PARTITIONING PAIRS OF COUNTABLE SETS OF ORDINALS

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1. Partitioning Pairs of Countable Ordinals

Before we proceed with the main results to be discussed we review some well known properties of stationary subsets on ω_1 (see [1] section II.6)

Lemma 1. *There are ω_1 disjoint stationary subsets of ω_1.*

Proof. Let $\mathrm{Cub}(\omega_1) = \{X \subseteq \omega_1 : \exists C \subseteq X(C \text{ is club on } \omega_1)\}$. Then $\mathrm{Cub}(\omega_1)$ is countably complete filter. Its dual ideal $\mathrm{Cub}^*(\omega_1) = \{X \subseteq \omega_1 : \exists X' \in \mathrm{Cub}(\omega_1)(X = \omega_1 \backslash X')\}$ is countably complete and contains all singletons and so all countable subsets of ω_1. Recall also that $X \subseteq \omega_1$ is stationary if and only if $X \notin \mathrm{Cub}^*(\omega_1)$.

For every $\rho < \omega_1$ let $f_\rho \colon \rho \to \omega$ be an injective mapping. Then $\forall \alpha < \omega_1 \ \forall n \in \omega$ let

$$X^n_\alpha = \{\rho < \omega_1 : \alpha < \rho \text{ and } f_\rho(\alpha) = n\}.$$

Note that if $\alpha \neq \beta$ then for every $n \in \omega$ we have $X^n_\alpha \cap X^n_\beta = \emptyset$ (otherwise $\exists \rho < \kappa$ greater than α, β such that $f_\rho(\alpha) = f_\rho(\beta) = n$ which is a contradiction to f_ρ being injective). Also for every $\alpha < \omega_1$

$$\cup_{n\in\omega} X^n_\alpha = \{\rho < \omega_1 : \alpha < \rho\} \in \mathrm{Cub}(\omega_1).$$

Since $\mathrm{Cub}^*(\omega_1)$ is countably complete ideal $\forall \alpha \in \omega_1 \exists h(\alpha) \in \omega$ such that $X^{h(\alpha)}_\alpha \notin \mathrm{Cub}^*(\omega_1)$ and so in particular $X^{h(\alpha)}_\alpha$ is stationary. But $h \colon \omega_1 \to \omega$ and so there is $n \in \omega$ such that $|h^{-1}(n)| = \omega_1$. Therefore $\{X^n_\alpha : h(\alpha) = n\}$ is an uncountable family of disjoint stationary subsets of ω_1. □

Corollary 1. *There is a mapping $g : \omega_1 \to \omega_1$ such that $\forall \alpha \in \omega_1$, $g^{-1}(\{\alpha\})$ is stationary.*

Proof. Let $\{X_\alpha : \alpha \in \omega_1\}$ be a family of disjoint stationary subsets of ω_1. Then $\forall \alpha < \omega_1$ define $g \upharpoonright X_\alpha = \alpha$ and $g \upharpoonright [\omega_1 \backslash (\cup_{\alpha<\omega_1} X_\alpha)] = 0$. □

Recall the following definitions:

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Definition 1. We say that

$$\omega_1 \to [\omega_1]^2_{\omega_1}$$

iff for every $f\colon [\omega_1]^2 \to \omega_1$ there is $A \in [\omega_1]^{\omega_1}$ s.t. $f"[A]^2 \neq \omega_1$.

Remark. Thus

$$\omega_1 \not\to [\omega_1]^2_{\omega_1}$$

iff there is a $f\colon [\omega_1]^2 \to \omega_1$ such that $\forall A \in [\omega_1]^{\omega_1}$, $f"[A]^2 = \omega_1$.

The following result is due to S. Todorcevic (see [3]).

Theorem 1. $\omega_1 \not\to [\omega_1]^2_{\omega_1}$.

Proof. We will find a function $f'\colon [\omega_1]^2 \to \omega_1$ such that for every uncountable $A \subseteq \omega_1$, $f"[A]^2 = \omega_1$. In fact we will find a function $f\colon [\omega_1]^2 \to \omega_1$ such that for every uncountable set A, $f"[A]^2$ contains a closed unbounded set. By Corollary 1 there is a function $g\colon \omega_1 \to \omega_1$ such that $g^{-1}(\{\alpha\})$ is stationary for every $\alpha \in \omega_1$. Then $f' = g \circ f$ is a witness to $\omega_1 \not\to [\omega_1]^2_{\omega_1}$.

