A CO-ANALYTIC MAXIMAL SET OF ORTHOGONAL MEASURES

VERA FISCHER AND ASGER TÖRNQUIST

Abstract. We prove that if $V = L$ then there is a Π^1_1 maximal orthogonal (i.e. mutually singular) set of measures on Cantor space. This provides a natural counterpoint to the well-known theorem of Preiss and Rataj [16] that no analytic set of measures can be maximal orthogonal.

§1. Introduction. Let X be a Polish space and let $P(X)$ be the associated Polish space of Borel probability measures on X (see e.g. [11, 17.E]). Recall that $\mu, \nu \in P(X)$ are said to be *orthogonal* (or *mutually singular*) if there is a Borel set $B \subseteq X$ such that $\mu(B) = 1$ and $\nu(B) = 0$. We will write $\mu \perp \nu$.

Preiss and Rataj proved in [16] that if X is an uncountable Polish space then no analytic set of measures can be maximal orthogonal, answering a question raised by Mauldin. Later Kechris and Sofronidis [12] gave a new proof of this result using Hjorth's theory of turbulence.

The purpose of this paper is to prove that Preiss and Rataj's result is in some sense optimal. Specifically, we will prove:

THEOREM 1.1. *If $V = L$ then there is a Π^1_1 maximal set of orthogonal measures in $P(2^\omega)$.*

The assumption that $V = L$ can of course be replaced by the assumption that all reals are constructible. Also, the proof easily relativizes to a parameter $x \in 2^\omega$: If $V = L[x]$ then there is a $\Pi^1_1(x)$ maximal orthogonal set of measures in $P(2^\omega)$.

Theorem 1.1 belongs to a line of results starting with Miller's paper [14]. Miller proved, among several other results, that assuming $V = L$ there is a Π^1_1 maximal almost disjoint family in $\mathcal{P}(\omega)$, there is a Π^1_1 Hamel

---

2000 *Mathematics Subject Classification.* 03E15.

Key words and phrases. Descriptive set theory; constructible sets.

The authors wish to thank the Austrian Science Fund FWF for post-doctoral support through grant no. P 19375-N18 (Törnquist) and P 20835-N13 (Fischer).

basis for $\mathbb{R}$ over $\mathbb{Q}$, and there is a Π^1_1 set meeting every line in $\mathbb{R}^2$ exactly twice.

More recently, Miller's technique has found use in the study of maximal cofinitary subgroups of the infinite symmetric group S_∞: Gao and Zhang showed in [3] that if $V = L$ then there is a maximal cofinitary group generated by a Π^1_1 subset of S_∞, and Kastermans in [8] improved this by showing that if $V = L$ then there is a Π^1_1 maximal cofinitary subgroup of S_∞.

The present paper is organized into four sections. In §2 we introduce the basic effective descriptive set-theoretic notions related to the space $P(2^\omega)$, in particular, we introduce a natural notion of a code for a measure on 2^ω. We also revisit a product measures construction due to Kechris and Sofronidis. Theorem 1.1 is proved in §3. The proof hinges on a method for coding a given real into a non-atomic measure while keeping the *measure class* of the original measure intact in the process; this is the content of the "Coding Lemma" 3.5. Finally in §4 we show that a maximal orthogonal family of continuous measures always has size continuum, and that if there is a Cohen real over L in V then there is no Π^1_1 maximal set of orthogonal measures.

Remark. In the present paper we have attempted to give a completely elementary account of Miller's technique as it applies to Theorem 1.1 above, and to provide the details of the argument while relying only on standard methods that can be found in places such as [6, §13] or [1, Ch. 5]. A somewhat different exposition of the details of Miller's technique can be found in Kastermans' thesis [7].

