Bachelorarbeit

Suslin's Problem

Verfasser

Joaquín Padilla Montani

angestrebter akademischer Grad

Bachelor of Science (BSc)

Wien, 2018

Studienkennzahl lt. Studienblatt:	A 033 621
Fachrichtung:	Mathematik
Betreuerin:	Vera Fischer, Privatdoz., PhD

Contents

Introduction

Motivation[1]

The first volume of the Polish journal *Fundamenta Mathematicae*, published in 1920, ended with a list of open problems. Problem 3 [17], posthumously attributed to the Russian mathematician Mikhail Yakovlevich Suslin (1894-1919), read as follows:

> Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénumerable, est-il nécessairement un continue linéaire (ordinaire)?

Which roughly translates to: "Is a (linearly) ordered set, with no jumps and no gaps, and such that every collection of its intervals (containing more than one element) not overlapping each other is at most countable, necessarily an (ordinary) linear continuum?". In modern terms, the absence of *jumps* refers to the ordering being dense, and the lack of *gaps* amounts to asking for the ordering to be complete. If we compare this to the usual characterization of $(\mathbb{R}, <_{\mathbb{R}})$ where the real numbers are that unique unbounded linearly ordered set which is dense in itself, complete and separable, *Suslin's Problem* asks whether the hypothesis of separability can be replaced by the topological *countable chain condition* (c.c.c.): that every collection of nonempty pairwise disjoint open intervals is at most countable.

It is not difficult to see that any separable linear ordering must satisfy the countable chain condition. In the opposite direction, the question about the existence of a *Suslin line*, that is, an unbounded linearly ordered set which is dense, complete and ccc, but not separable, proved to be much harder to settle. The existence of a Suslin line would constitute a negative answer to Suslin's Problem, and the assertion "there are no Suslin lines" came to be known as *Suslin's Hypothesis* or SH, although Suslin simply posed the question, without any discussion.

Other problems in that original list of 1920, mostly dealing with topics from the incipient field of descriptive set theory, would eventually be solved. Suslin's Problem, however, remained unsolved for half a century, and so it grew in importance through the years. Around 1940, the first advance towards settling Suslin's Problem was independently produced by Djuro Kurepa [11], Edwin Miller [14] and Wacław Sierpiński [15]. Their contribution reduced SH, an eminently topological problem, to a problem purely in the realm of combinatorial set theory. They found that the existence of a Suslin line is equivalent to the existence of a combinatorial object called a *Suslin tree*, a tree of height ω_1 without uncountable chains or antichains.

Much later, in a paper published in 1968 [18], Stanley Tennenbaum proved that $\neg$SH is consistent with ZFC. That is, he showed that there are models of set theory where a Suslin line exists, hence rendering impossible the task of proving Suslin's Hypothesis. His proof made use of Paul Cohen's celebrated forcing technique, and was done in the summer of 1963, only months after Cohen had developed his method for extending models of set theory. As such, after Cohen's initial applications of forcing (most notably on the non-provability of the Continuum Hypothesis) SH was one of the first problems which were solved using this groundbreaking technique.

[1] This historical outline is mostly based on [10] and [12].

Thomas Jech independently proved [8], a few years later, the same results on the non-provability of SH. His use of forcing was more complicated than Tennenbaum's and also made, along with SH, the Continuum Hypothesis (CH) true in that model, whereas Tennenbaum's technique could both produce models of $\neg$SH + CH and of $\neg$SH + $\neg$CH. It was, however, Jech's approach which would generalize to prove results on higher cardinality versions of Suslin trees.

Close in time to Tennenbaum and Jech's result on the non-provability of SH, another major discovery was made by Ronald Jensen [9]. He showed that in Kurt Gödel's famous Constructible Universe L, a model of ZFC where the Generalized Continuum Hypothesis holds true, a Suslin tree (and hence also Suslin line) can be found. So Gödel's L is a witness for the non-provability of SH. Jensen then extracted from his proof a combinatorial principle known as Diamond ($\diamondsuit$), and showed that $\diamondsuit$ implies that there is a Suslin tree. This initiated his study of combinatorial principles in L. Combinatorial principles are focal axioms which hold in one or another model of set theory, out of which one can draw, by purely classical methods, strong consequences unavailable in ZFC alone.

In a paper published in 1971 [16], Robert Solovay and Stanley Tennenbaum finally settled Suslin's Problem. They showed that SH is consistent with ZFC, meaning that there is a model of set theory were no Suslin line exists. Together with Tennenbaum's earlier result on the consistency of $\neg$SH, this proved that SH is independent of ZFC, i.e., using the usual axioms for set theory, SH can neither be proven nor disproven. Solovay and Tennenbaum's proof was actually developed in 1965, and again used the forcing technique, but this time in an iterative fashion. Nowadays a standard tool in set theory, iterated forcing is considered to have originated in this proof.

Donald Martin and Robert Solovay then showed in [13] that, using the same techniques as in [16], one can establish the consistency of a combinatorial principle nowadays known as Martin's Axiom, which has many consequences; among those is SH, if one takes the additional assumption that $2^{\aleph_0} > \aleph_1$, i.e. that CH does not hold.[2]

Suslin's Problem, for decades a famous open question, permeated the development of key areas in set theory, most notably in forcing and constructibility.

Outline

The aim of this thesis is to show, assuming the consistency of Jensen's Diamond Principle and of Martin's Axiom, the independence of Suslin's Hypothesis. Therefore the key results we want to prove are $\diamondsuit \to \neg$SH (Theorem 4.12) and MA($\aleph_1$) $\to$ SH (Theorem 6.11). We also present Gödel's constructible universe L and show that $\diamondsuit$ is true in L, therefore fully proving the non-provability of SH.

Section 1 discusses, via the Dedekind Completion, the characterization of $(\mathbb{R}, <_{\mathbb{R}})$ mentioned in the introduction. **Section 2** formally states Suslin's Problem, related definitions, and shows that the product of a Suslin line with itself is not ccc. **Section 3** reduces SH to a problem in infinitary combinatorics: the proof that a Suslin line exists if and only if a Suslin tree exists is presented. In **Section 4** the Diamond Principle is defined, and the section is dedicated to proving that $\diamondsuit$ implies $\neg$SH. **Section 5** deals with Gödel's Constructible Universe, and the main result is the proof that $\diamondsuit$ holds in L. **Section 6** discusses Martin's Axiom, and it is shown in two ways that under MA($\aleph_1$), SH is true.

[2] The consistency of CH+SH was established by Ronald Jensen shortly thereafter (c.f. [1]).

Prerequisites

The main prerequisite is familiarity with the basics of ordinals and cardinals, transfinite induction and transfinite recursion. Excellent references for this are [3], [4], [5] and [6]. For Section 5 on the Constructible Universe, model theory, especially in the context of set theory, is required; in particular the Downward Löwenheim-Skolem Theorem, Mostowski's Collapsing Lemma and absoluteness. For this, see [5] or [6]. Also, basic ideas from topology are needed in a few proofs.

1 A Characterization of $(\mathbb{R}, <_{\mathbb{R}})$

In the next section we will formally define Suslin's Problem. To do this, one starts with a characterization of $(\mathbb{R}, <_{\mathbb{R}})$ as the unique (up to isomorphism) linearly ordered set which is unbounded, dense in itself, complete and separable. The aim of this section is to formally define the notions involved in this characterization and prove it via Dedekind cuts, starting from any ordered set resembling the rationals.

Definition 1.1 (Partial/Linear Ordering)**.** Let P be a nonempty set. A *partial ordering* of P is a binary relation $<$ on P which is *irreflexive*, i.e. $\forall p \in P(p \not< p)$, and *transitive*, meaning $\forall pqr \in P(p < q \wedge q < r \rightarrow p < r)$. The pair $(P, <)$ is called a *partially ordered set*. If, moreover, $<$ satisfies *trichotomy*, that is $\forall pq \in P(p = q \vee p < q \vee q < p)$, we say that $<$ is a *linear ordering* of P and call $(P, <)$ a *linearly ordered set*. The notation $p \leq q$ is an abbreviation for $p < q \vee p = q$.

Definition 1.2 (Dense Linear Ordering)**.** Let $(P, <)$ be a linearly ordered set with at least two elements and let $D \subseteq P$. Then we say that
(i) D is a *dense subset* of P or, alternatively, D is *dense in* P iff $\forall pq \in P(p < q \rightarrow \exists d \in D(p < d < q))$;
(ii) $(P, <)$ is *dense* iff P is dense in itself, that is $\forall pq \in P(p < q \rightarrow \exists x \in P(p < x < q))$.
Note that the existence of some $D \subseteq P$ dense in P already ensures that $(P, <)$ is dense.

Remark 1.3. If $(P, <)$ is a dense linearly ordered set, then P is always infinite. Moreover, $D \subseteq P$ being dense in P in the sense defined above is equivalent to asking for D to be dense in P in the topological sense, given the order topology induced by $<$.

Definition 1.4. Let $(P, <)$ be a partially ordered set, and let $a \in X \subseteq P$. Then
(i) a is a *maximal* element of X iff $\forall x \in X(a \not< x)$;
(ii) a is the *greatest* element of X iff $\forall x \in X(x \leq a)$.
Similarly,
(iii) a is a *minimal* element of X iff $\forall x \in X(x \not< a)$;
(iv) a is the *least* element of X iff $\forall x \in X(a \leq x)$.

Definition 1.5 (Unbounded Linear Ordering)**.** Let $(P, <)$ be a linearly ordered set. We say that $(P, <)$ is *unbounded* or, alternatively, that $(P, <)$ has *no endpoints* iff P has neither a least nor a greatest element. In symbols: $\forall p \in P\, \exists a, b \in P(a < p < b)$.

It is well known that the rational numbers $(\mathbb{Q}, <_{\mathbb{Q}})$ are a *countable* linearly ordered set, that there is no greatest or least rational, and that between any two rationals we can always find another one, i.e. $(\mathbb{Q}, <_{\mathbb{Q}})$ is dense. Moreover, it turns out that any other countable dense unbounded linearly ordered set looks just like $(\mathbb{Q}, <_{\mathbb{Q}})$. To prove this, we first need a notion of isomorphism between ordered sets.

Definition 1.6. Let $(P, <)$ and $(Q, \prec)$ be two partially ordered sets. An *order-embedding* is a function $f : P \to Q$ such that $\forall p_1 p_2 \in P(p_1 < p_2 \leftrightarrow f(p_1) \prec f(p_2))$. If f is also bijective, then we call it an *order-isomorphism.* A function $g : X \to Q$ for $X \subsetneq P$ is called a *partial isomorphism* from P to Q, provided that g behaves as an order-embedding on its domain of definition.

Lemma 1.7. Let $(P, <)$ and $(Q, \prec)$ be two dense linearly ordered sets without endpoints. If f is a partial isomorphism from P to Q such that $\text{dom}(f)$ is finite, then for any choice of $p \in P$ and $q \in Q$ there is a partial isomorphism $f_{p,q} \supseteq f$ such that $p \in \text{dom}(f_{p,q})$ and $q \in \text{ran}(f_{p,q})$.

Proof. Let $p \in P$ and $q \in Q$. Let $f = \{(p_1, q_1), \dots, (p_k, q_k)\}$, where $p_1 < \dots < p_k$, and thus also $q_1 \prec \dots \prec q_k$.

We first extend f by p. If $p \in \text{dom}(f)$, there's nothing to do, so assume that $p \notin \text{dom}(f)$. If $p < p_1$, then we find $\tilde{q} \in Q$ such that $\tilde{q} \prec q_1$, which is possible since the orderings we are considering have no endpoints. Then define $f_p = f \cup \{(p, \tilde{q})\}$; clearly this is still a partial isomorphism. Similarly if $p_k < p$, we find $\tilde{q} \in Q$ such that $q_k \prec \tilde{q}$ and define f_p in the same way. If otherwise there is some i such that $p_i < p < p_{i+1}$, since our orderings are dense we can find some $\tilde{q} \in Q$ such that $q_i \prec \tilde{q} \prec q_{i+1}$ and define f_p again as before.

To now extend f_p by q, note that f_p^{-1} is a finite partial isomorphism in the opposite direction, so we can add q to the domain of f_p^{-1} by arguing in the same way as above. Hence we get a partial isomorphism $f_{p,q}$ extending f such that $p \in \text{dom}(f_{p,q})$ and $q \in \text{ran}(f_{p,q})$. □

Theorem 1.8. Any two countable dense linearly ordered sets without endpoints are isomorphic.

Proof. Let $(P, <)$ and $(Q, \prec)$ be two unbounded dense linearly ordered sets such that $P = \{p_n : n \in \omega\}$ and $Q = \{q_n : n \in \omega\}$. We construct a sequence of partial isomorphisms by recursion. Let $f_0 = \emptyset$ and given f_n, let $f_{n+1} = (f_n)_{p_n, q_n}$ using the previous lemma. Notice that $f_{n+1} \supseteq f_n$ for any $n \in \omega$, so the partial isomorphisms are compatible with each other, in the sense that given any $p \in P$, it is not possible to find two functions in our sequence which map p to different values in Q. For this reason we can now let $f = \bigcup_{n \in \omega} f_n$. It is immediate to check that $f : P \to Q$ is an isomorphism between $(P, <)$ and $(Q, \prec)$. □

Remark 1.9. In model theory books, the statement of Theorem 1.8 would read: "the theory DLO (the theory of dense linear orders without endpoints) is $\aleph_0$-categorical". The method of proof given above is commonly referred to as the *back-and-forth* method. Cantor was the first to prove this theorem, although using a different technique.

Definition 1.10. Let $(P, <)$ be a partially ordered set, $p \in P$, and let $\emptyset \neq X \subseteq P$. Then

(i) p is an *upper bound* of X iff $\forall x \in X(x \leq p)$;

(ii) p is the *supremum* of X iff p is the least upper bound of X, i.e. p is the least element of the set $\{\tilde{p} \in P : \tilde{p} \text{ is an upper bound of } X\}$.

Similarly,

(iii) p is a *lower bound* of X iff $\forall x \in X(p \leq x)$;

(iv) p is the *infimum* of X iff p is the greatest lower bound of X, i.e. p is the greatest element of the set $\{\tilde{p} \in P : \tilde{p} \text{ is a lower bound of } X\}$.

Definition 1.11 (Complete Linear Ordering)**.** A linearly ordered set $(P, <)$ is *complete* iff every nonempty subset of P which has an upper bound also has a supremum.

Remark 1.12. For a linearly ordered set $(P, <)$, being dense and complete as defined above is equivalent to asking for $(P, <)$ to be connected in its order topology.

