# NON-LINEAR ITERATIONS AND ALMOST DISJOINTNESS

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Abstract. Let $\kappa$ be an infinite regular cardinal, let $\mathfrak{a}_\kappa$, $\mathfrak{b}_\kappa, \mathfrak{d}_\kappa$ be the almost disjointness, bounding, and dominating numbers at $\kappa$, respectively, and let $\mathfrak{c}_\kappa = 2^\kappa$. Using a system of parallel non-linear iterations, we establish the consistency of $\mathfrak{b}_\kappa = \mathfrak{a}_\kappa < \mathfrak{d}_\kappa < \mathfrak{c}_\kappa$ where $\mathfrak{b}_\kappa, \mathfrak{d}_\kappa, \mathfrak{c}_\kappa$ are arbitrary subject to the known ZFC restrictions.

## 1. Introduction

The cardinal characteristics of the continuum occupy a central place in the study of the set theoretic properties of the real line, with many interesting research and survey articles, see [1], [9]. In the past decades, there has been an increased interest towards higher Baire spaces analogues of many of those characteristics. In this article we further examine the bounding, dominating and almost-disjointness numbers, denoted $\mathfrak{b}_\kappa, \mathfrak{d}_\kappa, \mathfrak{a}_\kappa$ respectively and show that subject to the known ZFC restrictions between these characteristics, consistently $\kappa^+ < \mathfrak{b}_\kappa = \mathfrak{a}_\kappa < \mathfrak{d}_\kappa < \mathfrak{c}_\kappa$ holds for $\kappa = \omega$ (which can be obtained also by other already existing methods) and more significantly for the current work, for $\kappa$ arbitrary regular uncountable cardinal.

Our result builds upon the methods of non-linear iterations of Cummings and Shelah from [4] and the method of matrix iterations as appearing in [2, 3]. Recall, that the method of matrix iteration was introduced by A. Blass and S. Shelah in 1989 to prove the relative consistency of $\mathfrak{u} < \mathfrak{d}$, where $\mathfrak{u}$ denotes the minimal size of a base for a non-principal ultrafilter on $\omega$. In [3] the method was further developed and systematized to establish the consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$, as well as $\mu < \mathfrak{b} = \kappa < \mathfrak{a} = \mathfrak{s} = \lambda$ above a measurable cardinal $\mu$, where $\mathfrak{s}$ denotes the splitting number. Of particular importance for the current work is the method of forcing with restricted Hechler posets along a matrix iteration introduced in the latter work. The method of non-linear iteration was introduced in [4] in order to (among others) simultaneously control the values of the generalized invariants $\mathfrak{b}_\kappa, \mathfrak{d}_\kappa$ and $\mathfrak{c}_\kappa$ at an arbitrary regular uncountable cardinal $\kappa$.

To obtain our main results, we merge the above techniques both in the countable and uncountable settings. The resulting forcing construction can be seen as a system of parallel non-linear iterations, which can be compared to the system of parallel (linear) matrix iterations given in [5]. Our main theorem states the following:

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**Theorem.** Let $\kappa$ be an infinite regular cardinal. If $\beta, \delta, \mu$ are infinite cardinals with $\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta) \leq \delta \leq \mu$ and $\mathrm{cof}(\mu) > \kappa$, then there is a cardinal preserving generic extension in which

$$\mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \leq \mathfrak{d}_\kappa = \delta \leq \mathfrak{c}_\kappa = \mu.$$

In addition, we outline a standard (linear) matrix iteration construction which gives an alternative proof of our main result for the special case in which $\mathfrak{d}_\kappa$ is regular and $\kappa$ is an arbitrary regular uncountable cardinal. To the best knowledge of the authors this is the first application of the method of matrix iterations in the context of higher Baire spaces. A key feature of our forcing construction is the fact that the iterands along relevant non-linear fragments are well-chosen, as indeed we make use only of suitable restricted Hechler forcings.

The paper is structured as follows: In Section 2 we revisit some basic notions and in Section 3, we introduce and study the properties of a well-founded index poset which plays a crucial role in our main forcing construction. In section 4 we, recursively along a suitable index poset, define the above mentioned forcing notion, establish its properties. In section 5 we study the preservation of a carefully chosen witness to $\mathfrak{a}_\kappa = \beta$ along this forcing construction. In Section 6 we complete the proof of the main theorem. In the final, Section 7, we give alternative proofs of the special case of the above theorem in which $\kappa = \omega$, as well as the special case in which $\kappa$ is regular uncountable and $\mathfrak{d}_\kappa$ is regular. We conclude the article, with some interesting remaining open questions, regarding (among others) the global behaviour or $\mathfrak{a}_\kappa$, $\mathfrak{b}_\kappa$, $\mathfrak{d}_\kappa$ and $\mathfrak{c}_\kappa$.

## 2. Preliminaries

Throughout $\kappa$ is a regular infinite cardinal.

**Definition 2.1.** Let $f$ and $g$ be functions from $\kappa$ to $\kappa$.

(1) Then $g$ eventually dominates $f$, denoted by $f <^* g$, if $\exists n < \kappa\ \forall m > n\ (f(m) < g(m))$.
(2) A family $\mathcal{F} \subseteq {}^\kappa\kappa$, is dominating if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ (g <^* f)$.
(3) A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is unbounded if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ (f \nless^* g)$.
(4) $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ denote the generalized bounding and dominating numbers respectively:

$$\mathfrak{b}_\kappa = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa, \mathcal{F} \text{ is unbounded}\},$$
$$\mathfrak{d}_\kappa = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa, \mathcal{F} \text{ is dominating}\}.$$

(5) Finally, $\mathfrak{c}_\kappa = 2^\kappa$.

**Definition 2.2.** Let $x, y \in [\kappa]^\kappa$.

(1) The sets $x$ and $y$ are almost disjoint if $|x \cap y| < \kappa$.
(2) A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is $\kappa$-almost disjoint if any two pairwise distinct elements in $\mathcal{A}$ are almost disjoint. An almost disjoint family is $\kappa$-maximal almost disjoint ($\kappa$-mad) if it is maximal with respect to inclusion.
(3) The almost disjointness number $\mathfrak{a}_\kappa$ is the minimal size of a $\kappa$-maximal almost disjoint family of cardinality at least $\kappa$ and is denoted $\mathfrak{a}_\kappa$.

Some of the well-known relations between the above mentioned invariants are as follows: $\kappa^+ \leq \mathfrak{b}_\kappa = \mathrm{cof}(\mathfrak{b}_\kappa) \leq \mathrm{cof}(\mathfrak{d}_\kappa) \leq \mathfrak{d}_\kappa \leq \mathfrak{c}_\kappa$, $\mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa$, $\mathrm{cof}(\mathfrak{c}_\kappa) > \kappa$. We will use the following notation: $\mathbb{1} = \{\varnothing\}$ denotes the trivial forcing and for a forcing notion $\mathbb{P}$, $\mathbb{1}_\mathbb{P}$ is the largest element of $\mathbb{P}$.

**Definition 2.3.** The Hechler forcing notion is defined as the set $\mathbb{H} = \{(s,f) : s \in \kappa^{<\kappa}, f \in {}^\kappa\kappa\}$ with extension relation given by: $(t,g) \leq_\mathbb{H} (s,f)$ iff $s \subseteq t$, $\forall n \in \kappa\ (g(n) \geq f(n))$ and $\forall i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)\ (t(i) > f(i))$. If $A \subseteq {}^\kappa\kappa$, then $\mathbb{H}(A) = \{(s,f) : s \in \kappa^{<\kappa}, f \in A\}$ equipped with the same extension relation is known as restricted Hechler forcing.