Let $\{r_\alpha : \alpha \in \omega_1\}$ be a family of $\aleph_1$ distinct functions in ${}^\omega 2$ and for every $\alpha \in \omega_1$ fix an injective mapping $e_\alpha\colon \alpha \to \omega$. Then for all $\alpha, \beta \in \omega_1$ let

$$\sigma(\alpha,\beta) = \sigma(r_\alpha, r_\beta) = \min\{n : r_\alpha(n) \neq r_\beta(n)\}$$

and let

$$\Delta_{\alpha,\beta} = \{\delta : \alpha \leq \delta < \beta \text{ and } e_\beta(\delta) \leq \sigma(\alpha,\beta)\}.$$

Then for all $\{\alpha,\beta\} \in [\omega_1]^2$ define $f(\alpha,\beta) = \min \Delta_{\alpha,\beta}$ if $\Delta_{\alpha,\beta}$ is non-empty and 0 otherwise.

Consider any uncountable subset A of ω_1 and for every function g in ${}^{<\omega}2 = \cup\{{}^n 2 : n \in \omega\}$ define $B_g = \{\alpha \in A : g \subseteq r_\alpha\}$. Let

$$C = \{\delta < \omega_1 : \forall g \in^{<\omega} 2 \text{ either } B_g \subseteq \delta \text{ or } |B_g| = \omega_1 \text{ and } \delta \in B'_g\}$$

where B'_g denotes the set of all limit points of B_g.

Claim. C is closed unbounded subset of ω_1.

Proof. Let $I = \{g \in {}^{<\omega}2 : |B_g| < \omega_1\}$. Then for every $g \in I$ there is $\alpha_g \in \omega_1$ such that $B_g \subseteq \alpha_g$. Let $\alpha = \sup_{g\in I} \alpha_g$. Then $C = (\bigcap_{g \in {}^{<\omega}2 \setminus I} B'_g) \setminus \alpha$ is closed unbounded subset of ω_1. □

Let $\delta \in C$. Since A is unbounded there is $\beta \in A$ such that $\delta < \beta$. Let $n = e_\beta(\delta)$ and $g = r_\beta \restriction n$. Then $\beta \in B_g$ and so by definition of C, B_g is uncountable. For every $\gamma \in B_g$ such that $\gamma > \beta$ let $m_\gamma = \sigma(\beta,\gamma)$ and $h_\gamma = r_\gamma \restriction m_\gamma + 1$. Since B_g is uncountable there is $m \in \omega$, $h\colon m+1 \to 2$ such that for uncountably many $\gamma \in B_g$, $m = m_\gamma$, $h = h_\gamma$. Then B_h is an uncountable subset of B_g such that for every $\gamma \in B_h$ the distance $\sigma(\beta, \gamma) = m \geq n$. We will find $\alpha \in B_h$ such that $f(\alpha, \beta) = \delta$.

Let $F = \{\gamma < \delta : e_\beta(\gamma) \leq m\}$. Since e_β is injective F is a finite subset of δ. By definition of C, δ is a limit point of B_h and so there is $\alpha \in B_h \cap \delta$ such that $F \subseteq \alpha$. Suppose γ is an ordinal such that $\alpha \leq \gamma < \beta$ and $e_\beta(\gamma) \leq \sigma(\alpha, \beta) = m$. Then $\gamma \notin F$ and so $\delta \leq \gamma$. Therefore $\delta = \min \Delta_{\alpha,\beta}$ and so $\delta = f(\alpha, \beta)$. □

2. Partitioning Pairs of Countable Sets of Ordinals

Definition 2. Let λ be an uncountable cardinal and let $\mathcal{P}_{\omega_1}(\lambda)$ be the family of all countable subsets of λ. Then

$$[\mathcal{P}_{\omega_1}(\lambda)]^2_\subset = \{(x, y) \in [\mathcal{P}_{\omega_1}(\lambda)]^2 : x \subseteq y\}.$$

Definition 3. Let λ be an uncountable cardinal. We say that

$$\mathcal{P}_{\omega_1}(\lambda) \to [\text{unbdd}]^2_\lambda$$

iff for every coloring $f\colon [\mathcal{P}_{\omega_1}(\lambda)]^2_\subset \to \lambda$ there is an unbounded set $A \subseteq \mathcal{P}_{\omega_1}(\lambda)$ such that $f``[A]^2_\subset \neq \lambda$.