Definition 1.13 (Dedekind Cut)**.** Let $(P, <)$ be a linearly ordered set. A *Dedekind cut* is a pair (A, B) of disjoint nonempty subsets of P satisfying the three following conditions:
(i) $A \cup B = P$;
(ii) A does not have a greatest element;
(iii) $\forall a \in A \, \forall b \in B (a < b)$.

Example 1.14. Given some $p \in P$, the pair $(\{\tilde{p} \in P \mid \tilde{p} < p\}, \{\tilde{p} \in P \mid \tilde{p} \geq p\})$ is a Dedekind cut; such a cut will be denoted by $[p]$.

Theorem 1.15 (Dedekind Completion)**.** Let $(P, <)$ be a dense linearly ordered set without endpoints. Then there is a complete linearly ordered set without endpoints $(C, \prec)$ and an order-embedding $\varphi : P \to C$ such that $\varphi[P] = \{c \in C \mid \exists p \in P(\varphi(p) = c)\}$ is dense in C. Moreover, $(C, \prec)$ is unique, in the sense that given further $(\tilde{C}, \sqsubset)$ and $\tilde{\varphi} : P \to \tilde{C}$ as before, there is an order-isomorphism $f : C \to \tilde{C}$ satisfying $f \circ \varphi = \tilde{\varphi}$. We call $(C, \prec)$ the *Dedekind completion* of $(P, <)$.

Proof. Existence

We define C to be the set of all Dedekind cuts of P. Given $(A_1, B_1) \in C$ and $(A_2, B_2) \in C$, the ordering $\prec$ is characterized by $(A_1, B_1) \prec (A_2, B_2)$ iff $A_1 \subsetneq A_2$.

First we will show that $(C, \prec)$ is a linearly ordered set. The irreflexivity of $\prec$ is immediate since the relation $\prec$ is defined in terms of strict inclusion, and transitivity follows from the transitivity of (strict) set inclusion. It remains to show that $\prec$ satisfies trichotomy. Let $(A_1, B_1) \in C$ and $(A_2, B_2) \in C$ with $(A_1, B_1) \neq (A_2, B_2)$. Without loss of generality we can assume that there is $a_1 \in A_1$ such that $a_1 \notin A_2$, and hence $a_1 \in B_2$. By Definition 1.13 it is clear that $A_2 \subseteq \{p \in P \mid p < a_1\} \subsetneq A_1$, therefore $(A_2, B_2) \prec (A_1, B_1)$, i.e. trichotomy holds.

Next consider the function $\varphi : P \to C$ defined by $p \mapsto [p]$. Note that for any $a, b \in P$, $a < b$ if and only if $\{p \in P \mid p < a\} \subsetneq \{p \in P \mid p < b\}$, which in turn holds iff $[a] \prec [b]$, which is equivalent to $\varphi(a) \prec \varphi(b)$, so φ is indeed an order-embedding.

Having defined the linearly ordered set $(C, \prec)$ and the order-embedding $\varphi : P \to C$, it remains to show that
(a) $\varphi[P]$ is dense in C, hence $(C, \prec)$ is dense;
(b) $(C, \prec)$ is unbounded;
(c) $(C, \prec)$ is complete.

For (a), let $(A_1, B_1) \in C$ and $(A_2, B_2) \in C$ with $(A_1, B_1) \prec (A_2, B_2)$, i.e. $A_1 \subsetneq A_2$. Then $A_2 \setminus A_1 \neq \emptyset$, so there is some $b \in A_2 \setminus A_1$. We can assume that b is not the least element of B_1, if not we could take $\tilde{b} > b$ with $\tilde{b} \in A_2$ by the fact that A_2 has no greatest element. It would still be the case that $\tilde{b} \notin A_1$, otherwise $\tilde{b} > b$ would be contradicting Definition 1.13. This guarantees that $(A_1, B_1) \prec [b]$. Moreover, since $b \in A_2$, we have that $p \in A_2$ for any $p \in P$ satisfying $p < b$, using Definition 1.13. This shows that $[b] \prec (A_2, B_2)$, and (a) is now proven.

For (b), fix $(A, B) \in C$ and take any $b \in B$ which is not the least element of B. Then we have that $(A, B) \prec [b]$, and of course $[b] \in C$. Now given any $a \in A$, it holds that $[a] \in C$ and $[a] \prec (A, B)$. This shows that C has no endpoints.

It remains to show (c), the completeness of $(C, \prec)$. Let D be a nonempty subset of C and let $(A_0, B_0) \in C$ be an upper bound for D, i.e. for any $(A, B) \in D$ we have that $A \subseteq A_0$. We need to come up with a supremum for D. To this end, let $A_D = \bigcup\{A \mid (A, B) \in D\}$ and $B_D = X \setminus A_D$. Notice that $B_D = \bigcap\{B \mid (A, B) \in D\}$ using De Morgan's laws.

We first verify that (A_D, B_D) is a Dedekind cut. By construction A_D and B_D are disjoint subsets of P. Moreover, since $D \neq \emptyset$, we know that $A_D \neq \emptyset$, and $B_D \neq \emptyset$ is clear from $B_0 \subseteq B_D$. It remains to check the three conditions from Definition 1.13. First, $A_D \cup B_D = P$ is clear by construction, and the same is true for condition (iii). Furthermore, A_D cannot have a greatest element, since none of the A's have one.

Now given that $A \subseteq A_D$ for any $(A, B) \in D$, we have that (A_D, B_D) is an upper bound of D. It remains to show that (A_D, B_D) is the supremum, i.e. that (A_D, B_D) is $\preceq$ than any other upper bound of D. For any upper bound $(\tilde{A}, \tilde{B}) \in C$ of D it holds that $A \subseteq \tilde{A}$ for any $(A, B) \in D$, so $A_D = \bigcup\{A \mid (A, B) \in D\} \subseteq \tilde{A}$; it follows that $(A_D, B_D) \preceq (\tilde{A}, \tilde{B})$, which is what we needed to show.

Uniqueness

Let $(C, \prec)$ and $\varphi : P \to C$ be the Dedekind completion of $(P, <)$ and its corresponding order-embedding as defined above. Let $(\tilde{C}, \sqsubset)$ be another complete and unbounded linearly ordered set, with $\tilde{\varphi} : P \to \tilde{C}$ an order-embedding and $\tilde{\varphi}[P]$ dense in $\tilde{C}$. We want to show that there is an order-isomorphism $f : C \to \tilde{C}$ satisfying $f \circ \varphi = \tilde{\varphi}$.

For a Dedekind cut $c = (A, B) \in C$ we define $f(c)$ as $\sup_{\tilde{C}}(\tilde{\varphi}[A])$, i.e. the supremum of $\tilde{\varphi}[A]$ as a subset of $\tilde{C}$. It is clear that f is a mapping from C to $\tilde{C}$, since for $(A, B) \in C$ we have that A is bounded above in P by any element of B, and since $\tilde{\varphi}$ is an order-embedding we have that $\tilde{\varphi}[A]$ also has an upper bound in $\tilde{C}$, so its supremum in $\tilde{C}$ exists (and is unique, as suprema always are).

It remains to check that f is an order-isomorphism and that $f \circ \varphi = \tilde{\varphi}$. To see that f is onto, let $\tilde{c} \in \tilde{C}$ be arbitrary. Notice that, by the density of $\tilde{\varphi}[P]$ in $\tilde{C}$ and the fact that $\tilde{\varphi}$ is an order-embedding, the pair

$$(A_{\tilde{c}}, B_{\tilde{c}}) := (\{p \in P \mid \tilde{\varphi}(p) \sqsubset \tilde{c}\}, \{p \in P \mid \tilde{\varphi}(p) \sqsupseteq \tilde{c}\})$$

is a Dedekind cut, and therefore an element of C. It is easy to see that then $f((A_{\tilde{c}}, B_{\tilde{c}})) = \sup_{\tilde{C}}(\tilde{\varphi}[A_{\tilde{c}}]) = \tilde{c}$, which shows that f is onto.

Now let $(A_1, B_1) \in C$ and $(A_2, B_2) \in C$ with $(A_1, B_1) \prec (A_2, B_2)$. By definition this is equivalent to $A_1 \subsetneq A_2$, which is equivalent to $\tilde{\varphi}[A_1] \subsetneq \tilde{\varphi}[A_2]$ since $\tilde{\varphi}$ is injective. We are done if we can prove that this is equivalent to $\sup_{\tilde{C}}\tilde{\varphi}[A_1] \sqsubset \sup_{\tilde{C}}\tilde{\varphi}[A_2]$, but this is true by the fact that A_1 and A_2 are both initial segments of P that don't have a greatest element. We conclude that f is an order-isomorphism.

Finally we have that

$$(f \circ \varphi)(p) = f([p]) = \sup_{\tilde{C}}\tilde{\varphi}[\{q \in P \mid q < p\}] = \tilde{\varphi}(p)$$

which proves that $f \circ \varphi = \tilde{\varphi}$. □

Definition 1.16. Let $(\mathbb{Q}, <_{\mathbb{Q}})$ be the rational numbers with their usual ordering. The completion of $(\mathbb{Q}, <_{\mathbb{Q}})$ is denoted by $(\mathbb{R}, <_{\mathbb{R}})$. The elements of $\mathbb{R}$ are called *real numbers*.

Definition 1.17 (Separable Linear Ordering)**.** A linearly ordered set $(P, <)$ is *separable* iff there is a countable $D \subseteq P$ that is dense in P.

We can now state and prove the characterization of $(\mathbb{R}, <_{\mathbb{R}})$ that we mentioned at the beginning of this section.

Theorem 1.18. Let $(L, <_L)$ be a dense linearly ordered set without endpoints which is complete and separable. Then $(L, <_L)$ is order-isomorphic to $(\mathbb{R}, <_{\mathbb{R}})$.

Proof. Let $(L, <_L)$ be a complete linearly ordered set without endpoints and let $D \subseteq L$ be countable and dense in L. Then $(D, <_L\restriction D)$ is a countable dense linearly ordered set without endpoints, so it is isomorphic to $(\mathbb{Q}, <_{\mathbb{Q}})$ by Theorem 1.8. By the uniqueness of the Dedekind completion (Theorem 1.15), $(L, <_L)$ is isomorphic to $(\mathbb{R}, <_{\mathbb{R}})$, the completion of $(\mathbb{Q}, <_{\mathbb{Q}})$. □

2 Suslin's Hypothesis

Using the characterization of $(\mathbb{R}, <_{\mathbb{R}})$ from the previous section, we will now discuss Suslin's Problem. Further, we will define the related concepts of a Suslin line and of Suslin's Hypothesis, and prove that a Suslin line can always be made to be dense, unbounded and complete. We will also show that the negation of Suslin's Hypothesis, i.e. the existence of a Suslin line, implies that the product of ccc topological spaces it not always ccc.

Definition 2.1 (Countable Chain Condition)**.** A topological space X has the *countable chain condition* (or the *ccc*) iff every family of disjoint non-empty open subsets of X is at most countable.

Remark 2.2. In the context of a linearly ordered set $(P, <)$, we will call $(P, <)$ ccc if it satisfies the countable chain condition given its order topology. This is equivalent to the statement: every family of pairwise disjoint non-empty open intervals of the form $(a, b) = \{p \in P \mid a < p < b\}$ is at most countable.

Let $(L, <_L)$ be a dense linearly ordered set without endpoints which is complete and satisfies the countable chain condition. Is $(L, <_L)$ order-isomorphic to $(\mathbb{R}, <_{\mathbb{R}})$? This is known as *Suslin's Problem*, i.e. asking if in Theorem 1.18 the condition of separability can be weakened to the countable chain condition. The next lemma shows that this condition is indeed weaker.

Lemma 2.3. Every separable linearly ordered set satisfies the ccc.

Proof. Let $(P, <)$ be a separable linearly ordered set, with $Q \subseteq P$ countable and dense in P. Let $\mathcal{A}$ be a family of pairwise disjoint open intervals in P. Consider the set $X \coloneqq Q \cap \bigcup \mathcal{A}$. Since Q is dense in P, every open interval $a \in \mathcal{A}$ contains at least one $x \in X$ and since the open intervals in $\mathcal{A}$ are pairwise disjoint, each $x \in X$ is an element of a unique open interval $a \in \mathcal{A}$. Hence the function $f : X \to \mathcal{A}$, which assigns to an element of X the unique open interval $a \in \mathcal{A}$ to which it belongs, is well-defined and onto. Since X is at most countable, we conclude that $\mathcal{A}$ is at most countable, witnessed by f. This shows that $(P, <)$ satisfies the ccc. □

Definition 2.4 (Suslin Line / Suslin's Hypothesis)**.** A linearly ordered set that satisfies the countable chain condition and is not separable is called a *Suslin line*. *Suslin's Hypothesis* (SH) is the statement, "there are no Suslin lines".

Thus SH is equivalent to a "yes" answer to Suslin's Problem, and the existence of a Suslin line, i.e. $\neg$SH, is equivalent to a "no" answer.

The given definition of a Suslin line omits some of the properties that are mentioned in Suslin's original question. The next two lemmas show that given a Suslin line, we can always find another one with these additional requirements.

Lemma 2.5. Assume that there exists a Suslin line $(L, <)$. Then there is a Suslin line $(L^*, \prec)$ that is dense, unbounded and where no non-empty open interval in L^* is separable.

Proof. Let $(L, <)$ be a Suslin line. We define a relation $\sim$ on L by: $x \sim y$ if and only if $x = y$ or the open interval $(\min(x, y), \max(x, y))$ is separable, meaning that it contains a countable dense subset. It is straightforward to check that $\sim$ is an equivalence relation. Let $L^* = L/{\sim}$ be the set of all equivalence classes. Note that every $I \in L^*$ is convex, in the sense that if $x, y \in I$ and $x < y$, then $I \supseteq (x, y) = \{\ell \in L \mid x < \ell < y\}$. Another easy observation is that under $\sim$, two points $x, y \in L$ such that $x < y$ and $(x, y) = \emptyset$ get identified to the same point in L^*.

Next we define a linear ordering $\prec$ on L^* as follows: $I \prec J$ iff $I \neq J$ and there is $x \in I$ and $y \in J$ with $x < y$. It is easy to check that $(L^*, \prec)$ is indeed a linear ordering, and $I \prec J$ implies that $x < y$ for *any* $x \in I$ and $y \in J$.