It is straightforward to check, that $\mathbb{H}(A)$ adjoins a $\kappa$-real eventually dominating the elements in $A$. The first coordinate $s$ of a condition $(s,f) \in \mathbb{H}(A)$ is called a stem. The poset given below is the generalization of what is known as the Hechler forcing for adjoining a mad family, see [6]:

**Definition 2.4.** Let $\lambda$ be an ordinal. Then $\mathbb{H}_\lambda$ consists of all partial functions $p : \lambda \times \kappa \to 2$, with $\mathrm{dom}(p) = F_p \times n_p$ where $F_p \in [\lambda]^{<\kappa}$, $n_p \in \kappa$ and extension relation is defined as follows: $q \leq p$ iff $p \subseteq q$ and $\forall i \in n_q \setminus n_p\ |q^{-1} \cap F_p \times \{i\}| \leq 1$.

If $G$ is a $\mathbb{H}_\lambda$-generic for an ordinal $\lambda$, then the family $\mathcal{A}_\lambda = \{A_\alpha : \alpha < \lambda\}$, where $A_\alpha = \{i : \exists p \in G\ p(\alpha, i) = 1\}$ is $\kappa$-almost disjoint. Moreover, if $\lambda \geq \kappa^+$ then $\mathcal{A}_\lambda$ is $\kappa$-maximal almost disjoint. If $\alpha \leq \beta$ are two ordinals, then $\mathbb{H}_\beta$ decomposes as follows: Let $G$ be a $\mathbb{H}_\alpha$-generic. In $V[G]$ let $\mathbb{H}_{[\alpha,\beta)}$ consist of pairs $(p, H)$, where $p: (\beta \setminus \alpha) \times \kappa \to 2$ has domain $\mathrm{dom}(p) = F_p \times n_p$, $H \in [\alpha]^{<\kappa}$ with $(p,H) \leq (q,K)$ iff $p \leq_{\mathbb{H}_\beta} q$, $K \subseteq H$ and for every $j \in F_q$, $k \in n_p \setminus n_q$ and $i \in K$, if $k \in A_i$, then $p(j,k) = 0$ holds. Then $\mathbb{H}_\beta \simeq \mathbb{H}_\alpha * \dot{\mathbb{H}}_{[\alpha,\beta)}$.

**Definition 2.5.** If $(\mathbb{Q}, \leq_\mathbb{Q}, \mathbb{1}_\mathbb{Q})$ and $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{1}_\mathbb{P})$ are forcing posets, then $i : \mathbb{Q} \to \mathbb{P}$ is called a complete embedding, denoted $\mathbb{Q} \lessdot \mathbb{P}$, if the following properties hold:

(1) $i(\mathbb{1}_\mathbb{Q}) = \mathbb{1}_\mathbb{P}$,
(2) $\forall q, q' \in \mathbb{Q} (q \leq_\mathbb{Q} q' \to i(q) \leq_\mathbb{P} i(q'))$,
(3) $\forall q, q' \in \mathbb{Q} (q \perp_\mathbb{Q} q' \leftrightarrow i(q) \perp_\mathbb{P} i(q'))$ and
(4) if $A \subseteq \mathbb{Q}$ is a maximal antichain in $\mathbb{Q}$, then $i(A)$ is a maximal antichain in $\mathbb{P}$.

We will make use of the following, which is a slightly modified version of [3, Lemma 13].

**Lemma 2.6.** Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions with $\mathbb{P} \lessdot \mathbb{Q}$. Suppose $\dot{\mathbb{A}}$ (resp. $\dot{\mathbb{B}}$) is a $\mathbb{P}$-name (resp. $\mathbb{Q}$-name) for a forcing poset, where in $V^\mathbb{Q}$ there is an embedding $i: \mathbb{A} \to \mathbb{B}$ with

- $i(\mathbb{1}_\mathbb{A}) = \mathbb{1}_\mathbb{B}$,
- $\forall p, p' \in \mathbb{A}\ \big(p \leq p' \to i(p) \leq i(p')\big)$,
- $\forall p, p' \in \mathbb{A}\ \big(p \perp p' \leftrightarrow i(p) \perp i(p')\big)$ and
- for every maximal antichain $A$ of $\dot{\mathbb{A}}$ in $V^\mathbb{P}$, $i(A)$ is a maximal antichain of $\dot{\mathbb{B}}$ in $V^\mathbb{Q}$.

Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$.

*Proof.* Let $j: \mathbb{P} \to \mathbb{Q}$ be a witness for $\mathbb{P} \lessdot \mathbb{Q}$. Define the following embedding: $k: \mathbb{P} * \dot{\mathbb{A}} \to \mathbb{Q} * \dot{\mathbb{B}}$, $k(p, \dot{q}) = (j(p), \dot{i(q)})$. Conditions (1), (2), (3) of Definition 2.5 are easily checked. We show property (4) of Definition 2.5. For suppose not and let $W = \{(p_\alpha, \dot{a}_\alpha) : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{P} * \dot{\mathbb{A}}$ and $(q, \dot{b}) \in \mathbb{Q} * \dot{\mathbb{B}}$ be incompatible with every condition in $k(W)$. Let $\dot{H}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter and let $\dot{I}$ be a $\mathbb{P}$-name with $\Vdash \dot{I} = \{\alpha : p_\alpha \in \dot{H}\}$.

We claim that $\Vdash$ "$\{\dot{a}_\alpha : \alpha \in \dot{I}\}$ is a maximal antichain of $\dot{A}$". Otherwise, we can find a $\mathbb{P}$-name $\dot{a}$ and $p \in \mathbb{P}$ such that

$$(*)\ p \Vdash \forall \alpha(\alpha \in \dot{I} \to \dot{a} \perp \dot{a}_\alpha).$$

Since $(p, \dot{a}) \in \mathbb{P} * \dot{\mathbb{A}}$ and $W$ is maximal, we can find $\alpha < \kappa$ and $(p', \dot{a})$ which is a common extension of $(p, \dot{a})$ and $(p_\alpha, \dot{a}_\alpha)$. Then $p' \Vdash \dot{a}' \leq \dot{a} \wedge \dot{a}' \leq \dot{a}_\alpha$ and $p' \Vdash \alpha \in \dot{I}$. Hence $p' \Vdash \alpha \in \dot{I} \wedge \dot{a}' \leq \dot{a} \wedge \dot{a}' \leq \dot{a}_\alpha$ which is a contradiction to $(*)$.

Now let $G$ be a $\mathbb{Q}$-generic filter containing $q$. As $\mathbb{P} \lessdot \mathbb{Q}$ we can find a $\mathbb{P}$-generic filter $H$ with $V[H] \subseteq V[G]$ (see [7, p. 270]). Let $b = \dot{b}[G]$, $a_\alpha = \dot{a}_\alpha[G] = \dot{a}_\alpha[H]$ and $I = \dot{I}[G] = \{\alpha < \kappa : p_\alpha \in H\}$. By the above $\{a_\alpha : \alpha \in I\}$ is a maximal antichain of $\mathbb{A}$ in $V[H] \subseteq V[G]$ and by assumption $\{i(a_\alpha) : \alpha \in I\}$ is a maximal antichain of $\mathbb{B}$ in $V[G]$. Thus $\exists \alpha \in I\ b \not\perp i(a_\alpha)$ and so $\exists q' \leq q, j(p_\alpha)$ such that $q' \Vdash \alpha \in \dot{I} \wedge \dot{b} \not\perp i(\dot{a}_\alpha)$. This further means that there is a $\mathbb{Q}$-name $\dot{r}$ with $q' \Vdash \dot{r} \leq \dot{b}, i(\dot{a}_\alpha)$, hence $(q', \dot{r})$ is a common extension of $(q, \dot{b})$ and $(j(p_\alpha), i(\dot{a}_\alpha))$, which is a contradiction. □

## 3. The index set

Bounding and dominating can be defined generally for arbitrary posets as follows:

**Definition 3.1** ([4]). Let $(P, \leq_P)$ be a partial order.

(1) We call $U \subseteq P$ *unbounded* if $\forall p \in P\ \exists q \in U\ (q \nleq_P p)$.
(2) $\mathfrak{b}(P) = \min\{|U| : U \subseteq P \text{ is unbounded}\}$.
(3) A subset $D \subseteq P$ is *dominating* if $\forall p \in P\ \exists q \in D\ (p \leq_P q)$.
(4) $\mathfrak{d}(P) = \min\{|D| : D \subseteq P \text{ is dominating}\}$.