§2. Preliminaries. For $s \in 2^{<\omega}$, let

$$N_s = \{x \in 2^\omega : s \subseteq x\},$$

the basic neighbourhood defined by s. Define

$$p(2^\omega) = \{f : 2^{<\omega} \to [0,1] : f(\emptyset) = 1 \wedge (\forall s \in 2^{<\omega}) f(s) = f(s^\frown 0) + f(s^\frown 1)\}.$$

Then $p(2^\omega) \subseteq [0,1]^{2^{<\omega}}$ is closed, and an easy application of Kolmogorov's consistency Theorem shows that for each $f \in p(2^\omega)$ there is a unique $\mu_f \in P(2^\omega)$ such that $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$, see [11, 17.17]. Conversely, if $\mu \in P(2^\omega)$ then $f(s) = \mu(N_s)$ defines $f \in p(2^\omega)$ such that $\mu_f = \mu$, thus $f \mapsto \mu_f$ is a bijection. We will call the element $f \in p(2^\omega)$ the *code* for μ_f.

Note that if s_n enumerates $2^{<\omega}$ and we let $h_n : 2^\omega \to \mathbb{R}$ be defined as follows:

$$h_n(x) = \begin{cases} 1 & \text{if } s_n \subseteq x \\ 0 & \text{otherwise,} \end{cases}$$

then the metric on $P(2^\omega)$ defined by

$$\delta(\mu,\nu) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{|\int h_n d\mu - \int h_n d\nu|}{\|h_n\|_\infty}$$

given in [11, 17.19] makes the map $f \mapsto \mu_f$ an isometric bijection if we equip $p(2^\omega)$ with the metric

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n-1} |f(s_n) - g(s_n)|.$$

Let $(q_i : i \in \omega)$ be a recursive enumeration of

$$\{q : 2^n \to \mathbb{Q} \cap [0,1] : n \in \omega \wedge \sum_{s \in \mathrm{dom}(q)} q(s) = 1\}.$$

For each q_i, let $\hat{q}_i \in p(2^\omega)$ be the unique element of $p(2^\omega)$ such that

$$(\forall s \in \mathrm{dom}(q_i))(\forall k) \hat{q}_i(s \frown 0^k) = q_i(s),$$

where 0^k denotes a sequence of zeros of length k. Clearly the sequence $(\hat{q}_i : i \in \omega)$ is dense in $p(2^\omega)$, and it is routine to see that the relations $P, Q \subseteq \omega^4$ defined by

$$P(i,j,m,k) \iff d(\hat{q}_i, \hat{q}_j) \leq \frac{m}{k+1}$$

$$Q(i,j,m,k) \iff d(\hat{q}_i, \hat{q}_j) < \frac{m}{k+1}$$

are recursive. Thus $(\hat{q}_i : i \in \omega)$ provides a recursive presentation (in the sense of [15, 3B]) of $p(2^\omega)$, and so $(\mu_{\hat{q}_i} : i \in \omega)$ provides a recursive presentation of $P(2^\omega)$. The map $f \mapsto \mu_f$ is then a recursive isomorphism between $p(2^\omega)$ and $P(2^\omega)$. So from a descriptive set-theoretic point of view there is really no difference between working with $P(2^\omega)$ or $p(2^\omega)$. In particular, it doesn't matter in hierarchy complexity calculations if we deal with the codes for measures, or with the measures themselves.

Remark 2.1. Although we could easily have given $P(2^\omega)$ a recursive presentation directly without the detour via $p(2^\omega)$, the space $p(2^\omega)$ will still be useful to us. Namely, elements of $P(2^\omega)$ are formally functions $\mu : \mathcal{B}(2^\omega) \to [0,1]$ defined on the Borel sets $\mathcal{B}(2^\omega)$, and so formally $\mu \notin L_\delta$ for any $\delta < \omega_1$. However, since codes are simply functions from $2^{<\omega}$ to $[0,1]$, the code for μ may be in L_δ for some $\delta < \omega_1$, even if $\mu \notin L_\delta$.