We claim that every $I \in L^*$ is separable as a subset of L. Indeed, let $\mathcal{A}$ be a $\subseteq$-maximal collection of disjoint non-empty open intervals of the form $(x, y) \subseteq L$ with $x, y \in I$. Since $(L, <)$ is a Suslin line, it has the ccc, hence $\mathcal{A}$ is at most countable and we can write $\mathcal{A} = \{(x_n, y_n) \mid n \in \omega\}$. For each $n \in \omega$ we have that $x_n \in I$ and $y_n \in I$, i.e. $x_n \sim y_n$ which by definition means that (x_n, y_n) is separable, so for each $n \in \omega$ let $D_n \subseteq (x_n, y_n)$ be a countable subset dense in (x_n, y_n). Then $D \coloneqq \bigcup_{n\in\omega} D_n$ is dense in $\bigcup_{n\in\omega}(x_n, y_n)$, and D is countable. Let $z \in I$ be arbitrary. If z is not the first or last element of I, then there is a non-empty open interval $(x, y) \subseteq I$ with $x, y \in I$ and $z \in (x, y)$. By the maximality of $\mathcal{A}$, the interval (x, y) intersects (x_n, y_n) for some $n \in \omega$. This implies that z is an element of the closure of $\bigcup_{n\in\omega}(x_n, y_n)$, and since D is dense in $\bigcup_{n\in\omega}(x_n, y_n)$, we have that z is an element of the closure of D. Therefore D together with the first and last element of I (if I has a first or last element) is a countable dense subset of I.

We list the facts we will prove next:

(a) $(L^*, \prec)$ is dense;
(b) no non-empty open interval in L^* is separable;
(c) $(L^*, \prec)$ satisfies the ccc.

We first show (a). Towards a contradiction, assume we have $I, J \in L^*$ with $I \prec J$ but $\{K \in L^* \mid I \prec K \prec J\} = (I, J) = \emptyset$. Fix $x \in I$ and $y \in J$. The assumption $(I, J) = \emptyset$ implies that $(x, y) = ((x, y) \cap I) \cup ((x, y) \cap J) \subseteq I \cup J$, and $I \cup J$ is separable by the previous paragraph, so (x, y) is also separable and we get $x \sim y$, which is a contradiction to $I \prec J$.

Moving on, we will now prove (b). Say it does not hold, i.e. there are $I, J \in L^*$ with $I \prec J$ such that (I, J) is separable. Let $\{K_n \in (I, J) \mid 2 \leq n < \omega\}$ be dense in (I, J), and set $K_0 = I$ and $K_1 = J$. Using the previous claim (the fact that every element of L^* is separable as a subset of L) we define, for each $n \in \omega$, D_n to be a countable dense subset of K_n. Then we have that $D \coloneqq \bigcup_{n\in\omega} D_n$ is dense in $\bigcup_{n\in\omega} K_n$, but also $\bigcup_{n\in\omega} K_n$ is dense in $\bigcup\{K \in L^* \mid I \preceq K \preceq J\}$, so D is dense in $\bigcup\{K \in L^* \mid I \preceq K \preceq J\}$ and D is countable. Using the definition of $\sim$, this is a contradiction to the fact that $I \prec J$, i.e. to $I \neq J$.

To prove (c), assume that we have an uncountable collection $\mathcal{A} = \{(I_\alpha, J_\alpha) \mid \alpha \in \omega_1\}$ of pairwise disjoint non-empty open intervals in L^*. For each $\alpha \in \omega_1$ pick $x_\alpha \in I_\alpha$ and $y_\alpha \in J_\alpha$. Then $\mathcal{B} = \{(x_\alpha, y_\alpha) \mid \alpha \in \omega_1\}$ would be a collection of pairwise disjoint non-empty open intervals in L, which contradicts the fact that $(L, <)$ is ccc.

To recapitulate, we have shown that $(L^*, \prec)$ is a dense linearly ordered set. What's more, property (b) implies that $(L^*, \prec)$ is not separable. This, together with (c), shows that $(L^*, \prec)$ is indeed a Suslin line. To finish the proof of our Lemma, it remains to argue that we can get $(L^*, \prec)$ to also be unbounded. This is not hard, since if L^* happens to have a least element or a greatest element (or both), we can just remove it from L^*, and all the properties we have shown so far would remain true. □

Lemma 2.6. Let $(P, <)$ be a dense linearly ordered set without endpoints. Let $(C, \prec)$ be its Dedekind completion and $\varphi : P \to C$ the corresponding order-embedding, as constructed in Theorem 1.15. If $(P, <)$ is a Suslin line, then $(C, \prec)$ is also a Suslin line.

Proof. We first show that if $(P, <)$ is not separable, then $(C, \prec)$ is also not separable. Assume it is, so there is $D \subseteq C$ countable and dense in C. Then $(C, \prec)$ is a dense linearly ordered set without endpoints that is complete and separable, so by Theorem 1.18 it is isomorphic to $(\mathbb{R}, <_\mathbb{R})$. Then, since P is an infinite set and φ is an order-embedding, we know that $\varphi[P]$ is an infinite subset of C, and using the fact that infinite subsets of $\mathbb{R}$ always have a countable dense subset we have that also $\varphi[P]$ has a countable dense subset, which would give us a countable dense subset for P, contradicting the fact that P is not separable.

Now assume that $(P, <)$ satisfies the ccc, but $(C, \prec)$ does not. So there is a family $\mathcal{A}$ of pairwise disjoint open intervals in C which is uncountable. Since $\varphi[P]$ is dense in C, this would immediately give us an uncountable family of pairwise disjoint open intervals in P, contradicting the fact that $(P, <)$ satisfies the ccc. □

To finish this section we show that ¬SH implies that the product of ccc topological spaces is not always ccc.

Lemma 2.7. If $(L, <)$ is a Suslin line, then $L \times L$ with the product topology is not ccc.

Proof. Given $x, y \in L$ with $x < y$, let (x, y) denote the open interval $\{\ell \in L \mid x < \ell < y\}$ as usual. Notice that (x, y) could be empty even though $x < y$, since $(L, <)$ is not necessarily dense.

By recursion on $\alpha < \omega_1$, we will construct sequences $\langle a_\alpha \mid \alpha \in \omega_1\rangle$, $\langle b_\alpha \mid \alpha \in \omega_1\rangle$ and $\langle c_\alpha \mid \alpha \in \omega_1\rangle$ of elements of L so that, for every $\alpha \in \omega_1$, the following properties hold:

(i) $a_\alpha < b_\alpha < c_\alpha$;
(ii) $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$;
(iii) $\beta < \alpha \to b_\beta \notin (a_\alpha, c_\alpha)$.

First, assume that we have already constructed such three sequences as above. Then, for every $\alpha \in \omega_1$, the open set $U_\alpha := (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \subseteq L \times L$ is non-empty by (ii). Moreover, given $\beta < \alpha < \omega_1$, we have that $U_\beta \cap U_\alpha = \emptyset$, since, by (iii), we have that either $b_\beta \leq a_\alpha$, in which case $(a_\beta, b_\beta) \cap (a_\alpha, b_\alpha) = \emptyset$, or $b_\beta \geq c_\alpha$, in which case $(b_\beta, c_\beta) \cap (b_\alpha, c_\alpha) = \emptyset$. Therefore $\{U_\alpha \mid \alpha \in \omega_1\}$ is an uncountable collection of pairwise disjoint non-empty open subsets of $L \times L$, i.e. $L \times L$ does not satisfy the ccc.

To construct the sequences $\langle a_\alpha \mid \alpha \in \omega_1\rangle$, $\langle b_\alpha \mid \alpha \in \omega_1\rangle$ and $\langle c_\alpha \mid \alpha \in \omega_1\rangle$, let W be the set of all isolated points of L, i.e. $W := \{\ell \in L \mid \{\ell\} \text{ is open}\}$. Since $(L, <)$ satisfies the ccc, we have that W is at most countable. Now assume we have picked

$a_\beta, b_\beta, c_\beta$ for all $\beta \in \alpha$ and choose a_α and c_α such that $a_\alpha < c_\alpha$ and $(a_\alpha, c_\alpha) \neq \emptyset$ and $(a_\alpha, c_\alpha) \cap (W \cup \{b_\beta \mid \beta < \alpha\}) = \emptyset$. This can be done since the set $W \cup \{b_\beta \mid \beta < \alpha\}$ is countable and $(L, <)$ is not separable. Then, since (a_α, c_α) is non-empty and contains no isolated points, it must be infinite and hence we can choose $b_\alpha \in (a_\alpha, c_\alpha)$ such that $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$. □

3 Trees

Trees are combinatorial objects that abound in set theory. The purpose of this section is to show a connection between Suslin's Hypothesis and a certain type of trees called Suslin trees.

Definition 3.1 (Tree)**.** A *tree* is a partially ordered set $(T, <)$ such that, given any $x \in T$, the set $x{\downarrow} := \{y \in T \mid y < x\}$ of all predecessors of x is well-ordered by $<$. Moreover, we say that

- the *height* $\mathrm{ht}(x)$ of an element $x \in T$ is the order type of $x{\downarrow}$;
- if $\mathrm{ht}(x)$ is a successor ordinal, then x is called a *successor node*; otherwise x is called a *limit node*;
- the set $\mathcal{L}_\alpha(T) = \{x \in T \mid ht(x) = \alpha\}$ is the *αth level* of T;
- the least ordinal α such that $\mathcal{L}_\alpha(T) = \emptyset$ is called the *height* $\mathrm{ht}(T)$ of the tree T;
- T is *rooted* iff $|\mathcal{L}_0(T)| = 1$, in which case the *root* of T is the element $\mathbb{1}_T \in \mathcal{L}_0(T)$;
- we write $T^{(\alpha)}$ for $\{x \in T \mid ht(x) \in \alpha\}$, which is the same as the set $\bigcup_{\beta\in\alpha} \mathcal{L}_\beta(T)$;
- a *branch* in T is a maximal chain (i.e., a maximal linearly ordered subset) of T;
- the *length* $\ell(b)$ of a branch $b \subseteq T$ is the order type of b;
- an *antichain* in T is a subset $A \subseteq T$ such that any two distinct elements $x, y \in A$ are incomparable, i.e., neither $x < y$ nor $y < x$;
- $y \in T$ is an *immediate successor* of $x \in T$ iff $x < y$ and $\mathrm{ht}(y) = \mathrm{ht}(x) + 1$;

Example 3.2. Every well-ordered set $(W, <)$ is a tree. For this reason trees can be seen as generalizations of well-ordered sets, or generalizations of ordinals. The height of W is the order type of W, and the only branch is W itself.

Example 3.3. Let α be an ordinal number and X a nonempty set. We denote by $X^{<\alpha}$ the set of transfinite sequences of length less than α, i.e. $X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta = \bigcup_{\beta<\alpha} \{f : \beta \to X\}$. We define an ordering $<$ on $X^{<\alpha}$ by $f < g :\Longleftrightarrow f \subsetneq g$, so $f < g$ iff g extends f. It is easy to verify that $(X^{<\alpha}, <)$ is a tree of height α.

Given a tree $(T, <)$, any subset $X \subseteq T$ is a tree itself under the induced order, but the levels of the T and X need not coincide. For this reason we define subtrees as follows:

Definition 3.4 (Subtree)**.** Let $(T, <)$ be a tree. A subset $S \subseteq T$ is a *subtree* iff $s{\downarrow} \subseteq S$ for any $s \in S$, i.e. iff for any $s \in S$ and $x \in T$, $x < s$ implies $x \in S$.

Example 3.5. Let $(T, <)$ be a tree and $\alpha \in \mathrm{ht}(T) + 1$ an ordinal. Then the set $T^{(\alpha)} = \bigcup_{\beta\in\alpha} \mathcal{L}_\beta(T)$ is a subtree of T and $\mathrm{ht}(T^{(\alpha)}) = \alpha$.

Having stated the basic definitions regarding trees, we can now define what Suslin trees are and proceed to show the relationship between Suslin lines and Suslin trees.

Definition 3.6 (Suslin Tree)**.** A tree $(T, <)$ is called a *Suslin tree* if it has the following properties:

(i) $\mathrm{ht}(T) = \omega_1$;
(ii) every branch in T is at most countable;
(iii) every antichain in T is at most countable.

Definition 3.7 (Normal α-Tree)**.** Let $\alpha \leq \omega_1$ be an ordinal number. A tree $(T, <)$ is called a *normal α-tree* if it has height α and satisfies the following properties:
(iv) T is rooted;
(v) if $\beta \in \mathrm{ht}(T)$ is a limit ordinal and $x, y \in \mathcal{L}_\beta(T)$ and if $x{\downarrow} = y{\downarrow}$, then $x = y$;
(vi) for every $x \in T$ there is some $y > x$ at each higher level less than $\mathrm{ht}(T)$;
(vii) the set of immediate successors of x, $\mathrm{succ}_T(x)$, is countably infinite for every $x \in T$.
(viii) each level of T is at most countable.

Lemma 3.8. If there exists a Suslin tree, then there exists a normal Suslin tree.

Proof. Let $(T, <)$ be a Suslin tree. Notice that, by property (iii), every level of T is countable, so property (viii) doesn't need to be discussed in the context of a Suslin tree. For $x \in T$, let $T_x := \{y \in T \,|\, y \geq x\}$. Now consider the tree $T_1 = \{x \in T \,|\, T_x \text{ is uncountable}\}$, i.e. we are using the induced ordering on the subset T_1. If $x \in T_1$ and α is such that $\mathrm{ht}(x) < \alpha < \mathrm{ht}(T_1) = \omega_1$, then there must be some $y \in \mathcal{L}_\alpha(T_1)$ with $y > x$: otherwise $T_x = \bigcup_{y \in \mathcal{L}_\alpha(T)} (\{t \in T \,|\, x \leq t \leq y\} \cup T_y)$ would be countable. Hence T_1 satisfies (vi) and is clearly still a Suslin tree.

Now we define T_2 by inserting nodes to T_1 as follows: for every limit node $y \in T_1$ we add an extra node a_y such that $x < a_y \iff x < y$ and $a_y < x \iff y \leq x$. We can think of a_y as a node between $y\downarrow$ and y. So we have that $T_2 := T_1 \cup \{a_y \,|\, y \in T_1 \text{ and } y \text{ is a limit node}\}$, and since every level of T_1 is countable and there are only ω_1 countable limit ordinals, T_2 is still a Suslin tree. Moreover, T_2 still satisfies (vi) and now also (v).

We call a node a *branching point* if it has at least two immediate successors. Since T_2 has no uncountable chain (by the fact that it has no uncountable branch), and since T_2 satisfies (vi), there must be uncountably many branching points in T_2. Define T_3 to be the set of all branching points of T_2. By the previous observation, $\mathrm{ht}(T_3) = \omega_1$, so T_3 is still a Suslin tree and it is not hard to see that it still satisfies (v) and (vi).

Let $T_4 = \{x \in T_3 \,|\, x \text{ is a limit node}\}$. Since each node of T_3 has at least two immediate successors, keeping only limit nodes ensures that each node has infinitely many immediate successors. By (iii), sets of immediate successors are in particular countably infinite. So T_4 satisfies (vii) and is still a Suslin tree satisfying (v) and (vi).

