$$r \restriction \alpha \Vdash q_1(\alpha) = r(\alpha) \leq p(\alpha) = q_2(\alpha).$$

At σ-centered stages we have $r \restriction \alpha \Vdash s(q_1(\alpha)) = s(q_2(\alpha)) = checkn$ for some n. Therefore we can choose $r(\alpha)$ to be a $\mathbb{P}_\alpha$-name for a common extension of $q_1(\alpha)$ and $q_2(\alpha)$ and so

$$r \restriction \alpha \Vdash (r(\alpha) \leq q_1(\alpha)) \wedge (r(\alpha) \leq q_2(\alpha)).$$

□

Lemma 5. *Let $\mathbb{P}_\alpha$ be a finite/countable support iteration with α a limit ordinal and let $\langle x_\xi : \xi < \lambda \rangle$ be a tower in $[\omega]^\omega$ for some regular λ. If there is an infinite $x \subseteq \omega$ in $V^{\mathbb{P}_\alpha}$ such that for all $\xi < \lambda$, $x \subseteq^* x_\xi$, then there is $\beta < \alpha$ and an infinite $y \subseteq \omega$ in $V^{\mathbb{P}_\beta}$ such that $\forall \xi < \lambda$, $y \subseteq^* x_\xi$.*

Theorem 3 (CH)**.** Let $\mathbb{P}_{\omega_2}$ be the ω_2-stage finite/countable factored Mathias iteration. Then, in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathfrak{h} = 2^{\aleph_0} = \aleph_2$ and there are no ω_2-towers in $[\omega]^\omega$.

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