Remark. Thus

$$\mathcal{P}_{\omega_1}(\lambda) \nrightarrow [\text{unbdd}]^2_\lambda$$

iff for every coloring $f\colon [\mathcal{P}_{\omega_1}(\lambda)]^2_\subset \to \lambda$ for every unbounded set $A \subseteq \mathcal{P}_{\omega_1}(\lambda)$ we have $f``[A]^2_\subset = \lambda$.

In 1990 D. Velleman obtained a generalization of Theorem 1 (see [4]) to pairs of countable sets of ordinals.

Theorem 2. Let λ be an uncountable cardinal. Suppose that there is a stationary subset S of $\mathcal{P}_{\omega_1}(\lambda)$ of cardinality λ. Then $\mathcal{P}_{\omega_1}(\lambda) \nrightarrow [\text{unbdd}]^2_\lambda$.

Remark. If GCH holds, then for every uncountable cardinal λ the cardinality of $\mathcal{P}_{\omega_1}(\lambda)$ is $\lambda^\omega = \lambda$ and so the hypothesis of Theorem 2 holds. Thus GCH implies $\mathcal{P}_{\omega_1}(\lambda) \nrightarrow [\text{unbdd}]^2_\lambda$ for every uncountable cardinal λ.

Proof. We will show that there is $f'\colon [\mathcal{P}_{\omega_1}(\lambda)]^2_\subset \to \lambda$ such that for every unbounded $A \subseteq \mathcal{P}_{\omega_1}(\lambda)$, $f``[A]^2_\subset = \lambda$. In fact we will find a function $f\colon [\mathcal{P}_{\omega_1}(\lambda)]^2_\subset \to \mathcal{P}_{\omega_1}(\lambda)$ such that for every unbounded set $A \subseteq \mathcal{P}_{\omega_1}(\lambda)$ there is a closed unbounded set $C = C_A$ on $\mathcal{P}_{\omega_1}(\lambda)$ such that $S \cap C \subseteq f``[A]^2_\subset$. Matsubara has shown (see [2]) that (under the hypothesis of the theorem) there is a function $g\colon S \to \lambda$ such that $(\forall \alpha \in \lambda) g^{-1}(\{\alpha\})$ is stationary. Then $f' = g \circ f\colon [\mathcal{P}_{\omega_1}(\lambda)]^2_\subset \to \lambda$ is the desired coloring.

Just as in Theorem 1 fix a family $\{r_\alpha : \alpha < \omega_1\}$ of $\aleph_1$ distinct functions in ${}^\omega 2$. For any two x, y countable subsets of λ let

$$\sigma(x, y) = \sigma(r_{\text{type}(x)}, r_{\text{type}(y)})$$

if $\text{type}(x) \neq \text{type}(y)$ and let $\sigma(x, y) = 0$ otherwise. Let $c \colon S \to \lambda$ be a bijection. For every $y \in \mathcal{P}_{\omega_1}(\lambda)$ let $Q_y = \{x \in S : x \subseteq y \text{ and } c(x) \in y\}$. Then $|Q_y| = \omega$ (since y is countable and c is injective) and so we can fix an injective mapping $e_y \colon Q_y \to \omega$. For any pair x, y of countable subsets of λ let

$$\Delta_{x,y} = \{d \in Q_y : x \subseteq d \text{ and } e_y(x) \leq \sigma(x, y)\}.$$

Then for every $(x, y) \in [\mathcal{P}_{\omega_1}(\lambda)]^2_{\subset}$ let $f(x, y) = \min_{\subset} \Delta_{x,y}$ if there is such a minimum (i.e. a smallest under inclusion element of $\Delta_{x,y}$) and let $f(x, y) = \emptyset$ otherwise. We claim that f is the desired coloring.