Finally, let T_5 be T_4 but with the initial level forced to have only one element. So we just pick one of the elements in $\mathcal{L}_0(T_4)$ and set $\mathcal{L}_0(T_5)$ to be the set consisting of solely that node. The rest of the levels of T_5 are the excepted restrictions of the levels of T_4. None of the relevant properties of T_4 are broken, and moreover now T_5 also satisfies (iv). □

The next theorem is the main purpose of this section. It reduces Suslin's Problem to a purely combinatorial question regarding trees.

Theorem 3.9. There exists a Suslin line if and only if there exists a Suslin tree.

Proof. ($\Rightarrow$) Let $(S, <)$ be a Suslin line. Assume our line has the properties given by Lemma 2.5 and Lemma 2.6. We will construct a Suslin tree T from certain subsets of

S. Namely, given $a, b \in S$ with $a < b$, we call $[a, b] := \{s \in S \mid a \leq s \leq b\} \subseteq S$, a *non-degenerate closed interval* in S. The intervals will be ordered by reverse inclusion, i.e. if $I, J \subseteq S$ are two such intervals, then $I \prec J$ iff $J \subsetneq I$. This is clearly a partial ordering.

The construction of our tree T will be done by recursion on $\alpha < \omega_1$. Let $I_0 = [a_0, b_0]$ be an arbitrary non-degenerate closed interval. Assume that for some $\alpha \in \omega_1$ we have already constructed non-degenerate closed intervals $I_\beta = [a_\beta, b_\beta]$ for every $\beta \in \alpha$. Consider the set $C_\alpha := \{a_\beta \mid \beta \in \alpha\} \cup \{b_\beta \mid \beta \in \alpha\}$ consisting of the endpoints of the intervals I_β for $\beta \in \alpha$. Note that, since α is countable, $C_\alpha \subseteq S$ must also be countable. Since S is a Suslin line, no countable subset of S is dense in S. In particular, C_α is not dense in S, so there exists a non-degenerate closed interval $I_\alpha = [a_\alpha, b_\alpha]$ disjoint from C_α. Define $T = \{I_\alpha \mid \alpha \in \omega_1\}$, which is clearly uncountable, and partially ordered by $\prec$ as defined above. To say that $(T, \prec)$ is a tree, we still need to check that for an element of T, the set of its predecessors is well-ordered by $\prec$. If $\alpha \in \beta \in \omega_1$, then by the previous construction either $I_\alpha \prec I_\beta$ (meaning $I_\beta \subsetneq I_\alpha$) or I_α and I_β are incomparable (i.e. $I_\alpha \cap I_\beta = \emptyset$). Hence for every $\alpha \in \omega_1$ we have that $I_\alpha{\downarrow} = \{I \in T \mid I \prec I_\alpha\}$ is well-ordered by $\prec$ and $(T, \prec)$ is a tree.

To prove that $(T, \prec)$ is a Suslin tree, it remains to show that

(a) every antichain in T is at most countable;
(b) every branch in T is at most countable;
(c) $\mathrm{ht}(T) = \omega_1$.

For (a) note that, by construction, $I, J \in T$ are incomparable if and only if they are disjoint intervals of S. Therefore, since S satisfies the ccc, every antichain in T is at most countable.

For (b), we argue by contradiction: if $b = \{[a_\alpha, b_\alpha] \mid \alpha \in \omega_1\} \subseteq T$ is a branch of T of length ω_1, then the left endpoints of these intervals form an increasing sequence $\langle a_\alpha \mid \alpha \in \omega_1 \rangle$ of points of S. Then $\{(a_\alpha, a_{\alpha+1}) \mid \alpha \in \omega_1\}$ is an uncountable collection of nonempty pairwise disjoint open intervals in S, which contradicts the fact that S is ccc. So every branch of T must be at most countable, i.e. T satisfies (b).

To show (c): since all branches of T are at most countable, $\mathrm{ht}(T)$ is at most ω_1 and since every level of T is an antichain (and hence at most countable), and T is uncountable, $\mathrm{ht}(T)$ is at least ω_1. These two observations show (c), i.e. that $\mathrm{ht}(T) = \omega_1$.

($\Leftarrow$) Let $(T, \prec)$ be a Suslin tree. By Lemma 3.8, we can assume that $(T, \prec)$ is a normal Suslin tree, i.e. T satisfies properties (i) to (viii) of Definitions 3.6 and 3.7. We will construct a Suslin line $(S, <)$.

Let $S := \{b \subseteq T \mid b \text{ is a branch in } T\}$. Having defined S, we now proceed to construct an ordering $<$ on its elements, one which resembles a lexicographical ordering. By property (vii), every $x \in T$ has countably many immediate successors, so for each $x \in T$ we can fix a bijection $\sigma_x : \mathrm{succ}_T(x) \to \mathbb{Q}$. Let $a, b \in S$, with $a \neq b$, $a = \{a_\xi \mid \xi \in \alpha\}$ and $b = \{b_\xi \mid \xi \in \beta\}$. Note that $a \subseteq b$ or $b \subseteq$ is impossible, since branches are maximal chains. By property (v), if $a \neq b$ and γ is the least level at which a and b differ, then γ is a successor ordinal and a_γ and b_γ are both successors of the same node $c := a_{\gamma-1} = b_{\gamma-1}$, accordingly we define

$$a < b :\Longleftrightarrow \sigma_c(a_\gamma) <_{\mathbb{Q}} \sigma_c(b_\gamma)$$

To show that $(S, <)$ is a Suslin line, it remains to prove that

(a) S is linearly ordered by $<$;
(b) $(S, <)$ satisfies the countable chain condition;
(c) $(S, <)$ is not separable.

We first show (a). As before, let $a, b \in S$, with $a \neq b$, $a = \{a_\xi \mid \xi \in \alpha\}$ and $b = \{b_\xi \mid \xi \in \beta\}$. Again we denote by γ the least level at which a and b differ, and set $c := a_{\gamma-1} = b_{\gamma-1}$. Then since $a_\gamma \neq b_\gamma$ and σ_c is injective, we have that $\sigma_c(a_\gamma) \neq \sigma_c(b_\gamma)$, i.e. either $\sigma_c(a_\gamma) <_{\mathbb{Q}} \sigma_c(b_\gamma)$ or $\sigma_c(b_\gamma) <_{\mathbb{Q}} \sigma_c(a_\gamma)$. This proves that S is linearly ordered by $<$. Moreover, it is easy to see that $(S, <)$ is also dense and unbounded by using the density and unboundedness of $(\mathbb{Q}, <_{\mathbb{Q}})$.

For (b), define $I_x := \{b \in S \mid x \in b\} \subseteq S$ for each $x \in T$. Given (a, b) an arbitrary nonempty open interval in S, i.e. $(a, b) = \{s \in S \mid a < s < b\}$, we claim that there is $x \in T$ such that $I_x \subseteq (a, b)$. To see this, again let γ denote the least level at which a and b differ, and set $c := a_{\gamma-1} = b_{\gamma-1}$. Since $a < b$, we have that $\sigma_c(a_\gamma) <_{\mathbb{Q}} \sigma_c(b_\gamma)$. By the density of $<_{\mathbb{Q}}$, we can find $x \in \mathrm{succ}_T(c)$ such that $\sigma_c(a_\gamma) <_{\mathbb{Q}} \sigma_c(x) <_{\mathbb{Q}} \sigma_c(b_\gamma)$, hence we have $I_x \subseteq (a, b)$. Now notice that if I_x and I_y for $x \neq y$ are such that $I_x \cap I_y = \emptyset$, then neither $x \prec y$ nor $y \prec x$, otherwise we would have a branch in T which passes through both x and y.

Let $\mathcal{A}$ be a family of pairwise disjoint nonempty open intervals in S. Using the observations from the above paragraph, we can obtain an antichain in T with the same cardinality as $\mathcal{A}$. Since $(T, \prec)$ is a Suslin tree, we conclude that $\mathcal{A}$ is at most countable, i.e. $(S, <)$ satisfies the ccc.

Finally let $C \subseteq S$ be a countable subset of S, i.e. C is a countable set of branches of T. Let $\alpha := \sup\{\ell(b) \mid b \in C\}$. Note that since T is a Suslin tree, $\ell(b) \in \omega_1$ for any $b \in C$. This together with the fact that C is countable and ω_1 is regular ensures that $\alpha \in \omega_1$. Therefore we can let $x \in \mathcal{L}_\alpha(T)$. Since x has infinitely many immediate successors, we can choose $a, b \in S$ such that $a \neq b$ and both contain x. Without loss of generality, we can assume that $a < b$. Then the nonempty open interval (a, b) is disjoint from C, and thus C is not dense in S. This proves (c). □

Definition 3.10 (Aronszajn Tree)**.** A tree $(T, <)$ is called an *Aronszajn tree* if it has the following properties:
(i) $\mathrm{ht}(T) = \omega_1$;
(ii) every branch in T is at most countable;
(iii) every level in T is at most countable.

Suslin trees are in particular also Aronszajn trees. We have discussed that the existence of a Suslin line cannot be proven or disproven, i.e., by the previous theorem, the existence of a Suslin tree cannot be proven or disproven. In contrast, the next result shows that Aronszajn trees can always be found.

Theorem 3.11. There exists an Aronszajn tree.

Proof. Let $\omega^{<\omega_1}$ denote the set of all transfinite sequences of elements of ω of length less than ω_1, or in symbols $\omega^{<\omega_1} = \bigcup_{\alpha\in\omega_1} \omega^\alpha = \bigcup_{\alpha\in\omega_1} \{s : \alpha \to \omega\}$. We start by setting $T := \{s \in \omega^{<\omega_1} \mid s \text{ is injective}\}$ and ordering the elements of T by inclusion as in Example 3.3. It is easy to see that T is a subtree of $\omega^{<\omega_1}$, since restricting the domain of an injective function always yields an injective function. Moreover, T still has height ω_1, since for every $\alpha \in \omega_1$ we can construct an injective function from α into ω, by the fact that either $|\alpha| \in |\omega|$ or $|\alpha| = |\omega|$. Moreover, if b would be an uncountable branch in T, then $\bigcup b$ would be an injective function from ω_1 to ω, which is impossible. Therefore branches in T are at most countable.

The previous observations show that T satisfies (i) and (ii) of the definition of Aronszajn tree; T is, however, not an Aronszajn tree since, for example, there are uncountably many functions (in fact, $\mathfrak{c}$ many) from ω to ω, so $\mathcal{L}_\omega(T)$ is uncountable. We will fix this by constructing a subtree S of T which *is* Aronszajn.

First, for $\alpha \in \omega_1$ and given $s, t \in \omega^\alpha$ we define $s =^* t$ iff the set $\{\beta \in \alpha \mid s(\beta) \neq t(\beta)\}$ is finite. Now consider a transfinite sequence $\langle s_\alpha \mid \alpha \in \omega_1 \rangle$ satisfying the following properties for every $\alpha, \beta \in \omega_1$:

(i) $s_\alpha \in \omega^\alpha = \{t : \alpha \to \omega\}$;
(ii) s_α is injective;
(iii) $\alpha \in \beta \to s_\alpha =^* s_\beta \restriction \alpha$;
(iv) $\omega \setminus \mathrm{ran}(s_\alpha)$ is infinite.

Suppose that such a sequence exists. Then we can define $S := \bigcup_{\alpha\in\omega_1} \{t \in \mathcal{L}_\alpha(T) \mid t =^* s_\alpha\}$. Note that $S \subseteq T$. Property (iii) above implies that S is a subtree of T. The definition of S and properties (i) and (ii) above mean that $s_\alpha \in S$ for every $\alpha < \omega_1$, so the levels of S are nonempty, i.e. S also has height ω_1. Since $S \subseteq T$ and we are simply considering the ordering of T restricted to S, the property that every branch is at most countable is still satisfied, but now S also has no uncountable levels, since $\{t \in \omega^\alpha \mid t =^* s_\alpha\}$ is countable for every $\alpha \in \omega_1$. Hence S is an Aronszajn tree.

To finish the proof, it remains to construct the sequence $\langle s_\alpha \mid \alpha \in \omega_1 \rangle$ described above. We will do this by recursion on $\alpha < \omega_1$. Start with $s_0 = \emptyset$. For the successor steps of the construction, assume that s_α is given. Then, by (iv), it is always possible to pick some $n \in \omega \setminus \mathrm{ran}(s_\alpha)$ and set $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$. For the limit steps, suppose we have constructed s_α for every $\alpha < \lambda$ with $\lambda < \omega_1$ a limit ordinal. Fix $\langle \alpha_n \mid n \in \omega \rangle$ a strictly increasing sequence of ordinals satisfying $\sup_{n\in\omega}\alpha_n = \lambda$. Using this, we define a sequence of functions $\langle t_n : \alpha_n \to \omega \mid n \in \omega \rangle$ as follows: let $t_0 = s_{\alpha_0}$ and having constructed t_n, choose t_{n+1} such that is is injective, $t_{n+1} =^* s_{\alpha_{n+1}}$ and $t_{n+1}\restriction \alpha_n = t_n$. This is possible by condition (iii) above. Now let $t = \bigcup_{n\in\omega} t_n$; then $t : \gamma \to \omega$ and t is injective. If we would set $s_\gamma = t$, then (i), (ii) would hold for $\alpha = \gamma$, and (iii) would hold for every $\alpha \in \beta = \gamma$, which is promising. However, condition (iv) could fail. To fix this, we will remove $\aleph_0$ elements from $\mathrm{ran}(t)$. Let $s_\gamma(\alpha_n) = t(\alpha_{2n})$ for every $n \in \omega$ and $s_\gamma(\xi) = t(\xi)$ for $\xi \in \gamma \setminus \{\alpha_n \mid n \in \omega\}$. Then we have that $\omega \setminus \mathrm{ran}(s_\gamma) \supseteq \{t(\alpha_{2n+1}) \mid n \in \omega\}$, so condition (iv) is preserved, and we have constructed $\langle s_\alpha \mid \alpha \in \omega_1 \rangle$ satisfying (i), (ii), (iii) and (iv) as desired. □

4 Jensen's Diamond

This section introduces the combinatorial principle $\Diamond$. The main objective is to prove that under $\Diamond$ we can always construct a Suslin line, i.e. we show that $\Diamond \to \neg\mathrm{SH}$.

Definition 4.1. Let α be a limit ordinal. A subset $X \subseteq \alpha$ is called *bounded in* α iff $\sup(X) < \alpha$, and *unbounded in* α iff $\sup(X) = \alpha$.

Remark 4.2. The condition that the set X is unbounded in α can be restated as: for every $\beta < \alpha$ there is some $\gamma \in X$ such that $\gamma > \beta$.