Recall from real analysis that if $\mu, \nu \in P(2^\omega)$, then μ is *absolutely continuous* with respect to ν, written $\mu \ll \nu$ if for all Borel subset B of 2^ω it holds that $\nu(B) = 0$ implies $\mu(B) = 0$. We say that $\mu, \nu \in P(2^\omega)$ are *absolutely equivalent*, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

LEMMA 2.2. *(a) The relations $\ll$, $\approx$ and $\perp$ are arithmetical.*
(b) The set

$$P_c(2^\omega) = \{\mu \in P(2^\omega) : \mu \textit{ is non-atomic}\}$$

is arithmetical.

PROOF. (a) To see that $\ll$ and $\approx$ are arithmetical, note that by [11, p. 105]

$$\mu \ll \nu \iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall B \subseteq 2^\omega \text{ Borel})(\nu(B) < \delta \to \mu(B) < \epsilon)$$

and so using [11, 17.10] this is equivalent to

$$\mu \ll \nu \iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall s_1, \ldots, s_n \in 2^{<\omega})(\nu(\bigcup_{i=1}^n N_{s_i}) < \delta \longrightarrow \mu(\bigcup_{i=1}^n N_{s_i}) < \epsilon).$$

To see that $\perp$ is arithmetical, note that

$$\mu \perp \nu \iff (\forall \epsilon > 0)(\exists s_1, \ldots, s_n \in 2^{<\omega})(\mu(\bigcup_{i=1}^n N_{s_i}) < \epsilon \wedge \nu(\bigcup_{i=1}^n N_{s_i}) > 1 - \epsilon)$$

(b) We claim that

$$\mu \in P_c(2^\omega) \iff (\forall \epsilon > 0)(\exists n)(\forall s \in 2^{<\omega})(\mathrm{lh}(s) = n \to \mu(N_s) < \epsilon).$$

The implication from right to left is clear. To see the reverse implication, note that the tree

$$\{s \in 2^{<\omega} : \mu(N_s) > \epsilon\}$$

is finite branching, so by König's Lemma it either has finite height or it has an infinite branch. The latter is the case if and only if μ has an atom of measure at least ϵ. ⊣

Remark 2.3. We let

$$p_c(2^\omega) = \{f \in p(2^\omega) : \mu_f \text{ is non-atomic}\},$$

which is arithmetical by the above.

We now recall a construction due to Kechris and Sofronidis [12, p. 1463f], which is based on a result of Kakutani [5] regarding the equivalence of product measures. For $x \in 2^\omega$, define $\alpha^x \in [0,1]^\omega$ by

$$\alpha^x(n) = \begin{cases} \frac{1}{4}(1+\frac{1}{\sqrt{n+1}}) & \text{if } x(n)=1 \\ \frac{1}{4} & \text{if } x(n)=0. \end{cases}$$

Then we let $\mu^x \in P(2^\omega)$ be the product measure on 2^ω defined by

$$\mu^x = \prod_{n=0}^{\infty}[\alpha^x(n)\delta_0 + (1-\alpha^x(n))\delta_1]$$

where δ_0, δ_1 are the point measures on $2 = \{0,1\}$. The function $x \mapsto \mu^x$ is continuous. The corresponding map $2^\omega \to p(2^\omega) : x \mapsto f^x$ such that $\mu^x = \mu_{f^x}$ for all x, is given by

$$f^x(s) = \prod_{k=0}^{\mathrm{lh}(s)}[(1-s(k))\alpha^x(k) + s(k)(1-\alpha^x(k))],$$

and is clearly recursive. For $x, x' \in 2^\omega$, let

$$xE_Ix' \iff \sum_{n=0}^{\infty}\frac{|x(n)-x'(n)|}{n+1} < \infty.$$

From Kakutani's theorem we obtain that if xE_Ix' then $\mu^x \approx \mu^{x'}$ and if $\neg(xE_Ix')$ then $\mu^x \perp \mu^{x'}$. (See [12, p. 1463].) For the next lemma it is worth recalling that $\mu, \nu \in P(2^\omega)$ are orthogonal if and only if

$$\neg(\exists \eta \in P(2^\omega))\eta \ll \mu \wedge \eta \ll \nu.$$

Moreover, $\ll$ has the *ccc below* property: For any $\mu \in P(2^\omega)$, any family of orthogonal measures $\ll$ below μ is countable, see e.g. the proof of [12, Theorem 3.1].