Note that $\leq^*$ is not antisymmetric. However the relation $=^*$ is an equivalence relation on ${}^\kappa\kappa$. Let $[f]_{=^*} = \{g \in {}^\kappa\kappa : f =^* g\}$ denote the equivalence class of $f$. The relation $\leq_{=^*}$ on the equivalence classes, given as $[f] \leq_{=^*} [g]$ iff $f \leq^* g$ is well-defined and a partial order. So $\mathfrak{b}_\kappa = \mathfrak{b}(\{[f]_{=^*} : f \in {}^\kappa\kappa\}, \leq_{=^*})$ and $\mathfrak{d}_\kappa = \mathfrak{d}(\{[f]_{=^*} : f \in {}^\kappa\kappa\}, \leq_{=^*})$.

**Lemma 3.2** ([4]). For any poset $P$ there is a well-founded and dominating subposet $P'$ of $P$.

*Proof.* Let $\tau = \langle p_\alpha : \alpha < \lambda\rangle$ be a maximal sequence such that $\forall \alpha < \lambda\ \forall \beta < \alpha\ (p_\alpha \nleq p_\beta)$. It is not difficult to check that $P'$ is dominating, as if not for any $p \in P$ such that $\forall \alpha < \lambda (p \nleq p_\alpha)$, the sequence $\langle p_\alpha : \alpha \leq \lambda\rangle$ contradicts the maximality of $\tau$, where $p_\lambda = p$. Take $P' = \{p_\alpha : \alpha < \lambda\}$. □

In the above Lemma $P'$ is clearly cofinal in $P$ and so $\mathfrak{d}(P) = \mathfrak{d}(P')$ and $\mathfrak{b}(P) = \mathfrak{b}(P')$.

For the purposes of the next lemma, let $(R, <_R)$ be a well-founded poset such that $|R| = \delta$, $\mathfrak{d}(R) = \delta$ and $\mathfrak{b}(R) = \beta$ for some cardinals $\beta$ and $\delta$. Further, for each $a \in R$, let $(L_a, <_{L_a})$ be a well-order of order type $\delta$ and let $L_a = \langle l_{a,\gamma} : \gamma < \delta\rangle$ where $l_{a,\gamma} \leq_{L_a} l_{a,\gamma'}$ iff $\gamma \leq \gamma'$. Let $Q$ be the disjoint union $Q = R \cup \bigcup\{L_a : a \in R\}$ and let $<_Q$ be the partial order on $Q$ defined as follows: $<_Q \restriction R \times R = <_R$, $\forall a \in R\ (<_Q \restriction L_a \times L_a = <_{L_a})$, $\forall a \in R\ (a <_Q l_{a,0})$ and $\forall a' \neq a \in R\ \forall \gamma \in \delta\ (a' <_R a \to l_{a',\gamma} <_Q l_{a,\gamma})$.

**Lemma 3.3.** If $(R, <_R)$, $\{L_a : a \in R\}$, and $(Q, <_Q)$ are given as above, then $\mathfrak{d}(Q) = \delta, \mathfrak{b}(Q) = \beta$, $|Q| = \delta$, $Q$ is well-founded and for each $b \in Q$, $|b \uparrow_Q| = \delta$.

*Proof.* For any element $q \in Q$, define the trace $q^R$ of $q$ in $R$ to be

$$q^R = \begin{cases} a & q \in L_a \\ q & q \in R \end{cases}$$

and for any subset $A \subseteq Q$, $A^R$ to be $\{a^R : a \in A\}$. Let $b \in Q$. Then $|b \uparrow_Q| = \delta$, as either $b = a$ for an $a \in R$ or $b = l_{a,\gamma}$ for an $a \in R$ and $\gamma < \delta$. In either case $|L_a \cap b \uparrow_Q| \geq \delta$. Also $|Q| = \delta$, because $|R| = \delta$ and $|L_a| = \delta$ for each $a \in R$ and $\delta$ is an infinite cardinal. As $Q$ is dominating and $|Q| = \delta$, we have $\mathfrak{d}(Q) \leq \delta$.

$\mathfrak{d}(Q) \geq \delta$: Let $A \subseteq Q$ and $|A| < \delta$. Then also $|A^R| < \delta$ and $A^R$ is not dominating in $R$. So $\exists b \in R\ \forall a \in A^R\ (b \nleq_R a)$. Then $b$ is also unbounded in $A$.

$\mathfrak{b}(Q) \geq \beta$: Let $A \subseteq Q$ and $|A| < \beta$. Then also $|A^R| < \beta$ and $A^R$ is not unbounded in $R$ and so $\exists d \in R\ \forall a \in A^R\ (a <_R d)$. For an ordinal $\alpha < \delta$, let $H_\alpha = \{l_{a,\alpha} : a \in R\}$. Let $\alpha' = \sup\{\gamma : A \cap H_\gamma \neq \emptyset\}$. By regularity of $\beta$, $\alpha' < \beta$. However $\delta \geq \beta > \alpha'$ and any $l_{d,\gamma}$ where $\alpha' < \gamma < \delta$ domintaes $A$.

$\mathfrak{b}(Q) \leq \beta$: Let $A \subseteq R$ be unbounded in $R$ with respect to $<_R$ and let $|A| = \beta$. Consider an arbitrary $q \in Q$. Note that if $a \in A$ is such that $a \nleq_R q^R$, then also $a \nleq_Q q$. Thus $A$ is an unbounded family of $Q$ with respect to $<_Q$.

Finally, to show that $Q$ is well-founded consider an arbitrary, non-empty $A \subseteq Q$. If $A \cap R \neq \emptyset$, then a minimal element of $A \cap R$ is also a minimal element of $A$. Otherwise let $m \in R$ be a minimal element of $A^R$. Let $\alpha' = \min\{\gamma : A \cap H_\gamma \neq \emptyset\}$. Then $l_{m,\alpha'}$ is a minimal element of $A$. □

We will make use of the following notation: Whenever $(X, <_X)$ is a well-founded poset, then for an arbitrary $y$ in $X$, let $X_y = \{x \in X : x <_X y\}$ and $y \uparrow_X = \{x \in X : y <_X x\}$.

**Corollary 3.4.** (GCH) Let $\kappa$ be a regular infinite cardinal and let $\beta, \delta$ be cardinals such that $\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta)$. There is a well-founded (index) partial order $(W, <_W)$ of cardinality $\delta$, which has a least and largest elements, denoted $c$ and $m$ respectively and such that for $Q = W \backslash \{m, c\}$, $<_Q = Q \times Q \cap <_W$ the following holds

$$\mathfrak{b}(Q) = \beta, \mathfrak{d}(Q) = \delta, \text{ and } \forall b \in Q\ (|b \uparrow_Q| \geq \delta).$$

*Proof.* Let $(Q, <_Q)$ be a well-founded suborder of $([\delta]^{<\beta}, \subseteq)$ having the same generalized bounding and dominating numbers as $([\delta]^{<\beta}, \subseteq)$ such that $\forall b \in Q\ (|b \uparrow_Q| \geq \delta)$. By Lemmas 3.2 and 3.3, such a $(Q, <_Q)$ exists. Now, let $W = \{c\} \dot\cup Q \dot\cup \{m\}$ be a disjoint union and let $<_W$ be defined as follows:

(1) for each $a \in Q$, $c <_W a$
(2) $<_W \restriction Q \times Q = <_Q$,
(3) for each $a \in \{c\} \dot\cup Q$, $a <_W m$.