Definition 4.3 (Closed Unbounded Set)**.** A set $C \subseteq \omega_1$ is *club* or *closed unbounded* if it satisfies the following properties:

(i) C is unbounded in ω_1, i.e. $\sup(C) = \omega_1$;
(ii) C is *closed*: for all limit $\alpha < \omega_1$, if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$.

Remark 4.4. Condition (ii) can be restated as: for every countable sequence $\langle c_n \,|\, n \in \omega \rangle$ of elements of C we have that $\sup\{c_n \,|\, n \in \omega\} \in C$.

Definition 4.5 (Stationary Set)**.** A set $S \subseteq \omega_1$ is said to be *stationary* iff for every closed unbounded set $C \subseteq \omega_1$ we have that $S \cap C \neq \emptyset$.

Lemma 4.6. Let $S \subseteq \omega_1$ be stationary. Then S is unbounded in ω_1.

Proof. Assume $S \subseteq \omega_1$ is stationary and consider the family of nonempty open intervals $\mathcal{A} = \{(\alpha, \omega_1) \subseteq \omega_1 \,|\, \alpha < \omega_1\}$. Each $a \in \mathcal{A}$ is closed unbounded, so $S \cap a \neq \emptyset$. Hence for every $\alpha < \omega_1$ we can find some $\beta \in S$ with $\beta > \alpha$. □

Definition 4.7 (Diamond Sequence)**.** A sequence of sets $\langle S_\alpha \,|\, \alpha < \omega_1 \rangle$ is called a $\diamondsuit$-*sequence* if it satisfies the following properties:
(i) $S_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$;
(ii) for every $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 \,|\, X \cap \alpha = S_\alpha\} \subseteq \omega_1$ is stationary.

Definition 4.8 (Jensen's $\diamondsuit$ Principle)**.** There exists a $\diamondsuit$-sequence.

Theorem 4.9 ($\diamondsuit \rightarrow CH$)**.** If $\diamondsuit$ holds, then $2^{\aleph_0} = \aleph_1$.

Proof. Let $\langle S_\alpha \,|\, \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. By definition, for every $X \subseteq \omega\ (\subseteq \omega_1)$ the set $\{\alpha \in \omega_1 \,|\, X \cap \alpha = S_\alpha\}$ is stationary. By Lemma 4.6, this set is unbounded, and so we have that for every such $X \subseteq \omega$ there exists an α with $\omega < \alpha < \omega_1$ satisfying $X = S_\alpha$. This induces a function $f : \mathcal{P}(\omega) \rightarrow \omega_1$, defined as $f(X) \coloneqq \min\{\omega < \alpha < \omega_1 \,|\, X = S_\alpha\}$. We claim that f is injective. Indeed, given $X, Y \subseteq \omega$, if $f(X) = f(Y)$ then $X = S_{f(X)} = S_{f(Y)} = Y$. Since we constructed an injective function from $\mathcal{P}(\omega)$ into ω_1, we conclude that $2^{\aleph_0} = \aleph_1$. □

Let $(T, <)$ be a tree and A an antichain in T. Then A is called a *maximal antichain* if there is no antichain B in T such that $A \subsetneq B$, hence every $x \in T$ is comparable with some $a \in A$. If A is a maximal antichain in T and $\tilde{T} \supseteq T$ is a tree extending T, then A is not necessarily maximal in $\tilde{T}$.

Lemma 4.10. Let $(T, <)$ be a normal ω_1-tree. Let A be a maximal antichain in T. Then the set $C \coloneqq \left\{\alpha \in \omega_1 \,|\, A \cap T^{(\alpha)} \text{ is a maximal antichain in } T^{(\alpha)}\right\}$ is closed unbounded.

Proof. Since A is a maximal antichain in T, for any $\alpha \in \omega_1$ the set $A \cap T^{(\alpha)}$ is an antichain in $T^{(\alpha)}$. We first show that C is unbounded. We construct a sequence $\langle \alpha_n \,|\, n \in \omega \rangle$ by recursion. Start with an arbitrary $\alpha_0 < \omega_1$. Then, since $T^{(\alpha_0)}$ is countable and every element of $T^{(\alpha_0)}$ is comparable to some element of A, there exists α_1 with $\alpha_0 < \alpha_1 < \omega_1$ such that every element of $T^{(\alpha_0)}$ is comparable with some element of $A \cap T^{(\alpha_1)}$. Then we find α_2 with $\alpha_1 < \alpha_2 < \omega_1$ such that every element of $T^{(\alpha_1)}$ is comparable with some element of $A \cap T^{(\alpha_2)}$, and so on. Setting $\alpha \coloneqq \sup\{\alpha_n \,|\, n \in \omega\} \in \omega_1$, we have that every element of $T^{(\alpha)}$ is comparable to some element of $A \cap T^{(\alpha)}$, therefore $A \cap T^{(\alpha)}$ is a maximal antichain in $T^{(\alpha)}$, i.e. $\alpha \in C$. Our choice of $\alpha_0 < \alpha$ was arbitrary, so we conclude that $C \subseteq \omega_1$ is unbounded.

Now we prove that C is closed. Let $\langle \alpha_n \,|\, n \in \omega \rangle$ be a countable sequence of elements of C. We want to show that $\alpha \coloneqq \{\alpha_n \,|\, n \in \omega\} \in C$. Any element of $T^{(\alpha)}$ belongs to $T^{(\alpha_n)}$ for some $n \in \omega$, so any element of $T^{(\alpha)}$ is comparable so some element of $A \cap T^{(\alpha_n)}$ for some $n \in \omega$. It follows that any element of $T^{(\alpha)}$ is comparable to some element of $A \cap T^{(\alpha)}$, which shows that $A \cap T^{(\alpha)}$ is a maximal antichain in $T^{(\alpha)}$, i.e. $\alpha \in C$. □

Lemma 4.11. Let $(T, <)$ be a tree where each node has at least two immediate successors. If T has no uncountable antichains then T has no uncountable branches.

Proof. Let $(T, <)$ be a tree where each node has at least two immediate successors. Let $\{x_\alpha \,|\, \alpha \in \omega_1\}$ be an uncountable chain in T, i.e. a linearly ordered subset of T. For each $\alpha \in \omega_1$, the set of immediate successors of x_α has at least two elements, so we can define $y_{\alpha+1}$ to be an immediate successor of x_α with $y_{\alpha+1} \neq x_{\alpha+1}$. Then $\{y_{\alpha+1} \,|\, \alpha \in \omega_1\}$ is an uncountable antichain in T. □

Theorem 4.12 ($\Diamond \to \neg$SH)**.** If $\Diamond$ holds, then Suslin lines exist.

Proof. We will use $\Diamond$ to construct a Suslin tree $(T, <)$. This implies the existence of a Suslin line, by Theorem 3.9.

We set $T = \omega_1$, i.e. the nodes of our tree are the countable ordinals. The construction will be done by recursion on the levels of T. Each $T^{(\alpha)} = \{x \in T \,|\, ht(x) \in \alpha\}$ will be some countable ordinal, i.e. an initial segment of $T = \omega_1$. So every time we add a node, it will be the least countable ordinal not yet used. Moreover, our construction will ensure that each $T^{(\alpha)}$ is a normal α-tree.

Let $\mathcal{L}_0(T) = \{0\} = T^{(1)}$. Now if α is a limit ordinal, then $T^{(\alpha)}$ will simply be the union of the trees $T^{(\beta)}$ for $\beta < \alpha$. If α is a successor ordinal, then $T^{(\alpha+1)}$ is obtained from $T^{(\alpha)}$ by adding $\aleph_0$ immediate successors to each node at level $\alpha - 1$. Recall that we always pick the new nodes in order, i.e. in such a way that each $T^{(\alpha)}$ for $\alpha < \omega_1$ is an initial segment of ω_1. It is easy to check that the steps of the construction described in this paragraph preserve the property that each $T^{(\alpha)}$ is an α-tree, in particular property (viii) is preserved because we are either taking a countable union of countable sets or adding countable successors to each node in a countable level.

We still need to describe the construction of $T^{(\alpha+1)}$ in the case that α is a limit ordinal. Let $\langle S_\alpha \,|\, \alpha \in \omega_1 \rangle$ be a $\Diamond$-sequence. Recall that $S_\alpha \subseteq \alpha$, and, moreover, $\alpha \subseteq T^{(\alpha)}$. If S_α is a maximal antichain in $T^{(\alpha)}$, then for each $x \in T^{(\alpha)}$ there is $a \in S_\alpha$ such that either $x < a$ or $a < x$. Since $T^{(\alpha)}$ is a normal α-tree, we can choose a branch b_x of height α that contains both x and a. Hence for each $x \in T$ we add one node at level α to lie above the elements of b_x. This means that S_α is still a maximal antichain in $T^{(\alpha+1)}$, and the construction ensures that $T^{(\alpha+1)}$ is a normal $(\alpha+1)$-tree. On the other hand, if S_α is not a maximal antichain in $T^{(\alpha)}$, we can extend $T^{(\alpha)}$ in any way such that $T^{(\alpha+1)}$ is a normal $(\alpha+1)$-tree, i.e. we simply choose for each $x \in T$ a branch b_x of height α and add one node at level α to lie above the elements of b_x.

The construction of our tree $T = \bigcup_{\alpha \in \omega_1} T^{(\alpha)}$ ensures that T is a normal ω_1-tree. We now want to argue that T is a Suslin tree. By Lemma 4.11, we only need to show that T has no uncountable antichains. Let $A \subseteq T = \omega_1$ be a maximal antichain in T. By Lemma 4.10, the set $C := \left\{\alpha \in \omega_1 \,|\, A \cap T^{(\alpha)} \text{ is a maximal antichain in } T^{(\alpha)}\right\}$ is closed unbounded. Moreover, the set of $\alpha \in C$ such that $T^{(\alpha)} = \alpha$ is also closed unbounded[3]. Now since $\langle S_\alpha \,|\, \alpha \in \omega_1 \rangle$ is a $\Diamond$-sequence, we find a limit $\alpha \in \omega_1$ such that $T^{(\alpha)} = \alpha$ and $A \cap T^{(\alpha)}$ is a maximal antichain in $T^{(\alpha)}$ and $A \cap \alpha = S_\alpha$. It follows that we find limit $\alpha \in \omega_1$ such that $A \cap T^{(\alpha)} = S_\alpha$ and $A \cap T^{(\alpha)}$ is a maximal antichain in $T^{(\alpha)}$. Our construction of T now ensures that $A \cap T^{(\alpha)}$ remains a maximal antichain in T, so it must be that $A = A \cap T^{(\alpha)}$. By fact that $T^{(\alpha)}$ is countable, we have that A is countable, which is what we wanted to show. □

[3]Closure is clear. For unboundedness, note that $\alpha \subseteq T^{(\alpha)}$ for any $\alpha \in \omega_1$, and construct a sequence $\langle \alpha_n \,|\, n \in \omega \rangle$ of elements of C such that $T^{(\alpha_n)} \subseteq \alpha_{n+1}$. This can be done since C is club, and we get equality $T^{(\alpha)} = \alpha$ at the supremum of the sequence.

5 Gödel's Constructible Universe

The aim of this section is to present Gödel's class L, also known as Gödel's Constructible Universe, and show that $\diamondsuit$ holds in L. This completes the proof of the non-provability of SH, since $\diamondsuit \to \neg$SH as we saw in Theorem 4.12. To do this, we will of course have to show that all the ZFC axioms hold in L. The proof that the Generalized Continuum Hypothesis holds in L will also be presented.

Definition 5.1. Let A be a set. We say that $X \subseteq A$ is *definable* over the model $(A, \in)$ iff there exists a formula φ in the language of set theory $\mathcal{L} = \{\in\}$ and some $a_1, \ldots, a_n \in A$ such that $X = \{x \in A \mid (A, \in) \models \varphi[x, a_1, \ldots, a_n]\}$. Then let $\text{def}(A) := \{X \subseteq A \mid X \text{ is definable over } (A, \in)\}$, i.e. $\text{def}(A)$ is the set of all subsets of A that are definable over $(A, \in)$.

Lemma 5.2. Let A be a set. Then we have that
(a) $A \in \text{def}(A)$;
(b) $\text{def}(A) \subseteq \mathcal{P}(A)$;
(c) A transitive $\to A \subseteq \text{def}(A)$;
(d) every finite subset of A is in $\text{def}(A)$;
(e) assuming AC, $|\text{def}(A)| = |A|$ whenever $|A| \geq \omega$.

Proof. (a) In Definition 5.1, take φ to be a tautology.
(b) Clear, since the elements of $\text{def}(A)$ are subsets of A, i.e. elements of $\mathcal{P}(A)$.
(c) Since A is transitive, for $a \in A$ we have that $a = \{x \in A \mid x \in a\} \in \text{def}(A)$.
(d) Let $a_1, \ldots, a_n \in A$ for some $n \in \omega$. We want to show that $\{a_1, \ldots, a_n\} \in \text{def}(A)$. This is immediate if we let φ in Definition 5.1 to be the formula $x = y_1 \vee \cdots \vee x = y_n$.
(e) Assume AC and $|A| \geq \omega$. On the one hand we have that $|\text{def}(A)| \geq |A|$, since $\{a\} \in \text{def}(A)$ for every $a \in A$ by part (d). On the other hand it is also true that $|\text{def}(A)| \leq |A|$, since there are only countably many formulas (since formulas are finite sets of symbols) and only countably many finite subsets of A to be used as parameters in a given formula. □

We can now define L. The definition is identical to the definition of the von Neumann hierarchy of sets $V(\alpha)$, except that we replace the power set operation '$\mathcal{P}$' by 'def'.

Definition 5.3. Define $L(\alpha)$ via transfinite recursion on $\alpha \in ON$ as:
- $L(0) = \emptyset$;
- $L(\alpha + 1) = \text{def}(L(\alpha))$;
- $L(\lambda) = \bigcup_{\alpha<\lambda} L(\alpha)$ when λ is a limit ordinal.

Then set $L = \bigcup_{\alpha \in ON} L(\alpha)$. We call the proper class L the *constructible universe*.

Below we show some basic properties of L.

Lemma 5.4. For every ordinal α we have that
(a) $L(\alpha) \subseteq V(\alpha)$;
(b) $L(\alpha)$ is transitive;
(c) $\forall \beta \geq \alpha (L(\alpha) \subseteq L(\beta))$;
(d) $L(\alpha) \cap ON = \alpha$, so $ON \subseteq L$;
(e) $L(\alpha) \in L(\alpha + 1)$, which implies $L(\alpha) \in L$;
(f) every finite subset of $L(\alpha)$ is in $L(\alpha + 1)$.