LEMMA 2.4. *(a) If $\mu \in P(2^\omega)$ then*

$$\{x \in 2^\omega : \mu^x \perp \mu\}$$

is comeagre.

(b) If (μ_n) is a finite or countable sequence of measures on 2^ω then there is $\nu \in P(2^\omega)$ such that

$$(\forall n)\nu \perp \mu_n$$

and ν is arithmetical in (μ_n).

PROOF. (a) Since

$$\{x \in 2^\omega : \neg\mu^x \perp \mu\}$$

is clearly E_I invariant, if it is non-meagre it must be comeagre. But then there must be an uncountable sequence $x_\alpha \in 2^\omega$, $\alpha < \omega_1$, such that if $\alpha \neq \beta$ then $\neg x_\alpha E_I x_\beta$ and $\mu^{x_\alpha} \not\perp \mu$, contradicting that $\perp$ is ccc below μ.

(b) The set $\{x \in 2^\omega : (\forall n)(\mu^x \perp \mu_n)\}$ is arithmetical in (μ_n), and by (a) it is comeagre. Thus the second claim follows from [10, 4.1.4]. $\dashv$

§3. Proof of the main theorem. In this section we prove Theorem 1.1. It is clearly enough to establish the following:

THEOREM 3.1. *If $V = L$ then there is a Π^1_1 maximal set of orthogonal measures in $P_c(2^\omega)$.*

Then Theorem 1.1 follows by taking a union of a Π^1_1 maximal set of orthogonal measures in $P_c(2^\omega)$ with the set of all point measures (Dirac measures), which clearly is a Π^0_1 set.

Our notation follows that of [6, p. 167ff.], with very few differences. For convenience we recall the definitions and facts that are most important for the present paper.

The canonical wellordering of L will be denoted $<_L$. The language of set theory (LOST) is denoted $\mathcal{L}_\epsilon$. If $x \in 2^\omega$ then we define a binary relation on ω by

$$m \, \epsilon_x \, n \iff x(\langle m, n\rangle) = 1,$$

where $\langle \cdot, \cdot \rangle$ refers to some standard Gödel pairing function of coding a pair of integers by a single integer. We let

$$M_x = (\omega, \epsilon_x),$$

the $\mathcal{L}_\epsilon$ structure coded by x. If M_x is wellfounded and extensional then we denote by $\mathrm{tr}(M_x)$ the transitive collapse of M_x, and by $\pi_x : M_x \to \mathrm{tr}(M_x)$ the corresponding isomorphism.

The following proposition encapsulates the basic descriptive set-theoretic correspondences between x, M_x and the satisfaction relation. We refer to [6, 13.8] and the remarks immediately thereafter for a proof.

PROPOSITION 3.2. *(a) If $\varphi(v_0, \dots, v_{k-1})$ is a LOST formula with all free variables shown then*

$$\{(x, n_0 \dots, n_{k-1}) \in 2^\omega \times \omega \times \cdots \times \omega : M_x \models \varphi[n_0, \dots, n_{k-1}]\}.$$

is arithmetical.

(b) For $x \in 2^\omega$ such that M_x is wellfounded and extensional, the relation

$$\{(m, f) \in \omega \times p(2^\omega) : \pi_x(m) = f\}$$

is arithmetical in x. The same holds if we replace $p(2^\omega)$ with ω^ω, 2^ω, or other reasonable Polish product spaces.

(c) There is a LOST sentence σ_0 such that if $M_x \models \sigma_0$ and M_x is wellfounded and extensional, then $M_x \simeq L_\delta$ for some limit ordinal $\delta < \omega_1$.

(d) There is a LOST formula $\varphi_0(v_0, v_1)$ which defines the canonical wellordering of L_δ for all $\delta > \omega$.