Then $(W, <_W)$ is a well-founded poset with the desired properties. □

## 4. The iteration and its properties

Now we are ready to construct our iteration, which is a slight modification of the non-linear iteration of Hechler forcing for adjoining a dominating real $D(\omega, Q)$ from [4]. From now on assume GCH in the ground model $V$ and we fix $\kappa$ a regular cardinal, $\beta, \delta$ infinite cardinals with

$\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta)$. Let $(W, <_W)$ and $(Q, <_Q)$ be the well-founded index posets defined in Corollary 3.4. Moreover, let $Q' = Q \cup \{m\}$, $<_{Q'} = Q' \times Q' \cap <_W$.

Fix a surjective book-keeping function $F : Q \to \beta$ such that for all $\alpha \in \beta$, $F^{-1}(\alpha)$ is cofinal in $Q$. That is $\forall \alpha < \beta\ \forall b \in Q\ (b \uparrow_Q \cap F^{-1}(\alpha) \neq \emptyset)$. Such a $F$ exists, since $|Q| = \delta \geq \beta$ and $\forall b \in Q\ \big(|b \uparrow_Q| \geq \delta\big)$. In addition, for each $\gamma \leq \beta$, let $J^\gamma = \{a \in Q : F(a) \geq \gamma\}$.

In the following, we consider $(\beta + 1) \times W$ with the inherited lexicographic order $<_{lex}$ and the product order $<$ where $(\alpha_0, a_0) < (\alpha_1, a_1)$ iff $\alpha_0 \in \alpha_1$ and $a_0 <_W a_1$, or $\alpha_0 = \alpha_1$ and $a_0 <_W a_1$.

**Definition 4.1.** For each $(\alpha, a)$ in $(\beta + 1) \times W$ we will define recursively on $<_{lex}$ a forcing notion $P_{\alpha,a}$ and take $V_{\alpha,a} = V^{P_{\alpha,a}}$. For each $\alpha \leq \beta$ let $P_{\alpha,c} = \mathbb{H}_\alpha$. Let $(\alpha, a) \in (\beta + 1) \times Q'$ and suppose:

1. for each $(\gamma, b) <_{lex} (\alpha, a)$ the poset $P_{\gamma,b}$ has been defined;
2. in case $b \neq c$, also a $P_{\gamma,c}$-name $\dot{T}_{\gamma,b}$ for a forcing notion is given so that $P_{\gamma,b} = P_{\gamma,c} * \dot{T}_{\gamma,b}$;
3. whenever $(\alpha_0, a_0) < (\alpha_1, a_1) < (\alpha, a)$, $c \neq a_0$ then $\Vdash_{P_{\alpha_1,c}} \dot{T}_{\alpha_0,a_0} \lessdot \dot{T}_{\alpha_1,a_1}$.

Then, in particular, for each $(\alpha_0, a_0) < (\alpha_1, a_1) \leq (\alpha, a)$, $P_{\alpha_0,a_0} \lessdot P_{\alpha_1,a_1}$ (see Lemma 4.3).

We proceed to define $P_{\alpha,a}$. Since for each $b \in Q'_a \setminus J^\alpha$, $F(b) < \alpha$ and so $(F(b), b) < (\alpha, b)$, in $V_{\alpha,c}$ we can fix a $T_{\alpha,b}$-name $\dot{H}^\alpha_b$ for $V^{F(b),b} \cap {}^\kappa\kappa$. Now, in $V_{\alpha,c}$ let $T_{\alpha,a}$ be the poset of all functions $p$ such that $\mathrm{dom}(p) = Q'_a$ and

1. for each $b \in Q'_a \cap J^\alpha$, $p(b)$ is a $T_{\alpha,b}$-name for an element in the trivial poset;
2. for each $b \in Q'_a \setminus J^\alpha$, $\Vdash_{T_{\alpha,b}} p(b) \in \mathbb{H}(\dot{H}^\alpha_b)$;
3. for $\mathrm{supp}(p) = \{b \in Q'_a \setminus J^\alpha : \Vdash_{T_{\alpha,b}} p(b) \neq \mathbb{1}_{\mathbb{H}(\dot{H}^\alpha_b)}\}$ we have $|\mathrm{supp}(p)| < \kappa$.

The extension relation of $T_{\alpha,a}$ is defined as follows: $p \leq q$ iff $\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$ and for each $b \in \mathrm{supp}(q)$, if $b \in Q'_a \setminus J^\alpha$ then $p \upharpoonright b \Vdash_{T_{\alpha,b}} p(b) \leq_{\mathbb{H}(\dot{H}^\alpha_b)} q(b)$, where $p \upharpoonright b$ abbreviates $p \upharpoonright Q'_b$. For $b \in Q'_a \setminus J^\alpha$, w.l.o.g. we assume that $p(b) = (s^p_b, \dot{f}^p_b)$ where the stem $s^p_b$ is in the ground model and $\dot{f}^p_b$ is a nice $T_{\alpha,b}$-name for a $\kappa$-real in $V^{P_{F(b),b}} \cap {}^\kappa\kappa$. Let $P_{\alpha,a} = P_{\alpha,c} * \dot{T}_{\alpha,a}$.

**Lemma 4.2.** For any $\alpha \leq \alpha' \leq \beta$ and $a \in Q'$, $V_{\alpha',c} \vDash T_{\alpha,a} \lessdot T_{\alpha',a}$.

*Proof.* Consider in $V_{\alpha',c}$ the mapping $i : T_{\alpha,a} \to T_{\alpha',a}$ where $\mathrm{supp}(i(p)) = \mathrm{supp}(p)$ and for each $b \in \mathrm{supp}(i(p))$, $\Vdash_{T_{\alpha',b}} i(p)(b) = (s^{i(p)}_b, \dot{f}^{i(p)}_b)$, where $s^{i(p)}_b = s^p_b$ and $\dot{f}^{i(p)}_b$ is a $T_{\alpha',b}$-name for the $\kappa$-real named by $\dot{f}^p_b$. The mapping $i$ witnesses that $T_{\alpha,a} \lessdot T_{\alpha',a}$ in $V_{\alpha',c}$, by making crucial use of $J^{\alpha'} \subseteq J^\alpha$. If $b \in \mathrm{supp}(p) \subseteq Q'_a \setminus J^\alpha$, then (by $J^{\alpha'} \subseteq J^\alpha$) $b \in \mathrm{supp}(i(p)) \subseteq Q'_a \setminus J^{\alpha'}$. In this case, $F(b) < \alpha$ and $\dot{H}^\alpha_b$ is a $T_{\alpha,b}$-name for $V^{P_{F(b),b}} \cap {}^\kappa\kappa$. But $F(b) < \alpha'$ holds also and $\dot{H}^{\alpha'}_b$ is a $T_{\alpha',b}$-name for $V^{P_{F(b),b}} \cap {}^\kappa\kappa$ as well. As the second coordinates refer to the same set of $\kappa$-reals, compatibility and incompatibility depends on the stems at $\mathrm{supp}(p)$. □

**Lemma 4.3.** $\forall b \in W\ \forall \alpha < \alpha' \leq \beta\ (P_{\alpha,b} \lessdot P_{\alpha',b})$.

*Proof.* Proceed inductively on $W$. If $b = c$ and $\alpha \leq \beta$, then the Lemma holds by the product-like property of the forcing in Definition 2.4. For $b \in Q'$ the claim holds by Lemmas 4.2 and 2.6. □

**Remark 4.4.** All together we have $\forall \alpha, \alpha' \leq \beta\ \forall a, b \in W\ (\alpha \leq \alpha' \wedge a <_W b \to P_{\alpha,a} \lessdot P_{\alpha',b})$.