Proof. (a) Follows easily by transfinite induction, using the fact that $\mathrm{def}(A) \subseteq \mathcal{P}(A)$, shown in Lemma 5.2 (b).

(b) Use transfinite induction on α. The steps where α is 0 or a limit ordinal are easy. Now assume $L(\alpha)$ is transitive and note that $L(\alpha) \subseteq L(\alpha + 1)$, since any $a \in L(\alpha)$ can be written as $a = \{x \in L(\alpha) \mid x \in a\} \in L(\alpha + 1)$. Then, since $L(\alpha + 1) \subseteq \mathcal{P}(L(\alpha))$, we have that $x \in L(\alpha + 1) \to x \subseteq L(\alpha) \subseteq L(\alpha + 1)$, which concludes the proof.

(c) We will prove the statement for an arbitrary fixed α by transfinite induction on β. The cases where β is α or β is a limit ordinal are immediate, and the remaining case is covered by $L(\alpha) \subseteq L(\beta) \subseteq L(\beta + 1)$, where the second inclusion was proven in part (b).

(d) Once again we prove the statement using transfinite induction on α. The step $\alpha = 0$ is trivial, and the step where α is a limit is also easy. Therefore we assume that $L(\alpha) \cap ON = \alpha$, and attempt to show $L(\alpha+1) \cap ON = \alpha+1$. But $L(\alpha) \subseteq L(\alpha+1)$ using part (c), and $L(\alpha + 1) = \mathrm{def}(L(\alpha)) \subseteq \mathcal{P}(L(\alpha))$ using Lemma 5.2 (b), so $\alpha = L(\alpha) \cap ON \subseteq L(\alpha+1) \cap ON \subseteq V(\alpha+1) \cap ON = \alpha \cup \{\alpha\}$, which means that we only need to show that $\alpha \in L(\alpha+1)$. Using the fact that being an ordinal can be written as a Δ_0 formula and part (b) which says that $L(\alpha)$ is transitive, we have that $\alpha = L(\alpha) \cap ON = \{x \in L(\alpha) \mid x \text{ is an ordinal}\} = \left\{x \in L(\alpha) \mid (x \text{ is an ordinal})^{L(\alpha)}\right\} \in \mathrm{def}(L(\alpha)) = L(\alpha + 1)$.

(e) $L(\alpha) = \left\{x \in L(\alpha) \mid (x = x)^{L(\alpha)}\right\} \in \mathrm{def}(L(\alpha)) = L(\alpha + 1)$.

(f) This follows directly from Lemma 5.2 (d). □

Given $x \in L$, by the definition of L it is clear that the least α such that $x \in L(\alpha)$ is a successor ordinal. This justifies the following definition.

Definition 5.5. Let $x \in L$. Define $\rho(x)$, the *L-rank* of x, as the least ordinal β such that $x \in L(\beta + 1)$.

Lemma 5.6. For any ordinal α the following properties hold:

(a) $L(\alpha) = \{x \in L \mid \rho(x) < \alpha\}$;

(b) $L(\alpha + 1) \setminus L(\alpha) = \{x \in L \mid \rho(x) = \alpha\}$;

(c) $\rho(L(\alpha)) = \rho(\alpha) = \alpha$.

Proof. (a) For $x \in L$, $\rho(x) < \alpha$ iff $\exists \beta < \alpha (x \in L(\beta + 1))$ iff $x \in L(\alpha)$.

(b) Follows easily from (a).

(c) $L(\alpha) \notin L(\alpha)$ is clear, and in the previous lemma we showed that $L(\alpha) \in L(\alpha + 1)$, hence $\rho(L(\alpha)) = \alpha$. Also by the previous lemma, $\alpha \notin L(\alpha)$ and $\alpha \in L(\alpha + 1)$, i.e. $\rho(\alpha) = \alpha$. □

Lemma 5.7.

(a) $|L(\alpha)| = |\alpha|$ for every $\alpha \geq \omega$;

(b) $L(\alpha) = V(\alpha)$ for every $\alpha \leq \omega$;

(c) $L(\omega + 1) \subsetneq V(\omega + 1)$.

Proof. (a) By Lemma 5.4 (d) we have that $\alpha \subseteq L(\alpha)$, so $|L(\alpha)| \geq |\alpha|$. We prove $|L(\alpha)| = |\alpha|$ by transfinite induction on $\alpha \geq \omega$. If $\alpha = \omega$, we just need to show that $L(\omega)$ is countable, but this is clear since $L(\omega)$ is a countable union of finite sets. If $|L(\alpha)| = |\alpha|$, then $|L(\alpha + 1)| = |L(\alpha)| = |\alpha| = |\alpha + 1|$ using 5.2 (e), which covers successor steps. Finally if $|L(\alpha)| = |\alpha|$ for all α with $\omega \leq \alpha < \beta$ with β a limit ordinal, then $|L(\alpha)| \leq |\beta|$ for all such α, therefore we have that $L(\beta)$ is a union of $|\beta|$-many sets of cardinality $\leq |\beta|$, so $|L(\beta)| \leq |\beta|$ using AC, hence $|L(\beta)| = |\beta|$.

(b) Using Lemma 5.4 (f), it is immediate to prove $L(n) = V(n)$ for any $n \in \omega$ using induction on n. Then $L(\omega) = V(\omega)$ is clear since $L(\omega) = \bigcup_{n\in\omega} L(n)$ and $V(\omega) = \bigcup_{n\in\omega} V(n)$.

(c) Note that $|V(\omega+1)| = |\mathcal{P}(V(\omega))|$ which is uncountable, whereas $L(\omega+1) = \mathrm{def}(L(\omega))$ is countable using (a) and Lemma 5.2 (e). □

Theorem 5.8. All axioms of ZF hold in L.

For proving Theorem 5.8, we will make use of two results, which we state here without further discussion. For detailed proofs, see [5].

Lemma 5.9. Suppose that M is a transitive class such that the Comprehension Axiom holds in M, and assume that for every subset $x \subseteq M$, there is a set $y \in M$ such that $x \subseteq y$. Then all the ZF axioms hold in M.

Theorem 5.10 (Reflection)**.** Let $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ be any list of formulas for the language $\mathcal{L} = \{\in\}$. Assume that B is a non-empty class and $A(\xi)$ is a set for each $\xi \in ON$, and assume that:

(i) $\xi < \eta \to A(\xi) \subseteq A(\eta)$.

(ii) $A(\eta) = \bigcup_{\xi<\eta} A(\xi)$ for limit η.

(iii) $B = \bigcup_{\xi\in ON} A(\xi)$.

Then $\forall\xi\exists\eta > \xi(A(\eta) \neq \emptyset \wedge \bigwedge_{i<n}(A(\eta) \preccurlyeq_{\varphi_i} B) \wedge \eta$ is a limit ordinal).

Proof of Theorem 5.8. In Lemma 5.4 (b) we showed that each $L(\alpha)$ is transitive, so we know that L is a transitive class. By Lemma 5.9, we only need to verify that the Comprehension Axiom holds in L, and that for every subset $x \subseteq L$ there is a $y \in L$ such that $x \subseteq y$. This last condition is straightforward to prove, since given $x \subseteq L$ we can let $y = L(\alpha)$ for $\alpha = \sup\{\rho(z) + 1 \,|\, z \in x\}$.

Now we show that Comprehension Axiom holds in L. Fix a formula φ without y among its free variables. The formula φ may have x, z and also possibly other variables $v_0, \ldots, v_{n-1}$ free, therefore we write it as $\varphi(x, z, v_0, \ldots, v_{n-1})$. The Comprehension Axiom reads in this case as:

$$\forall z, v_0, \ldots, v_{n-1} \in L\, \exists y \in L\, \forall x \in L(x \in y \;\leftrightarrow\; x \in z \wedge \varphi^L(x, z, \vec{v}))$$

We begin by fixing arbitrary $z, v_0, \ldots, v_{n-1} \in L$, and let $y = \left\{x \in z \,|\, \varphi^L(x, z, \vec{v})\right\}$. The task is now to verify that $y \in L$. Since $z, v_0, \ldots, v_{n-1} \in L$, we can fix α such that $z, v_0, \ldots, v_{n-1} \in L(\alpha)$. By Theorem 5.10 (Reflection), there is $\beta > \alpha$ such that $L(\beta) \preccurlyeq_\varphi L$. Then $y = \left\{x \in L(\beta) \,|\, \psi^{L(\beta)}(x, z, \vec{v})\right\} \in \mathrm{def}(L(\beta)) = L(\beta+1) \subseteq L$, where $\psi(x, z, \vec{v})$ is the formula $\varphi(x, z, \vec{v}) \wedge x \in z$, so $y \in L$. □

Definition 5.11. The *Axiom of Constructibility*, denoted $V = L$, is the statement that every set is constructible, i.e. a member of L. In symbols: $\forall x\exists\alpha(x \in L(\alpha))$.

Lemma 5.12. The function $\alpha \mapsto L(\alpha)$ is absolute for transitive $M \models ZF \setminus (Pow)$.

Proof sketch. It is a basic fact that the notion of ordinal is absolute for transitive models of $ZF \setminus (Pow)$. Moreover, it can be shown that recursively defined notions such as "φ is an $\mathcal{L}$-formula" and "$\mathfrak{A} \models \varphi$" are absolute for transitive models of $ZF \setminus (Pow)$ (see, e.g., [5, pp. 124–125]). Therefore 'def(A)' is absolute.

The theorem of transfinite recursion only requires $ZF \setminus (Pow)$ to work, and this suffices for the definition of $L(\alpha)$. So $(L(\alpha))^M$ is defined for ordinals $\alpha \in M$, and the equality $(L(\alpha))^M = L(\alpha)$ can then be proven by transfinite induction using the absoluteness of def(A) discussed above. □

Corollary 5.13. The Axiom of Constructibility holds in L.

Proof. We need to check $(V = L)^L$, i.e. $(\forall x \exists \alpha (x \in L(\alpha)))^L$, which is the same as $\forall x \in L \exists \alpha \in L (x \in L(\alpha))^L$. Now using, from Lemma 5.4 (d), that $ON \subseteq L$, the absoluteness of being an ordinal and Lemma 5.12 this reduces to $\forall x \in L \exists \alpha \in ON (x \in L(\alpha))$, which is clear from $L = \bigcup_{\alpha \in ON} L(\alpha)$. □

Definition 5.14. Let M be a transitive set. Define $\theta(M) \coloneqq M \cap ON$, i.e. $\theta(M)$ is the set of ordinals in M, or, equivalently, the first ordinal not in M.

Remark 5.15. Note that $\theta(M)$ is always a limit ordinal whenever M is a model of $ZF \setminus (Pow)$, since $\alpha \in M \to (\alpha + 1) = \alpha \cup \{\alpha\} \in M$.

Lemma 5.16. Let M be a transitive set. Assume that M is a model of $ZF \setminus (Pow)$. Then M is a model of $V = L$ if and only if $M = L(\theta(M))$.

Proof. Let M be a transitive set with $M \models ZF \setminus (Pow)$. Let $\gamma = \theta(M)$. By Lemma 5.12, we have that $L(\alpha) \in M$ for every $\alpha < \gamma$. Also recall from Remark 5.15 that γ is a limit ordinal, so $L(\gamma) = \bigcup_{\alpha < \gamma} L(\alpha)$. This two observations, together with the transitivity of M, imply that $L(\gamma) \subseteq M$.

The statement $(V = L)^M$, which is $(\forall x \exists \alpha (x \in L(\alpha)))^M$, reduces in this case to $\forall x \in M \exists \alpha \in \gamma (x \in L(\alpha))$, and this is equivalent to $M \subseteq L(\gamma) = \bigcup_{\alpha < \gamma} L(\alpha)$, i.e. equivalent to $M = L(\gamma)$. Therefore we have shown that $(V = L)^M$ is equivalent to $M = L(\gamma) = L(\theta(M))$. □

Theorem 5.17. There is a relation $<_L$ that well-orders L. Therefore, assuming $V = L$, the relation $<_L$ well-orders V and AC holds.

Proof sketch. For a complete formal proof see [4, pp. 188–190] or [5, pp. 139–140]. We sketch here the main ideas involved in the construction of $<_L$.

First, we argue why def(A) can be well-ordered, given a well-order $<_A$ of A. Using this fact, by recursion we piece together a well-order of $L(\alpha)$, for each α. Finally we use these to construct a well-order on L.

Recall, from Definition 5.1, that $X \in \text{def}(A)$ iff $X = \{x \in A \mid (A, \in) \models \varphi[x, a_1, \ldots, a_n]\}$ for a formula φ and some $a_1, \ldots, a_n \in A$. First, notice that there are only countably many possible formulas φ in the language of set theory $\{\in\}$, and we can well-order them using the standard Gödel numbering. What's more, the set $A^{<\omega}$ of all possible tuples of parameters from A can be well-ordered, assuming $<_A$ well-orders A, as follows: for $(a_1, \ldots, a_n) \in A^{<\omega}$ and $(a'_1, \ldots, a'_m) \in A^{<\omega}$, let $(a_1, \ldots, a_n) < (a'_1, \ldots, a'_m)$ iff $n < m$ or, $n = m$ and $(a_1, \ldots, a_n) <_{\text{lex}} (a'_1, \ldots, a'_n)$, where $<_{\text{lex}}$ is the lexicographic ordering based on $<_A$.

Now given $X, Y \in \text{def}(A)$, we can let φ_X and φ_Y be the least formulas, in the above sense, such that there are some parameters witnessing $X, Y \in \text{def}(A)$. Having fixed φ_X and φ_Y, let $\vec{a}_X$ and $\vec{a}_Y$ be the least parameters, in the above sense, witnessing

$X, Y \in \mathrm{def}(A)$ for φ_X and φ_Y. Then we can give a well-ordering for def(A) as follows: $X <_{\mathrm{def}(A)} Y$ iff $\varphi_X < \varphi_Y$ or, $\varphi_X = \varphi_Y$ and $\vec{a}_X < \vec{b}_X$.

Proceeding with the construction we can now define, by recursion on α, a well-ordering $\lhd_\alpha \subseteq L(\alpha) \times L(\alpha)$ as follows: $x \lhd_\alpha y$ iff $\rho(x) < \rho(y)$ or, $\rho(x) = \rho(y)$ and $x <_{\mathrm{def}(L(\rho(x)))} y$. Finally we define the relation $<_L$ on L by: $x <_L y$ iff $\rho(x) < \rho(y)$ or, $\rho(x) = \rho(y)$ and $x \lhd_{\rho(x)+1} y$. □

The above argument actually shows that $V = L$ implies *global choice*, i.e. there is one relation $<_L$ that well-orders everything.

Lemma 5.18. Let κ be a regular uncountable cardinal. Then $ZF \setminus (Pow)$ and $V = L$ hold in $L(\kappa)$.