Remark. For $x \in 2^\omega$ and $n_0, n_1 \in \omega$ it will be convenient to write $n_0 <^x_{\varphi_0} n_1$ as an abbreviation of $M_x \models \varphi_0[n_0, n_1]$. By (a) in the previous proposition $n_0 <^x_{\varphi_0} n_1$ is arithmetical uniformly in x.

As motivation for the proof of Theorem 3.1, we first prove the following easier result:

PROPOSITION 3.3. *If $V = L$ then there is a Δ^1_2 maximal set of orthogonal measures.*

PROOF. Work in L. The construction is done by induction on ω_1. We choose a sequence $\langle \mu_\beta : \beta < \omega_1 \rangle$ such that at each $\alpha < \omega_1$ it is the case that μ_α is the $<_L$-least measure such that $\mu_\alpha \perp \mu_\beta$ for all $\beta < \alpha$. That μ_α always exists follows from Lemma 2.4. Then it is easy to see that

$$A = \{\mu_\alpha : \alpha < \omega_1\}$$

is a maximal orthogonal set of measures. To see that A is Δ^1_2, define a relation $P \subseteq p(2^\omega)^{\leq\omega} \times 2^\omega$ by letting $P(s, x)$ if and only if

1. M_x is wellfounded and transitive, $M_x \models \sigma_0$, and for some $m \in \omega$ we have $\pi_x(m) = s$.
2. $\{\mu_{s(n)} : n < \mathrm{lh}(s)\}$ is a set of orthogonal measures.
3. For all $n < \mathrm{lh}(s)$ it holds that $s(n)$ is the $<_L$-smallest code for a measure orthogonal to all $s(k)$ for which $s(k) <_L s(n)$.

Condition (1) is clearly Π^1_1, and (2) is arithmetical. Finally, if (1) and (2) hold then (3) may be expressed by saying

$$(\forall n < \mathrm{lh}(s))(\forall f \in p(2^\omega))(\forall n_1)[(\exists n_0) n_0 <^x_{\varphi_0} n_1 \wedge \pi_x(n_0) = f \wedge \pi_x(n_1) = s(n)]$$
$$\longrightarrow [(\exists l)(\exists l') \neg s(l) \perp f \wedge \pi_x(l') = s(l) \wedge l' <^x_{\varphi_0} n_1].$$

Note that $P(s, x)$ holds if and only if s is a sequence of codes for the measures in some initial segment $\{\mu_\alpha : \alpha < \beta\}$, and that the inductive construction of this initial segment is witnessed in $L_\delta \simeq M_x$, for some limit $\delta < \omega_1$. It then follows that

$$\begin{aligned}\mu \in A &\iff (\exists s)(\exists x)[P(s,x) \wedge (\exists n)\mu_{s(n)} = \mu]\\ &\iff (\forall f)(\forall s)(\forall x)[(P(s,x) \wedge \mu = \mu_f) \longrightarrow\\ &\qquad ((\forall l)s(l) <_L f \vee (\exists l)s(l) = \mu)].\end{aligned}$$

Since the reference to $<_L$ can be replaced by $<^x_{\varphi_0}$, this shows that A is Δ^1_2. ⊣

To prove Theorem 3.1 we will use the technique developed by Miller in [14]. The idea is to replace P in the previous proof with a Π^1_1 relation $\hat{P} \subseteq p_c(2^\omega) \times 2^\omega$ with the property that for all $f \in p_c(2^\omega)$ if

$$(\exists x)\hat{P}(f,x)$$

then f "codes" some witness $x \in 2^\omega$ to this fact, more precisely we will have

$$(\exists x)\hat{P}(f,x) \iff (\exists x \in \Delta^1_1(f))\hat{P}(f,x).$$

Our maximal orthogonal set of measures will then be

$$\hat{A} = \{\mu_f \in P_c(2^\omega) : (\exists x)\hat{P}(f,x)\} = \{\mu_f \in P_c(2^\omega) : (\exists x \in \Delta^1_1(f))\hat{P}(f,x)\},$$

which will be Π^1_1 since $(\exists x \in \Delta^1_1(f))$ may be replaced by a universal quantification, see e.g. [13, 4.19]. What $f \in p_c(2^\omega)$ specifically will code is on the one hand the part of the inductive construction witnessing that $f \in \hat{A}$, and on the other hand $x \in 2^\omega$ such that $M_x \simeq L_\delta$ for some limit $\delta < \omega_1$ in which the inductive construction takes place. We need the following facts for which B. Kastermans' thesis [7] is an excellent reference; see also [9, §3].