**Remark 4.5.** Note that $J^0 = Q$, so at the bottom "plane" we iterate with trivial forcing only. Also $J^\beta = \emptyset$, so at the top "plane" we have no trivial forcings, but only restricted Hechlers.

**Example 4.6.** Working in $V_{\alpha,c}$ observe the following: Let $p,q \in T_{\alpha,a}$ for some $a \in Q'$ be such that for each $b \in \text{supp}(q) \cap \text{supp}(p)$, $s_b^p \subseteq s_b^q \vee s_b^p \supseteq s_b^q$. Then $p,q$ are compatible, with a common extension $r \in T_{\alpha,a}$ defined as follows: $\text{supp}(r) = \text{supp}(p) \cup \text{supp}(q)$ and

- $\Vdash_{T_{\alpha,b}} r(b) = p(b)$ if $b \in \text{supp}(p) \setminus \text{supp}(q)$
- $\Vdash_{T_{\alpha,b}} r(b) = q(b)$ if $b \in \text{supp}(q) \setminus \text{supp}(p)$
- $\Vdash_{T_{\alpha,b}} r(b) = (s_b^r, \dot{f}_b^r)$ if $b \in \text{supp}(p) \cap \text{supp}(q)$, where $s_b^r = s_b^p \cup s_b^q$ and $\dot{f}_b^r$ is a $T_{\alpha,b}$-name for the pointwise maximum of $\dot{f}_b^q$ and $\dot{f}_b^p$.

**Lemma 4.7.** For any $\alpha \leq \beta$ and $a \in W$, the forcing $P_{\alpha,a}$ is $\kappa^+$-c.c. and is $\kappa$-closed.

*Proof.* If $a = c$, then $P_{\alpha,a}$ equals $\mathbb{H}_\alpha$ which has the $\kappa^+$-c.c. and is $\kappa$-closed.

If $a \neq c$, then $P_{\alpha,a} = P_{\alpha,c} * \dot{T}_{\alpha,a}$. Since $P_{\alpha,c} = \mathbb{H}_\alpha$ has the $\kappa^+$-c.c., it is sufficient to show that for any $\mathbb{H}_\alpha$-generic $G$, $V[G] \models$ "$T_{\alpha,a}$ has the $\kappa^+$-c.c.". In $V[G]$, consider any $S = \{p_\alpha : \alpha < \kappa^+\}$ a family of conditions in $T_{\alpha,a}$ of size $\kappa^+$. We will show that $S$ is not an antichain. Since the support of each condition is of size less than $\kappa$, and $\kappa^{<\kappa} = \kappa$, we can apply the $\Delta$-System-Lemma to $\{\text{supp}(p_\alpha) : \alpha < \kappa^+\}$ to get a $Y \in [S]^{\kappa^+}$ such that $\{\text{supp}(p_\alpha) : p_\alpha \in Y\}$ forms a $\Delta$-System with root $R$. Again since $\kappa^{<\kappa} = \kappa$, $|Y| = \kappa^+$ and $|R| < \kappa$, we can assume that if $b \in R$ and $p_\alpha \in Y$ then $p_\alpha(b) = (t_b, \dot{f}_b^\alpha)$ where $t_b$ is the same stem for each $p_\alpha \in Y$. Now, for $p_\alpha, p_\beta \in Y$ one can define a common extension $q$ as follows: $\text{supp}(q) = \text{supp}(p_\alpha) \cup \text{supp}(p_\beta)$; if $b \in R$ then $q(b) = (t_b, \dot{f}_b)$ where $\dot{f}_b$ is the pointwise maximum of $\{\dot{f}_b^\alpha, \dot{f}_b^\beta\}$. If $b \in \text{supp}(p_\alpha) \setminus \text{supp}(p_\beta)$ then $q(b) = p_\alpha(b)$ and if $b \in \text{supp}(p_\beta) \setminus \text{supp}(p_\alpha)$ then $q(b) = p_\beta(b)$.

Again as $P_{\alpha,c} = \mathbb{H}_\alpha$ is $\kappa$-closed, it is sufficient to show that for any $\mathbb{H}_\alpha$-generic $G$, $V[G] \models$ "$T_{\alpha,a}$ is $\kappa$-closed". Consider in $V[G]$ a decreasing sequence $(p_\alpha : \alpha < \gamma)$ of conditions, where $\gamma < \kappa$. We will define a common extension $p$, by using the fact that the forcing in Definition 2.3 is $\kappa$-closed. Proceed as follows. Let $\text{supp}(p) = \bigcup_{\alpha<\gamma} \text{supp}(p_\alpha)$. Then $|\text{supp}(p)| < \kappa$ by regularity of $\kappa$. If for any $\alpha < \gamma$ and $b \in \text{supp}(p_\alpha)$ we have $p_\alpha(b) = (t_\alpha(b), \dot{f}_\alpha(b))$, then let $p(b) = (t, \dot{f})$ where $t = \bigcup\{t_\alpha(b) : b \in \text{supp}(p_\alpha)\}$ and $\dot{f}$ is a $T_{\alpha,b}$-name for the pointwise supremum of the second coordinates $\{\dot{f}_\alpha(b) : b \in \text{supp}(p_\alpha)\}$. Then $p$ is as desired. $\square$

The next Lemma is analogous to Lemma 15 in [3].

**Lemma 4.8.** Suppose $b \in W$, then the following two properties hold:
(a) Any condition $p \in P_{\beta,b}$ is already in $P_{\alpha,b}$ for some $\alpha < \beta$.
(b) If $\dot{f}$ is a $P_{\beta,b}$-name for a $\kappa$-real then it is a $P_{\alpha,b}$-name for some $\alpha < \beta$.

*Proof.* We show (a) and (b) simultaneously by transfinite induction on $b \in W$, the well-founded poset. Because $P_{\beta,b}$ has the $\kappa^+$-c.c. property and $\beta$ is such that $\text{cof}(\beta) > \kappa$, we can easily see that (a) implies (b) if we pass over to a nice name of the $\kappa$-real at hand.

Now we begin the induction by letting $b = c$: Properties (a) and (b) for $b = c$ are both true as $\beta$ is regular, above $\kappa$ and the domain of a condition in $\mathbb{H}_\beta$ is of size less than $\kappa$. Hence this stage does not add new $\kappa$-reals.

Let $b \neq c$ and let $p \in P_{\beta,b} = P_{\beta,c} * \dot{T}_{\beta,b}$. Then $p$ is of the form $(p_0, \dot{p}_1)$, where $p_0 \in P_{\beta,c}$ and $\Vdash_{P_{\beta,c}} \dot{p}_1 \in \dot{T}_{\beta,b}$. For $p_0 \in P_{\beta,c}$ the induction hypothesis on (a) holds. So there is a $\alpha_0 < \beta$ such that $p_0 \in P_{\alpha_0,c}$. Since $\Vdash_{T_{\beta,b}} |\mathrm{supp}(\dot p_1)| < \kappa$, $\dot p_1$ involves less than $\kappa$-many names for $\kappa$-reals (the second coordinate of the restricted Hechler forcing). This gives an object of size at most $\kappa$, and we can use the induction hypothesis on (b) in order to find an $\alpha_1 < \beta$ such that $\dot p_1$ is a $P_{\alpha_1,c}$-name. Then $p = (p_0, \dot p_1)$ belongs to $P_{\alpha,b}$, where $\alpha = \max\{\alpha_0, \alpha_1\}$. So (a) is true for stages with $b \neq c$ and implies (b) for stages with $b \neq c$, because a nice name for a $\kappa$-real involves at most $\kappa$-many conditions and $\mathrm{cof}(\beta) = \beta > \kappa$. □

## 5. Preserving a witness for $\mathfrak{a}_\kappa$

Recall [3] §2 (Adding a mad family).