Proof sketch. That V=L holds is a consequence of Lemma 5.16. Extensionality, Foundation, Pairing, Union and Infinity are not hard to show, and we skip them here. For a proof see [5]. For Comprehension, the proof is almost the same as the argument used in Theorem 5.8, but replacing L with $L(\kappa)$, and with α and β now ranging over κ instead of over ON. Also, instead of using the Reflection Theorem, a "set version" of it is used, where ON is replaced by the regular cardinal κ. We show Replacement in detail below.

Replacement: Let $A \in L(\kappa)$ and suppose that

$$\forall x \in L(\kappa)(x \in A \to \exists! y \in L(\kappa)\, \varphi^{L(\kappa)}(x, y)).$$

We need to produce $B \in L(\kappa)$ such that $(\forall x \in A\, \exists y \in B\, \varphi(x, y))^{L(\kappa)}$. Say $A \in L(\alpha)$ for $\alpha < \kappa$. Then, using Lemma 5.7 (a), $|A| \leq |L(\alpha)| = |\alpha| < \kappa$. Write f for the function with $\mathrm{dom}(f) = A$ such that $f(x)$ is the unique $y \in L(\kappa)$ such that $\varphi^{L(\kappa)}(x, y)$. Then $\rho(f(x)) < \kappa$ for each $x \in A$, so $\beta := \sup\{\rho(f(x)) + 1 \mid x \in A\} < \kappa$ since κ is regular and $|A| < \kappa$. We can now let $B = L(\beta)$, and we get $B \in L(\kappa)$ because $\rho(B) = \beta + 1 < \kappa$. □

Lemma 5.19. Assume $V = L$. Then $L(\kappa) = H(\kappa)$ for all cardinals $\kappa \geq \omega$.

Proof. For any cardinal κ, recall that $H(\kappa)$ denotes the set of all x such that $|\mathrm{trcl}(x)| < \kappa$. In Lemma 5.7 (b) we showed that $L(\omega) = V(\omega)$, and it is a standard fact that $H(\omega) = V(\omega)$, so we only need to consider $\kappa > \omega$.

We first show that $L(\kappa) \subseteq H(\kappa)$ for all cardinals κ. Let $x \in L(\kappa)$ for some cardinal $\kappa > \omega$. Since κ is a cardinal, we can fix α with $x \in L(\alpha)$ and $\omega \leq \alpha < \kappa$. Then $\mathrm{trcl}(x) \subseteq L(\alpha)$ since $L(\alpha)$ is transitive, so $|\mathrm{trcl}(x)| \leq |L(\alpha)| = |\alpha| < \kappa$ and we conclude that $x \in H(\kappa)$.

Note that, in order to show $L(\kappa) = H(\kappa)$, it suffices to only consider successor cardinals of the form λ^+ for some infinite cardinal λ, since for an uncountable limit cardinal κ we have $L(\kappa) = \bigcup_{\lambda<\kappa} L(\lambda^+) = \bigcup_{\lambda<\kappa} H(\lambda^+) = H(\kappa)$.

By the above remarks, we are left with showing $H(\lambda^+) \subseteq L(\lambda^+)$ for all infinite cardinals λ. Let $b \in H(\lambda^+)$ and let $T = \mathrm{trcl}(\{b\})$. So we have $b \in T$ and $|T| \leq \lambda$. Since we are assuming $V = L$, we can consider $\rho(T)$ and fix some regular uncountable cardinal $\gamma > \rho(T)$. Then, since $T \in L(\gamma)$ and $L(\gamma)$ is transitive, $T \subseteq L(\gamma)$ holds. Also, using Lemma 5.18, $L(\gamma) \models ZF \setminus (Pow) + V = L$. By the Downward Löwenheim-Skolem Theorem and the fact that $|T| \leq \lambda$, we can fix A such that $A \preccurlyeq L(\gamma)$ and $T \subseteq A$ and $|A| \leq \lambda$. Then also $A \models ZF \setminus (Pow) + V = L$ since $A \preccurlyeq L(\gamma)$.

Now let mos be the Mostowski collapsing function on A, and let $B = \mathrm{ran}(\mathrm{mos})$ the collapse of A. Then mos is an isomorphism from $(A, \in)$ to $(B, \in)$ and B is transitive, and $\text{mos}(x) = x$ for all $x \in T$ (since T is transitive). Hence $b = \text{mos}(b) \in B$ since $b \in T$. Moreover, since mos is an isomorphism, $B \models ZF \setminus (Pow) + V = L$. This yields, by Lemma 5.16 and the transitivity of B, that $B = L(\beta)$ for $\beta = \theta(B)$. Then, using Lemma 5.7 (a), $|\beta| = |L(\beta)| = |B| = |A| \leq \lambda$, so $\beta < \lambda^+$, which implies that $b \in B = L(\beta) \subseteq L(\lambda^+)$. □

Theorem 5.20. Assume $V = L$. Then GCH holds.

Proof. Assume $V = L$. By Theorem 5.17, AC holds. Let λ be an infinite cardinal. The inclusion $\mathcal{P}(\lambda) \subseteq H(\lambda^+)$ is clear from the definitions. By Lemma 5.19, $H(\lambda^+) = L(\lambda^+)$, so $\mathcal{P}(\lambda) \subseteq L(\lambda^+)$ and we get that $2^\lambda \leq |L(\lambda^+)| = \lambda^+$ using Lemma 5.7 (a), which required AC. Therefore $2^\lambda = \lambda^+$, since $2^\lambda \geq \lambda^+$ by Cantor's diagonal argument. □

Our aim is now to show that under $V = L$ the Diamond Principle is true. For this purpose we state the following Lemma. For a proof see [5].

Lemma 5.21. Let γ be an uncountable cardinal, and let M be countable with $M \preccurlyeq H(\gamma)$. Then the following statements are true:
(a) If $a \in M$ and a is countable, then $a \subseteq M$.
(b) $M \cap \omega_1$ is a countable limit ordinal.
(c) If $\gamma = \omega_1$, then M is transitive.
(d) If $\gamma > \omega_1$, then $\omega_1 \in M$ and $\omega_1 \subsetneq M$, and if also $\beta = M \cap \omega_1$ and mos is the Mostowski isomorphism from M onto a transitive T, then $\text{mos}(\omega_1) = \beta$ and $\text{mos}(\xi) = \xi$ for all $\xi < \beta$. Moreover, $T \models ZFC \setminus (Pow)$ if γ is regular, and $\beta = (\omega_1)^T$.

Theorem 5.22. Assume $V = L$. Then $\diamondsuit$ is true.

Proof. Assume $V = L$ throughout. Recall, from Theorem 5.17, that there is a relation $<_L$ that well-orders L. Using similar methods to the ones used in the proof of Lemma 5.12, it can be seen that $<_L$ is absolute for transitive models of $ZF \setminus (Pow) + V = L$.

By recursion we construct a sequence of pairs (S_α, C_α). For α a limit ordinal, let

$$P(\alpha, S, C) \iff S \subseteq \alpha \wedge C \subseteq \alpha \wedge C \text{ is club in } \alpha \wedge \neg \exists \xi \in C (S \cap \xi = S_\xi)$$

and let (S_α, C_α) be the $<_L$-least pair such that $P(\alpha, S, C)$. If α is not a limit ordinal, or if no such pair exists, then let $S_\alpha = C_\alpha = \emptyset$. Here we are using a generalized version of Definition 4.3 of club set, were we replace ω_1 for an arbitrary limit ordinal. Note that (S_α, C_α) is defined for all ordinals α, and the definition is absolute for transitive models of $ZF \setminus (Pow) + V = L$.

We will show that $\langle S_\alpha | \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. Assume it is not. By Definition 4.5 and Definition 4.7, this is equivalent to the existence of sets S, C such that $P(\omega_1, S, C)$ holds. Therefore $P(\omega_1, S_{\omega_1}, C_{\omega_1})$ holds, by the construction of (S_α, C_α).

Recall, from Lemma 5.19, that under $V = L$ we have that $L(\kappa) = H(\kappa)$ for all cardinals $\kappa \geq \omega$, so $L(\omega_2) = H(\omega_2)$ and using the Downward Löwenheim-Skolem Theorem we can get a countable $M \preccurlyeq H(\omega_2) = L(\omega_2)$. Now let $\beta = M \cap \omega_1$, which is some countable limit ordinal by Lemma 5.21 (b). Let mos be the Mostowski isomorphism from M onto some transitive model T. Then, using Lemma 5.21 (c), $\text{mos}(\omega_1) = \beta$, $\text{mos}(\xi) = \xi$ for all $\xi < \beta$, and $\beta = (\omega_1)^T$. Also, combining $M \preccurlyeq H(\omega_2) = L(\omega_2)$ with Lemma 5.18 and the regularity of ω_2, we have that T is a countable transitive model of $ZF \setminus (Pow)$ and of $V = L$. Therefore, by Lemma 5.16, $T = L(\gamma)$ for some countable γ.

By the absoluteness of $<_L$ and of $P(\alpha, S, C)$, and by $M \preccurlyeq L(\omega_2) = H(\omega_2)$ we have that, for each $\alpha \in M$, $S_\alpha \in M$ and $C_\alpha \in M$ and $\text{mos}(S_\alpha) = S_{\text{mos}(\alpha)}$ and $\text{mos}(C_\alpha) = C_{\text{mos}(\alpha)}$. So $S_{\omega_1} \in M$ and $C_{\omega_1} \in M$ and $\text{mos}(S_{\omega_1}) = S_\beta$ and $\text{mos}(C_{\omega_1}) = C_\beta$. By the definition of mos, we also have $\text{mos}(S_{\omega_1}) = \{\text{mos}(\xi) \mid \xi \in S_{\omega_1} \cap M\} = \{\xi \mid \xi \in S_{\omega_1} \cap M\} = S_{\omega_1} \cap \beta$. Therefore $S_\beta = S_{\omega_1} \cap \beta$. Analogously, $\text{mos}(C_{\omega_1}) = C_\beta = C_{\omega_1} \cap \beta$.

Proceeding similarly as above, by $P(\omega_1, S_{\omega_1}, C_{\omega_1})$ we know that C_{ω_1} is club in ω_1, so $\text{mos}(C_{\omega_1})$ is club in $\text{mos}(\omega_1)$, i.e. $C_{\omega_1} \cap \beta$ is club in β. Then $\beta \in C_{\omega_1}$ since C_{ω_1} is closed. Now $\beta \in C_{\omega_1}$ and $S_\beta = S_{\omega_1} \cap \beta$ is in direct contradiction with $P(\omega_1, S_{\omega_1}, C_{\omega_1})$. □

6 Martin's Axiom

In this section we will present a second combinatorial principle called Martin's Axiom and show that one of its consequences is the nonexistence of Suslin lines.

Up to this point, our partial orders have been strict, i.e. taking $<$ instead of $\leq$ as basic. In the context of Martin's Axiom, it is more common to use the non-strict form $(P, \leq)$. Moreover, in this section we will actually work with pre-orders, which are not required to be anti-symmetric.

Definition 6.1. (Pre-Order) Let P be a non-empty set. A *pre-order* of P is a binary relation $\leq$ on P which is *reflexive*, i.e. $\forall p \in P(p \leq p)$, and *transitive*, meaning $\forall pqr \in P(p \leq q \wedge q \leq r \to p \leq r)$. We call the pair $(P, \leq)$ a pre-ordered set.

Definition 6.2. Let $(P, \leq)$ be a pre-ordered set. Two elements $p_1, p_2 \in P$ are called *compatible* (denoted $p_1 \not\perp p_2$) iff there exists $q \in P$ such that $p_1 \leq q \geq p_2$[4]; they are called *incompatible* (denoted $p_1 \perp p_2$) otherwise. An *antichain* in P is a subset $A \subseteq P$ such that elements of A are pairwise incompatible, i.e. $\forall pq \in P(p \neq q \to p \perp q)$. If every antichain in P is at most countable, we say that $(P, \leq)$ satisfies the *countable chain condition* or the *ccc* for short.

Remark 6.3. In the context of trees, we used the term *antichain* to refer to a subset of *incomparable* elements, whereas here we are talking about *incompatible* elements. These are two different notions in general. It is, however, not hard to see that when talking about trees, these two definitions are equivalent, i.e. two elements of a tree are comparable iff they are compatible, since the set of predecessors of any node is linearly ordered.

Remark 6.4. Any linearly ordered set trivially satisfies the countable chain condition as defined in the current section, hence this definition and the one given previously for topological spaces are not equivalent.

Definition 6.5. Let $(P, \leq)$ be a pre-ordered set. Let $\emptyset \neq C \subseteq P$. Then we say that

(i) C is *cofinal* iff $\forall p \in P\, \exists q \in C(q \geq p)$;
(ii) C is *downwards closed* iff $\forall p \in C\, \forall q \in P(q \leq p \to q \in C)$;
(iii) C is *directed* iff $\forall p_1 p_2 \in C\, \exists q \in C(p_1 \leq q \geq p_2)$;
(iv) C is a *filter* iff C is directed and downwards closed.

Definition 6.6. Let $(P, \leq)$ be a pre-ordered set. Let $G \subseteq P$ be a filter and let $\mathcal{C}$ be a family of cofinal subsets of P. Then G is called $\mathcal{C}$-*generic* iff $G \cap C \neq \emptyset$ for every $C \in \mathcal{C}$.

[4] This is called the *Jerusalem* convention. Many texts define compatibility in terms of $p_1 \geq q \leq p_2$, which is called the *American* convention.

Lemma 6.7. (Rasiowa–Sikorski) Let $(P, \leq)$ be a pre-ordered set and let $\mathcal{C}$ be a collection of cofinal subsets of P with $|\mathcal{C}| \leq \aleph_0$. Then for every $p \in P$ there exists a $\mathcal{C}$-generic filter G such that $p \in G$.

Proof. Let $\mathcal{C} = \{C_n \,|\, n \in \omega\}$. We construct a sequence $\langle p_n \,|\, n \in \omega \rangle$ by recursion. Let $p_0 = p$. Given p_n, define $p_{n+1} = q$ for some $q \in C_n$ such that $p_n \leq q$; this is possible since C_n is cofinal. Now we define $G \coloneqq \{r \in P \,|\, r \leq p_n \text{ for some } n \in \omega\} \ni p$. Then G is clearly directed and downwards closed, i.e. it is a filter. Moreover, $G \cap C \neq \emptyset$ for every $C \in \mathcal{C}$, so G is also $\mathcal{C}$-generic. □

The previous lemma shows that it is only interesting to consider whether $\mathcal{C}$-generic filters exist in the case of uncountable $\mathcal{C}$. The next lemma shows that, in that case, such filters do not always exist.