Consider any unbounded subset A of $\mathcal{P}_{\omega_1}(\lambda)$ and for every g in ${}^{<\omega}2$ let $B_g = \{x \in A : g \subseteq r_x\}$. Let C be the set of all $d \in \mathcal{P}_{\omega_1}(\lambda)$ such that for every g in ${}^{<\omega}2$ the following holds: either there is no $x \in B_g$ such that $d \subseteq x$ or B_g is unbounded and for every finite subset w of d there is $x \in B_g$ such that $w \subseteq x \subseteq d$.

Claim. C is a closed unbounded subset of $\mathcal{P}_{\omega_1}(\lambda)$.

Proof. Let $I = \{g \in^{<\omega} 2 : B_g \text{ is not unbounded}\}$. Then for every $g \in I$ there is a countable subset x_g of λ such that for no $x \in B_g (x_g \subseteq x)$. Let $x_0 = \cup_{g \in I} x_g$.

Consider any $d \in \mathcal{P}_{\omega_1}(\lambda)$ and let $d_0 = d \cup x_0$. Let d_1 be a common limit point of $\langle B_g : g \in {}^{<\omega}2 \backslash I\rangle$ above d_0, i.e. $d_0 \subseteq d_1$ and for all $g \in {}^{<\omega}2 \backslash I$ there is an increasing sequence $\langle x^n_g : n \in \omega\rangle \subseteq B_g$ such that $d_1 = \cup_{n \in \omega} x^n_g$. To see that d_1 is an element of C consider any $g \in {}^{<\omega}2$. If B_g is bounded then there is no x in B_g covering d_1 since $x_g \subseteq x_0 \subseteq d_1$. If B_g is unbounded and w is a finite subset of d_1 then there is some $m \in \omega$ such that $w \subseteq x^m_g \subseteq d_1$.

To show that C is closed consider any increasing sequence $\langle d_n : n \in \omega\rangle$ of elements of C and let $d = \cup_{n \in \omega} d_n$. Let $g \in {}^{<\omega}2$. If $g \in I$ then since $d_0 \in C$ there is no $x \in B_g$ which covers d_0 and so there is no $x \in B_g$ which covers d. Otherwise B_g is unbounded. But then if w is a finite subset of d, there is some d_n such that $w \subseteq d_n$ and since $d_n \in C$ there is an element $x \in B_g$ for which $w \subseteq x \subseteq d_n \subseteq d$. □

Let $d \in S \cap C$. Since A is unbounded there is $y \in A$ such that $d \cup \{c(d)\} \subseteq y$. But then $d \in Q_y$ and so $n = e_y(d)$ is defined. Let $g = r_y \restriction n$. Since $y \in B_g$ covers d, by definition of C we obtain that B_g is unbounded. Then for every $z \in B_g$ such that $y \subseteq z$ and $\text{type}(y) \neq \text{type}(z)$ let $m_z = \sigma(y, z)$ and let $h_z = r_z \restriction m_z + 1$. Again since B_g is unbounded there is $m \in \omega$ and $h\colon m+1 \to 2$ such that for unboundedly many $z \in B_h$, $m_z = m$ and $h_z = h$. Then B_h is an unbounded subset of B_g and for every $z \in B_h$ the distance $\sigma(z, y) = m \geq n$. We will find a set $x \in B_h$ such that $f(x, y) = d$.

Let $F = \{q \in Q_y : e_y(q) \leq m \text{ and } d \nsubseteq q\}$. Since e_y is injective, F is finite. For every $q \in F$ let $\alpha_q \in d \backslash q$. Then $w = \{\alpha_q : q \in F\}$ is a finite subset of d and since B_h is unbounded and $d \in C$, there is $x \in B_h$ such that $w \subseteq x \subseteq d$. We claim that $f(x, y) = d$. Consider any $z \in Q_y$ such that $x \subseteq z$ and $e_y(z) \leq m$. Then $z \notin F$ and so $d \subseteq z$. Therefore d is the minimum (under inclusion) of $\Delta_{x,y}$ and so $f(x, y) = d$. □

References

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