**Definition 5.1.** ([3]) Let $M \subseteq N$ be models of ZFC, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\kappa]^\kappa$ and $A \in N \cap [\kappa]^\kappa$. Then we say $☆(M, N, \mathcal{B}, A)$ is true, if for every $h \in M \cap {}^{\kappa\times[\gamma]^{<\kappa}}\kappa$ and $m \in \kappa$ we can find $n \geq m$, $F \in [\gamma]^{<\kappa}$ satisfying $[n, h(n, F))\setminus \bigcup_{\alpha\in F} B_\alpha \subseteq A$.

**Lemma 5.2.** ([3]) Suppose $☆(M, N, \mathcal{B}, A)$ is true and let $I(\mathcal{B})$ be the $\kappa$-complete ideal generated by $\mathcal{B}$ and the sets of size less than $\kappa$. Then for $B \in M \cap [\kappa]^\kappa$, $B \notin I(\mathcal{B})$ we have $|A \cap B| = \kappa$.

*Proof.* For suppose not and let $A \cap B \subseteq n \in \kappa$. Let $m' \geq n$, $F' \in [\gamma]^{<\kappa}$. Since $Y \subseteq^* X \in I(\mathcal{B})$ implies $Y \in I(\mathcal{B})$ and $\bigcup_{\alpha\in F'} B_\alpha \in I(\mathcal{B})$ and $B \notin I(\mathcal{B})$, we must have $B \not\subseteq^* \bigcup_{\alpha\in F'} B_\alpha$. So there is $k^{F'}_{m'}$ such that $m' < k^{F'}_{m'} \in B\setminus \bigcup_{\alpha\in F'} B_\alpha$. Now for all $m \geq n$ and $F \in [\gamma]^{<\kappa}$ we define $h(m, F) = k^F_m + 1$ and $h(m, F) = 0$ if $m < n$. As $h$ is defined in $M$ and $[m, h(m, F))\setminus \bigcup_{\alpha\in F} B_\alpha \notin A$ for all $m \geq n, F \in [\gamma]^{<\kappa}$, we contradict $☆(M, N, \mathcal{B}, A)$. □

The family $\mathcal{A}_\gamma$ added by $\mathbb{H}_\gamma$ (Definition 2.4) satisfies the ☆-property in the following sense.

**Lemma 5.3.** ([3]) If $G_{\gamma+1}$ is $\mathbb{H}_{\gamma+1}$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{H}_\gamma$ and $\mathcal{A}_\gamma = \{A_\alpha\}_{\alpha<\gamma}$ where as above $A_\alpha = \{i : \exists p \in G_{\gamma+1}\ p(\alpha, i) = 1\}$ for each $\alpha \leq \gamma$, then we have $☆(V[G_\gamma], V[G_{\gamma+1}], \mathcal{A}_\gamma, A_\gamma)$.

*Proof.* Let $h \in V[G_\gamma] \cap {}^{\kappa\times[\gamma]^{<\kappa}}\kappa$, $(p, H) \in \mathbb{H}_{[\gamma,\gamma+1)}$ and $m \in \kappa$ be arbitrary. By the definition of $\mathbb{H}_{[\gamma,\gamma+1)}$ we have $\mathrm{dom}(p) = \{\gamma\} \times n_p$ for some $n_p \in \kappa$. Now we define the following extension $(q, K)$ of $(p, H)$. Let $n \in \kappa$ be above $n_p$ and $m$, and let $n_q = h(n, H)$. Define $\mathrm{dom}(q)$ to be $\{\gamma\} \times n_q$. Let $K = H$ and

$$q(\gamma, i) = \begin{cases} p(\gamma, i) & \text{if } i < n_p \\ 0 & \text{if } i \in [n_p, n) \\ 1 & \text{if } i \in [n, n_q) \wedge i \notin \bigcup_{\alpha\in H} A_\alpha \\ 0 & \text{if } i \in [n, n_q) \wedge i \in \bigcup_{\alpha\in H} A_\alpha \end{cases}$$

Then $(q, K)$ extends $(p, H)$ and $(q, K) \Vdash [n, h(n, H))\setminus \bigcup_{\alpha\in H} A_\alpha \subseteq A_\gamma$ and we are done. □

**Lemma 5.4.** ([3]) Let $M \subseteq N$ be models of ZFC, $P \in M$ a forcing poset such that $P \subseteq M$, $G$ a $P$-generic filter over $N$ (hence also $P$-generic over $M$). Then the following holds: If $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\kappa]^\kappa$ and $A \in N \cap [\kappa]^\kappa$ and $☆(M, N, \mathcal{B}, A)$ holds, then $☆(M[G], N[G], \mathcal{B}, A)$.

*Proof.* For suppose not and let $h \in M[G] \cap {}^{\kappa\times[\gamma]^{<\kappa}}\kappa$, $m \in \kappa$ be such that $\forall n \geq m\ \forall F \in [\gamma]^{<\kappa}\ N[G] \vDash [n, h(n, F))\setminus \bigcup_{\alpha\in F} B_\alpha \notin A$. Then there are $p \in G$, a $P$-name $\dot h \in M$ for $h$ and $m \in \kappa$ with $p \Vdash_N \forall n \geq m\ \forall F \in [\gamma]^{<\kappa}\ [n, h(n, F))\setminus \bigcup_{\alpha\in F} B_\alpha \notin A$.

Now in $M$, for $\dot{h}$ let $p_n^F \in G$ be a condition extending $p$ and deciding the value of $h$ at point $(n,F)$, i.e. $p_n^F \Vdash \dot{h}(n,F) = k_n^F$. Then $p_n^F \Vdash_N [n,k_n^F))\setminus\bigcup_{\alpha\in F} B_\alpha \notin A$, so $N \vDash [n,k_n^F))\setminus\bigcup_{\alpha\in F} B_\alpha \notin A$. However, the function

$$h'(n,F) = \begin{cases} 0 & \text{if } n < m \\ k_n^F & \text{else} \end{cases}$$

is in $M$ and contradicts $\text{☆}(M,N,\mathcal{B},A)$. □

**Lemma 5.5.** $\forall b \in W\ \forall \alpha < \beta\ (\text{☆}(V_{\alpha,b},V_{\alpha+1,b},\mathcal{A}_\alpha,A_\alpha))$.

*Proof.* Proceed inductively on $W$. If $b = c$ and $\alpha \le \beta$, then the statement $\text{☆}(V_{\alpha,c},V_{\alpha+1,c},\mathcal{A}_\alpha,A_\alpha)$ holds by Lemma 5.3. Suppose next that $b \in Q'$. Note that $\text{☆}(V_{\alpha,c},V_{\alpha+1,c},\mathcal{A}_\alpha,A_\alpha)$ holds, $T_{\alpha,b} \in V_{\alpha,c} \subseteq V_{\alpha',c}$ and $V_{\alpha',c} \vDash T_{\alpha,b} \lessdot T_{\alpha',b}$ (Lemma 4.2). So any $V_{\alpha',c}$-generic subset of $T_{\alpha',b}$ is also $V_{\alpha',c}$-generic subset of $T_{\alpha,b}$. Consequently, by Lemma 5.4, $\text{☆}(V_{\alpha,b},V_{\alpha+1,b},\mathcal{A}_\alpha,A_\alpha)$. □

## 6. The result

The next theorem gives us the consistency result.

**Theorem 6.1.** $V_{\beta,m} \vDash \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \le \mathfrak{d}_\kappa = \delta$.