Lemma 6.8. There exists a non ccc pre-ordered set $(P, \leq)$ and an uncountable family $\mathcal{C}$ of cofinal subsets of P such that no filter $G \subseteq P$ is $\mathcal{C}$-generic.

Proof. Let $P \coloneqq \{p \subset \omega \times \omega_1 \,|\, p \text{ is a function with finite domain}\}$. Clearly $(P, \subseteq)$ is a pre-ordered set, and the ordering is also anti-symmetric. Note also that P does not satisfy the ccc, since, for example, $\{\{(0, \alpha)\} \,|\, \alpha \in \omega_1\}$ is an uncountable antichain.

Define, for each $\alpha \in \omega_1$, the set $C_\alpha \coloneqq \{p \in P \,|\, \alpha \in \operatorname{ran}(p)\}$. We claim that each C_α is cofinal in P. Indeed, if $p \in P$ and $\alpha \in \operatorname{ran}(p)$, then $p \in C_\alpha$ and we are done. On the other hand, if $\alpha \notin \operatorname{ran}(p)$, then we have $p \subseteq p \cup \{(n, \alpha)\} \in C_\alpha$, where we pick n such that $n \notin \operatorname{dom}(p)$ (this is possible since $\operatorname{dom}(p)$ is finite).

Let $G \subseteq P$ be a $\mathcal{C}$-generic filter for $\mathcal{C} \coloneqq \{C_\alpha \,|\, \alpha \in \omega_1\}$. Note that $\mathcal{C}$ is uncountable. Then G is a family of functions with finite domain, and, being directed, any two elements of G are compatible. For this reason $f = \bigcup G$ is a function from a subset of ω into ω_1. Since $\operatorname{ran}(f)$ is at most countable, there exists some $\beta \in \omega_1 \setminus \operatorname{ran}(f)$. This contradicts the fact that $G \cap C_\beta \neq \emptyset$. □

Definition 6.9 (Martin's Axiom)**.** Let κ be a cardinal.

- $\mathrm{MA}(\kappa)$ is the statement: If $(P, \leq)$ is a ccc pre-ordered set, and $\mathcal{C}$ is a family of cofinal subsets of P with $|\mathcal{C}| \leq \kappa$, then there is a filter $G \subseteq P$ which is $\mathcal{C}$-generic.
- MA is the statement: $\mathrm{MA}(\kappa)$ holds for each $\kappa < 2^{\aleph_0}$.

Having stated Martin's Axiom, in the next lemma we show a few easy results related to it. Most importantly, we illustrate that MA is only interesting when CH fails.

Lemma 6.10.

(a) If $\lambda < \kappa$, then $\mathrm{MA}(\kappa) \to \mathrm{MA}(\lambda)$.
(b) $\mathrm{MA}(2^{\aleph_0})$ is false;
(c) $\mathrm{MA}(\kappa)$ is true whenever $\kappa \leq \aleph_0$;
(d) $\mathrm{CH} \to \mathrm{MA}$;
(e) $\mathrm{MA}(\kappa) \to \kappa < \mathfrak{c} = 2^{\aleph_0}$;
(f) $\mathrm{MA}(\aleph_1) \to \neg\mathrm{CH}$.

Proof. Statement (a) is obvious, and (c) is immediate from Lemma 6.7, where we also didn't make use of the ccc hypothesis. We prove (b).

Let $P \coloneqq \{p \subset \omega \times 2 \,|\, p \text{ is a function with finite domain}\}$. Similarly to Lemma 6.8, we have that $(P, \subseteq)$ is a pre-ordered set. It is easy to see that $(P, \subseteq)$ satisfies the ccc, since P is countable.

Define, for each $n \in \omega$, the set $C_n := \{p \in P \mid n \in \operatorname{dom}(p)\}$. It is easy to see that, given $n \in \omega$, any $p \in P$ can be extended to include n in its domain, so every C_n is cofinal in P. Given $h : \omega \to 2$, define $E_h := \{p \in P \mid \exists n \in \operatorname{dom}(p)(p(n) \neq h(n))\}$. Since h is defined on all of ω, any $p \in P$ can be extended in such a way that it disagrees somewhere with h, so any E_h is cofinal.

Let $\mathcal{C} = \{C_n \mid n \in \omega\} \cup \{E_h \mid h : \omega \to 2\}$. Then $\mathcal{C}$ is a family of cofinal subsets of P, and $|\mathcal{C}| = 2^{\aleph_0}$. Let $G \subseteq P$ be a $\mathcal{C}$- generic filter. Again, as in Lemma 6.8, $f = \bigcup G$ is a function, since G is a filter. Having nonempty intersection with every C_n ensures that $\operatorname{dom}(f) = \omega$. Now, since for every $h : \omega \to 2$ we have that $G \cap E_h \neq \emptyset$, $f \neq h$. Therefore f is a function from ω to 2 which is different from every function from ω to 2. This contradiction concludes the proof of (b).

For (d), assume CH. The condition $\kappa < 2^{\aleph_0}$ reduces to $\kappa \leq \aleph_0$, so the result follows from part (c). Part (e) is an immediate consequence of (a) and (b). Part (f) is immediate from (e). □

We can now show that, under MA($\aleph_1$), no Suslin lines exist. The next theorem proves this directly.

Theorem 6.11 (MA($\aleph_1$) $\to$ SH)**.** Assume MA($\aleph_1$); then there are no Suslin lines.

Proof. Assume that there is a Suslin line. Then, using Theorem 3.9 and Lemma 3.8, there exists a normal Suslin tree $(T, <)$. Note that $(T, \leq)$ is, in particular, a ccc pre-ordered set, since being Suslin doesn't allow uncountable antichains.

For each $\alpha \in \omega_1$, let $D_\alpha := \{x \in T \mid \operatorname{ht}(x) > \alpha\}$. By construction, and the fact that $(T, <)$ is a normal Suslin tree, the set D_α is cofinal in T for any $\alpha \in \omega_1$. Let $\mathcal{C} := \{D_\alpha \mid \alpha \in \omega_1\}$. Clearly $|\mathcal{C}| = \aleph_1$. Let $G \subseteq T$ be $\mathcal{C}$-generic. Then G is a branch in T of length ω_1, which is a contradiction to the assumption that $(T, <)$ is a Suslin tree. □

Lemma 6.12. Assume MA($\aleph_1$). Let X be a ccc topological space and $\{U_\alpha \mid \alpha \in \omega_1\}$ a family of nonempty open subsets of X. Then there is an uncountable $I \subseteq \omega_1$ such that $\{U_\alpha \mid \alpha \in I\}$ has the *finite intersection property*, meaning: if $J \subseteq I$ is a finite subset of I, then $\bigcap_{\alpha \in J} U_\alpha \neq \emptyset$.

Proof. Consider the sequence $\{V_\alpha \mid \alpha < \omega_1\}$ of nonempty open subsets of X given by $V_\alpha = \bigcup_{\gamma > \alpha} U_\gamma$. Clearly $V_\beta \subseteq V_\alpha$ for every $\alpha < \beta$. We claim that there is an α_0 such that $\overline{V}_\beta = \overline{V}_{\alpha_0}$ for every $\beta > \alpha_0$. Assume towards a contradiction that no such α_0 exists. Then we could, by recursion, find a strictly increasing sequence of ordinals $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ such that $\overline{V}_{\alpha_{\xi+1}} \neq \overline{V}_{\alpha_\xi}$ for every $\xi < \omega_1$, which implies that $V_{\alpha_\xi} \setminus \overline{V}_{\alpha_{\xi+1}} \neq \emptyset$ for every $\xi < \omega_1$. Now the set $\left\{V_{\alpha_\xi} \setminus \overline{V}_{\alpha_{\xi+1}} \mid \xi < \omega_1\right\}$ is an uncountable family of nonempty subsets of X. Moreover, the elements of the family are open, since each is the difference of an open and a closed set, and pairwise disjoint by construction. Hence this uncountable family of nonempty, pairwise disjoint subsets of X contradicts the fact that X satisfies the topological ccc.

By the argument above, we can fix $\alpha_0 < \omega_1$ such that $\overline{V}_\beta = \overline{V}_{\alpha_0}$ for every β with $\alpha_0 < \beta < \omega_1$. Let $P = \{p \subseteq V_{\alpha_0} \mid p \text{ is nonempty and open}\}$. For $pq \in P$, define $p \leq q$ iff $q \subseteq p$, i.e. we order P by reverse inclusion. Clearly $(P, \leq)$ is a pre-ordered set. We claim that $(P, \leq)$ satisfies the ccc. Indeed, consider $pq \in P$ such that $p \perp q$, i.e. such that there is no element of P included in both p and q. This implies that $p \cap q = \emptyset$, so any antichain in the pre-ordered set P is also a family of pairwise disjoint nonempty open subsets of X, so the topological ccc-ness of X implies that $(P, \leq)$ is a ccc pre-ordered set.

For each $\beta < \omega_1$, let $C_\beta \coloneqq \{p \in P \,|\, p \subseteq U_\gamma \text{ for some } \gamma > \beta\}$. We claim that C_β is cofinal in P for each $\beta < \omega_1$. Indeed, fix $\beta < \omega_1$; by the choice of α_0 we have that $\overline{V}_{\alpha_0} \subseteq \overline{V}_\beta$, so if $p \in P$ then $p \cap V_\beta \neq \emptyset$, since p is open. Therefore $p \cap U_\gamma \neq \emptyset$ for some $\gamma > \beta$, by the definition of V_β, and we have that $p \leq p \cap U_\gamma \in C_\beta$, so C_β is cofinal in P.

Let $\mathcal{C} \coloneqq \{C_\beta \,|\, \beta < \omega_1\}$. Using MA($\aleph_1$), let $G \subseteq P$ be a $\mathcal{C}$-generic filter. Define $A_G \coloneqq \{\gamma < \omega_1 \,|\, \exists p \in G(p \subseteq U_\gamma)\}$. Then $\{U_\gamma \,|\, \gamma \in A_G\}$ has the finite intersection property, since G has it by virtue of being a filter. It remains to argue that A_G is uncountable. For any $\beta < \omega_1$ we have that $G \cap C_\beta \neq \emptyset$, so A_G contains some ordinal $\gamma > \beta$. Since $\beta < \omega_1$ was arbitrary, this shows that A_G is uncountable. □

Theorem 6.13. Assume MA($\aleph_1$). If X and Y are topological spaces satisfying the ccc, then $X \times Y$ with the product topology is also ccc.

Proof. Let X and Y be ccc and towards a contradiction suppose that $X \times Y$ is not ccc. Let $\{W_\alpha \,|\, \alpha \in \omega_1\}$ be an uncountable family or pairwise disjoint nonempty open subsets of $X \times Y$. For each $\alpha \in \omega_1$, pick $U_\alpha \subseteq X$ and $V_\alpha \subseteq Y$ nonempty open subsets such that $U_\alpha \times V_\alpha \subseteq W_\alpha$. Using Lemma 6.12 there is an uncountable $I \subseteq \omega_1$ such that $\{U_\alpha \,|\, \alpha \in I\}$ has the finite intersection property. Then given $\alpha, \beta \in I$ with $\alpha \neq \beta$ we have that $U_\alpha \cap U_\beta \neq \emptyset$ but $(U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = \emptyset$, so it must be that $V_\alpha \cap V_\beta = \emptyset$. Hence the set $\{V_\alpha \,|\, \alpha \in I\}$ contradicts the fact that Y is a ccc topological space. □

Remark 6.14. In Theorem 6.11, we showed directly that MA($\aleph_1$) $\to$ SH. The above result, i.e. Theorem 6.13, together with Lemma 2.7 gives an alternative proof of the same fact.

References

Books

[1] Devlin, K. J. and Johnsbråten, H. *The Souslin Problem.* Lecture Notes in Mathematics. Vol. 405. Springer, 1974, pp. viii+132.

[2] Halbeisen, L. J. *Combinatorial Set Theory. With a Gentle Introduction to Forcing.* Second Edition. Springer Monographs in Mathematics. Springer, 2017, pp. xvi+594.

[3] Hrbacek, K. and Jech, T. J. *Introduction to Set Theory. Third Edition, Revised and Expanded.* Monographs and Textbooks in Pure and Applied Mathematics. Vol. 220. CRC Press, 1999, pp. xii+291.

[4] Jech, T. J. *Set Theory. The Third Millennium Edition, Revised and Expanded.* Springer Monographs in Mathematics. Springer, 2003, pp. xiv+769.

[5] Kunen, K. *Set Theory.* Studies in Logic. Mathematical Logic and Foundations. Vol. 34. College Publications, 2011, pp. viii+401.

[6] Kunen, K. *The Foundations of Mathematics.* Studies in Logic. Mathematical Logic and Foundations. Vol. 19. College Publications, 2009, pp. viii+251.

[7] Weaver, N. *Forcing for Mathematicians.* World Scientific Publishing Co. Pte. Ltd., 2014, pp. x+142.

Articles

[8] Jech, T. J. "Non-provability of Souslin's hypothesis". In: *Commentationes Mathematicae Universitatis Carolinae* 8 (1967), pp. 291–305.

[9] Jensen, R. B. "Souslin's Hypothesis is incompatible with $V = L$". In: *Notices of the American Mathematical Society* 15 (1968), p. 935.

[10] Kanamori, A. "Historical remarks on Suslin's problem". In: *Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies.* Ed. by Kennedy, J. and Kossak, R. Lecture Notes in Logic. Cambridge University Press, 2011, pp. 1–12.

[11] Kurepa, D. R. "H'hypothèse de ramification". In: *Comptes rendues hebdomadaires des séances de l'Académie des Sciences* 202 (1936), pp. 185–187.

[12] Malykhin, V. I. "The Suslin hypothesis and its significance for set-theoretic mathematics". In: *Russian Mathematical Surveys* 51 (1996), pp. 419–437.

[13] Martin, D. A. and Solovay, R. M. "Internal Cohen extensions". In: *Annals of Mathematical Logic* 2 (1970), pp. 143–178.

[14] Miller, E. W. "A note on Souslin's Problem". In: *American Journal of Mathematics* 65 (1943), pp. 673–678.

[15] Sierpiński, W. F. "Sur un problème de la théorie générale des ensembles équivalent au problème de Souslin". In: *Fundamenta Mathematicae* 35 (1948), pp. 165–174.

[16] Solovay, R. M. and Tennenbaum, S. "Iterated Cohen extensions and Souslin's problem". In: *Annals of Mathematics* 94 (1971), pp. 201–245.

[17] Suslin, M. Y. "Problème 3". In: *Fundamenta Mathematicae* 1 (1920), p. 223.

[18] Tennenbaum, S. "Souslin's Problem". In: *Proceedings of the National Academy of Sciences of the United States of America* 59 (1968), pp. 60–63.