Lemma 3.4. *(a) There are unboundedly many limit ordinals $\delta < \omega_1$ such that there is $x \in L_{\delta+\omega} \cap 2^\omega$ such that $M_x \simeq L_\delta$.*

(b) If $M_x \simeq L_\delta$ for some limit $\delta < \omega_1$ then there is $x' \in \Delta^1_1(x)$ such that $M_{x'} \simeq L_{\delta+\omega}$.

Remark. (a) follows from [9, Lemma 3.6], (b) from [9, Lemma 3.5].

Coding a real into a measure. We now describe a way of coding a given real $z \in 2^\omega$ into a measure $\mu \in P_c(2^\omega)$. Given $\mu \in P_c(2^\omega)$ and $s \in 2^{<\omega}$ we let $t(s,\mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_{s^\frown 0}) > 0$ and $\mu(N_{s^\frown 1}) > 0$ if it exists, and otherwise we let $t(s,\mu) = \emptyset$.

Define inductively $t^\mu_n \in 2^{<\omega}$ by letting $t^\mu_0 = \emptyset$ and

$$t^\mu_{n+1} = t(t^\mu_n{}^\frown 0, \mu).$$

Note that since μ is non-atomic we have that $\mathrm{lh}(t^\mu_{n+1}) > \mathrm{lh}(t^\mu_n)$; we let $t^\mu_\infty = \bigcup_{n=0}^\infty t^\mu_n$. For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ we will write t^f_n for $t^{\mu_f}_n$. Clearly the sequence $(t^f_n : n \in \omega)$ is recursive in f. Define $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows:

$$R(f,z) \iff (\forall n \in \omega)[z(n) = 1 \leftrightarrow (f(t^f_n{}^\frown 0) = \frac{2}{3} f(t^f_n) \wedge f(t^f_n{}^\frown 1) = \frac{1}{3} f(t^f_n))$$
$$\wedge\, z(n) = 0 \leftrightarrow f(t^f_n{}^\frown 0) = \frac{1}{3} f(t^f_n) \wedge f(t^f_n{}^\frown 1) = \frac{2}{3} f(t^f_n)]$$

The Coding Lemma 3.5. *Given $z \in 2^\omega$ and $f \in p_c(2^\omega)$ there is $g \in p_c(2^\omega)$ such that $\mu_f \approx \mu_g$ and $R(g,z)$. Moreover, g may be found in a recursive way given f and z: There is a recursive function $G : p_c(2^\omega) \times 2^\omega \to p_c(2^\omega)$ such that $\mu_{G(f,z)} \approx \mu_f$ and $R(G(f,z),z)$ for all $f \in p_c(2^\omega)$, $z \in 2^\omega$.*

Proof. We define $G(f,z)$ inductively. Suppose $G(f,z) \restriction 2^{<n}$ has been defined. Then for $s \in 2^n$ we let

$$G(f,z)(s^\frown i) = \begin{cases} \frac{2}{3} G(f,z)(s) & \text{if } s = t^f_k \text{ for some } k \in \omega, \text{ and} \\ & z(k) = 1, i = 0; \\ \frac{1}{3} G(f,z)(s) & \text{if } s = t^f_k \text{ for some } k \in \omega, \text{ and} \\ & z(k) = 1, i = 1; \\ \frac{1}{3} G(f,z)(s) & \text{if } s = t^f_k \text{ for some } k \in \omega, \text{ and} \\ & z(k) = 0, i = 0; \\ \frac{2}{3} G(f,z)(s) & \text{if } s = t^f_k \text{ for some } k \in \omega, \text{ and} \\ & z(k) = 0, i = 0; \\ 0 & \text{if } f(s) = 0; \\ \dfrac{G(f,z)(s)}{f(s)} f(s^\frown i) & \text{otherwise.} \end{cases}$$