*Proof.* $\mathfrak{a}_\kappa \le \beta$: The family $\mathcal{A}_\beta = \{A_\alpha : \alpha < \beta\}$ added in the first column is a $\kappa$-mad family in the model $V_{\beta,m}$. If this was not the case, then $\exists x \in V_{\beta,m} \cap [\kappa]^\kappa\ \forall A_\alpha \in \mathcal{A}_\beta\ (|x \cap A_\alpha| < \kappa)$. By Lemma 4.8, we have $\exists \alpha < \beta\ (x \in V_{\alpha,m} \cap [\kappa]^\kappa)$. However by Lemma 5.4, $\text{☆}(V_{\alpha,m},V_{\alpha+1,m},\mathcal{A}_\alpha,A_\alpha)$ holds and so $|A_\alpha \cap x| = \kappa$ by Lemma 5.2.

$\mathfrak{b}_\kappa \ge \beta$: Let $B \subseteq V_{\beta,m} \cap {}^\kappa\kappa$ be such that $|B| < \beta$. By $\mathfrak{b}(Q) = \beta$ and by Lemma 4.8, we have $\exists b \in Q\ \exists \alpha < \beta\ (B \subseteq V_{\alpha,b} \cap {}^\kappa\kappa)$. As $\forall \gamma < \beta\ \forall c \in Q\ (c \uparrow_Q \cap F^{-1}(\gamma) \neq \varnothing)$ we can find an element $b' \in Q$ with $b < b'$ and $F(b') = \alpha$. Then the poset $P_{\alpha+1,b'}$ adds, among other things, a dominating $\kappa$-real over $V_{\alpha,b'} \cap {}^\kappa\kappa \supseteq V_{\alpha,b} \cap {}^\kappa\kappa$, hence $B$ is not unbounded.

By the previous paragraphs we have $V_{\beta,m} \vDash \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta$, as $\mathfrak{b}_\kappa \le \mathfrak{a}_\kappa$ is provable in ZFC.

$\delta \ge \mathfrak{d}_\kappa$: Let $\dot{f}$ be a $P_{\beta,m}$-name for a $\kappa$-real. By the previous Lemma 4.8, the property $\mathfrak{b}(Q) = \beta \ge \kappa^+$ and the regularity of $\beta$, there is a $b \in Q$ and an $\alpha < \beta$ such that $f \in V_{\alpha,b} \cap {}^\kappa\kappa$. Let $D \subseteq Q$ be a dominating family of size $\delta$ and let $d \in D$ be such that $b <_Q d$. As $\forall \gamma < \beta\ \forall c \in Q\ (c \uparrow_Q \cap F^{-1}(\gamma) \neq \varnothing)$, we can find an element $d_{\alpha,b} \in Q$ with $d_{\alpha,b} > d$ and $F(d_{\alpha,b}) = \alpha$. Then $P_{\alpha+1,d_{\alpha,b}}$ adds a dominating real over the model $V_{\alpha,d_{\alpha,b}} \supseteq V_{\alpha,b}$, call it $g^{d_{\alpha,b}}$. Hence the arbitrary $f$ is dominated by the set $\{g^{d_{\alpha,b}} : d \in D, \alpha \in \beta\}$ which is of size $\delta \cdot \beta = \delta$.

Now, for each $a \in Q$ and $P_{\beta,m}$-generic filter $G$, let $f_G^a = \bigcup\{t_a : \exists p \in G\ (p(a) = (t_a,\dot{f}_a))\}$ and let $\dot{f}_G^a$ be a $P_{\beta,m}$-name for $f_G^a$.

**Claim 6.2.** If $g \in V_{F(a),a}$ and $b \nleq_Q a$, then $V_{\beta,m} \vDash f_G^b \nleq^* g$.

*Proof.* Let $p$ be an arbitrary condition in $T_{\beta,m}$ (in $V_{\beta,c}$), $n \in \kappa$ and let $\dot{g}$ be a $T_{\beta,a}$-name for $g$. We will find an extension of $p$ which forces $\dot{f}_G^b(k) \ge \dot{g}(k)$ for some $k \ge n$. Let $p(a) = (t,\dot{g}')$ and $p(b) = (s,\dot{h})$. Let $\dot{f}$ be a $T_{\beta,a}$-name for the pointwise maximum of $\dot{g}'$ and $\dot{g}$. Now define the condition $p_0$ as follows: $\text{supp}(p_0) = \text{supp}(p)$ and $p_0(e) = p(e)$ for each $e \neq a$, and $p_0(a) = (t,\dot{f})$.

Clearly $p_0 \leq p$. Now let $k \in \kappa$ be large enough such that $\{\mathrm{dom}(t), \mathrm{dom}(s), n\} \subset k$. Next let $q \in T_{\beta,a}$ extend $p_0 \restriction a$ and $q$ decide the value of $\dot{f}$ up to $k$. Now define the extension $p_1$ of $p_0$ by setting $p_1(e) = p_0(e)$ for each $e \not<_Q a$ and $p_1(e) = q(e)$ for each $e <_Q a$. So $p_1$ is an extension of $p_0$ carrying the information on the values of $\dot{f}$ up to $k$; and now we do the same for $b$ and $p_1$, so we let $r \in T_{\beta,b}$ with $r \leq p_1 \restriction b$ and $r$ decides the values of $\dot{h}$ up to $k$. We define the extension $p_2$ as $p_2(e) = p_1(e)$ for each $e \not<_Q b$ and $p_2(e) = r(e)$ for each $e <_Q b$. Now $p \geq p_0 \geq p_1 \geq p_2$ and $p_2(a) = p_0(a)$ and $p_2(b) = p(b)$. Now we extend $p_2$ as desired: First find an end-extension $t' \supseteq t$ such that $\mathrm{dom}(t') = k+1$ and for $\mathrm{dom}(t) \leq i < \mathrm{dom}(t')$, $t'(i) > \dot{f}(i)$. Then find an end-extension $s' \supseteq s$ such that $\mathrm{dom}(s') = k+1$ and for $\mathrm{dom}(s) \leq i < k+1$ $(s'(i) > \max\{\dot{h}(i), t'(i)\})$. Then any further extension $p_2'$ of $p_2$ satisfying $s_b^{p_2'} = s'$ forces $\dot{f}_G^b(k) > \dot{f}(k)$ which gives the claim. □

$\delta \leq \mathfrak{d}_\kappa$: Let $F \subseteq V_{\beta,m} \cap {}^\kappa\kappa$ be a family of size less than $\delta$. As in the previous paragraph we can find for every single $f \in F$ a stage $a_f \in Q$ such that $f \in V_{F(a_f),a_f} \cap {}^\kappa\kappa$. Now $|\{a_f : f \in F\}| < \delta$, so $\{a_f : f \in F\}$ is not dominating in $Q$. Hence $\exists u \in Q\ \forall f \in F\ (u \not<_Q a_f)$. Then by Claim 6.2 we have $\forall f \in F\ (f_G^u \not<^* f)$. Hence $F$ is not dominating. □

**Theorem 6.3.** *If $\beta, \delta, \mu$ are infinite cardinals with $\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta) \leq \delta \leq \mu$ and $\mathrm{cof}(\mu) > \kappa$, then there is a $\kappa^+$-c.c. and $\kappa$-closed generic extension in which $\mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta$, $\mathfrak{d}_\kappa = \delta$ and $\mathfrak{c}_\kappa = \mu$.*

*Proof.* In the above construction replace the underlying poset $(Q, <_Q)$ by the following poset $(R, <_R)$: $R$ consists of pairs $(p, i)$ such that either $i = 0 \wedge p \in \mu$ or $i = 1 \wedge p \in Q$. The order relation is defined as $(p,i) <_R (q,j)$ iff $i = 0 \wedge j = 1$ or $i = j = 1 \wedge p <_Q q$ or $i = j = 0 \wedge p < q$ in $\mu$. Then $\mathfrak{b}(R) = \mathfrak{b}(Q) = \beta$ and $\mathfrak{d}(R) = \mathfrak{d}(Q) = \delta$ as the map $i : Q \to R$ defined as $b \mapsto (1, b)$ is a cofinal embedding from $Q$ into $R$. The bottom part $(\mu, \in)$ of $R$ ensures that in the final model $\mathfrak{c}_\kappa \geq \mu$ holds. By a standard argument of counting nice names $\mathfrak{c}_\kappa \leq \mu$ in $V_{\beta,m}$. □

## 7. Further Remarks

We also want to point out that the model in [3, §4] is an alternative witness for the constellation we showed here in the case of $\kappa = \omega$, namely $\mathfrak{b} = \mathfrak{a} < \mathfrak{d} < \mathfrak{c}$. Recall the construction in [3] forcing $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$: Let $\kappa < \lambda$ be fixed regular uncountable cardinals. First introduce a surjective book-keeping function $f : \{\nu < \lambda : \nu \equiv 1 \bmod 2\} \to \kappa$ where $\forall \alpha < \kappa\ \big(f^{-1}(\alpha)$ is cofinal in $\lambda\big)$. The matrix is defined recursively and consists of finite support iterations $\langle\langle P_{\alpha,\xi} : \alpha \leq \kappa, \xi \leq \lambda\rangle, \langle \dot{Q}_{\alpha,\xi} : \alpha \leq \kappa, \xi \leq \lambda\rangle\rangle$ where:

(1) If $\xi = 0$, then for each $\alpha \leq \kappa$, $P_{\alpha,0}$ is Hechler's poset from Definition 2.4 which adds an almost disjoint family $\mathcal{A}_\alpha = \{A_\beta\}_{\beta<\alpha}$ which is m.a.d. in $V_{\alpha,0}$ if $\alpha \geq \omega_1$.

(2) If $\xi = \mu + 1 \equiv 1 \bmod 2$, then for each $\alpha \leq \kappa$, $\Vdash_{P_{\alpha,\mu}} \dot{Q}_{\alpha,\mu} = \mathbb{M}(\dot{U}_{\alpha,\mu})$ while $\dot{U}_{\alpha,\mu}$ is a $P_{\alpha,\mu}$-name for an ultrafilter with the property that for $\alpha < \beta \leq \kappa$, $\Vdash_{P_{\beta,\mu}} \dot{U}_{\alpha,\mu} \subseteq \dot{U}_{\beta,\mu}$. This helps to evaluate the splitting number in the final model.

(3) If $\xi = \mu + 1$ and $\xi \equiv 0 \bmod 2$, then for each $\alpha \leq f(\mu)$ $\dot{Q}_{\alpha,\mu}$ is a $P_{\alpha,\mu}$-name with $\Vdash_{P_{\alpha,\mu}}$ "$\dot{Q}_{\alpha,\mu}$ is the trivial forcing"; and if $\alpha > f(\mu)$ then $\dot{Q}_{\alpha,\mu}$ is the $P_{\alpha,\mu}$-name for adding a dominating real over the model $V_{f(\mu),\mu}$.

(4) If $\xi$ is a limit ordinal, then for each $\alpha \leq \kappa$, $P_{\alpha,\xi}$ is the direct limit of the previous $P_{\alpha,\mu}$.

For suitable cardinals $\kappa, \lambda, \mu$ in the final model $V_{\kappa,\lambda}$ one can witness $\mathfrak{a} = \mathfrak{b} = \kappa < \lambda = \mathfrak{d} (= \mathfrak{s}) < \mathfrak{c} = \mu$: Proceed with a finite support iteration of Cohen forcings of length $\mu$ in order to get an intermediate stage (model $V_0$) where $\mathfrak{c} = \mu$ holds. Over $V_0$ perform the above described construction. It is not difficult to check that in the resulting model $\mathfrak{a} = \mathfrak{b} = \kappa < \lambda = \mathfrak{s}$. Next, we show that in the model also $\mathfrak{d} = \lambda$:

$\mathfrak{d} \leq \lambda$: Let $f \in V_{\kappa,\lambda} \cap {}^{\omega}\omega$ be an arbitrary real. By Lemma [3, Lemma 15] and the regularity of $\lambda$ we have $\exists \alpha < \kappa, \xi < \lambda \ (x \in V_{\alpha,\xi} \cap {}^{\omega}\omega)$ such that $\xi = \eta + 1 \equiv 1 \bmod 2$. As $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in $\lambda$ we can find a $\xi < \xi' \equiv 0 \bmod 2$ with $f(\xi') = \alpha$. Then the poset $P_{\alpha+1,\xi'+1}$ adds a Hechler real over the model $V_{\alpha,\xi'} \cap {}^{\omega}\omega \supseteq V_{\alpha,\xi} \cap {}^{\omega}\omega$, and the $\lambda$-many (restricted) Hechler reals in the construction build a dominating family.

$\mathfrak{d} \geq \lambda$: Let $B \subseteq V_{\kappa,\lambda} \cap {}^{\omega}\omega$ be such that $|B| < \lambda$. By the regularity of $\lambda$ we have $\exists \xi < \lambda \ (x \in V_{\kappa,\xi} \cap {}^{\omega}\omega)$. As the remaining part is a finite support iteration of non-trivial forcings, limit stages with countable cofinality add a Cohen real which is unbounded. Hence $B$ is not dominating.

We further point out that the consistency of $\kappa^+ \leq \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \leq \mathfrak{d}_\kappa = \mathfrak{c}_\kappa = \delta$ can be shown by a (linear) matrix iteration: Assume in the construction of Section 4 additionally that $\delta$ is regular and replace $Q$ by the well-order $(\delta, \in)$. The final model of this matrix, which is of height $\beta$ and width $\delta$, satisfies $\kappa^+ \leq \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \leq \mathfrak{d}_\kappa = \mathfrak{c}_\kappa = \delta$. If we additionally want to separate $\mathfrak{d}_\kappa$ and $\mathfrak{c}_\kappa$, e.g. to force $\mathfrak{c}_\kappa = \mu$, we can add $\mu$-many Cohen $\kappa$-reals before the above described iteration. However, by arguing with a (linear) matrix iteration, we have to require that $\delta$ is regular, leaving the case $\mathfrak{d}_\kappa$ singular unsettled. To force $\kappa^+ \leq \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \leq \mathfrak{d}_\kappa = \delta \leq \mathfrak{c}_\kappa = \mu$ for a singular $\delta$ one has to take the more general approach given in Section 4.

**Question 7.1.** It is open whether four cardinal characteristics (among other natural candidates), namely $\mathfrak{a}, \mathfrak{s}, \mathfrak{r}$ and $\mathfrak{u}$, can be controlled strictly between $\mathfrak{b}$ and $\mathfrak{d}$. Is either of the following constellations consistent: $\mathfrak{b} < \mathfrak{a} < \mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{s} < \mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{r} < \mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{u} < \mathfrak{d} < \mathfrak{c}$?

Since $\mathfrak{b}_\kappa = \kappa^+$ implies that $\mathfrak{a}_\kappa = \kappa^+$ for $\kappa$ regular uncountable (see [8]), the main result of [4] implies that for a given suitable set $C$ of regular uncountable cardinals, it is consistent that $\mathfrak{b}_\lambda = \mathfrak{a}_\lambda = \lambda^+ < \mathfrak{d}_\lambda = \mathfrak{c}_\lambda$ holds simultaneously for all $\lambda \in C$. This naturally leads to the following:

**Question 7.2.** Given a set $C$ of regular uncountable cardinals is it consistent that

$$\lambda^+ < \mathfrak{b}_\lambda = \mathfrak{a}_\lambda < \mathfrak{d}_\lambda < \mathfrak{c}_\lambda$$

for all $\lambda \in C$ simultaneously?

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