Define for $s \in 2^{<\omega}$

$$\theta(s) = \frac{G(f,z)(s)}{f(s)}$$

whenever $f(s) \neq 0$, and let $\theta(s) = 0$ otherwise. Note that if $x \neq t^f_\infty$ and $s \in 2^{<\omega}$ is the longest sequence such that $s \subseteq x$ and $s \subseteq t^f_\infty$ then $\theta(x \restriction n)$ is constant for $n > \mathrm{lh}(s)$. Let $(s_i : i \in \omega)$ enumerate the set

$$\{s \in 2^{<\omega} : s \not\subseteq t^f_\infty \wedge s \restriction \mathrm{lh}(s) - 1 \subseteq t^f_\infty\}.$$

Then for any Borel set B,

$$\mu_{G(f,z)}(B) = \sum_{i=0}^\infty \theta(s_i) \mu_f(B \cap N_{s_i}),$$

and since $\theta(s_i) = 0$ if and only if $f(s_i) = 0$ this shows that $\mu_{G(f,z)}(B) = 0$ if and only if $\mu_f(B) = 0$. Thus $\mu_g \approx \mu_f$, as required. In fact, if we define

$$\hat{\theta}(x) = \lim_{n\to\infty} \theta(x \upharpoonright n)$$

if $x \neq t^f_\infty$ and $\hat{\theta}(t_\infty) = 0$ then

$$\frac{d\mu_g}{d\mu_f} = \hat{\theta}.$$

Finally, it is clear from the definition of G that $R(G(f,z),z)$, and that G is recursive. ⊣

Remark 3.6. The relation $R(f,z)$ may be read as "f codes z". The set

$$\mathrm{dom}(R) = \{f \in p_c(2^\omega) : (\exists z)R(f,z)\}$$

is Π^0_1 since deciding whether a given $f \in p_c(2^\omega)$ codes *some* $z \in 2^\omega$ only requires us to check for all n that either $f(t^f_n{}^\frown i) = \frac{1}{3}f(t^f_n)$ or $f(t^f_n{}^\frown i) = \frac{2}{3}f(t^f_n)$ holds, for $i = 0$ and $i = 1$. Thus we may define a Π^0_1 function r on $\mathrm{dom}(R)$ by

$$r(\mu) = z \iff R(\mu, z).$$

Proof of Theorem 3.1. Work in L. We first define a maximal set of orthogonal measures by induction on ω_1, and then subsequently see that this set is Π^1_1.

Let $\langle \mu_\alpha : \alpha < \omega_1\rangle$ be the sequence defined in 3.3. We will define a new sequence of measures $\langle \nu_\alpha : \alpha < \omega_1\rangle$ such that $\mu_\alpha \approx \nu_\alpha$, but where the resulting maximal orthogonal set

$$\hat{A} = \{\nu_\alpha : \alpha < \omega_1\}$$

is Π^1_1. Suppose $\langle \nu_\alpha \in P_c(2^\omega) : \alpha < \beta\rangle$ has been defined for some $\beta < \omega_1$. Let $s_0 \in p_c(2^\omega)^{\leq\omega}$ be $<_L$ least such that

$$\{\mu_\alpha : \alpha \leq \beta\} = \{\mu_{s_0(n)} : n \in \mathrm{lh}(s_0)\},$$

and $s_0(0)$ is the $<_L$ largest element of $\{s_0(n) : n \in \mathrm{lh}(s_0)\}$. Let $x_0 \in 2^\omega$ be $<_L$ least such that $M_{x_0} \simeq L_\delta$ for some limit $\delta < \omega_1$ and $s_0 \in L_\delta$, and $x_0 \in L_{\delta+\omega}$. That x_0 exists follows from Lemma 3.4. We let $\nu_\beta = \mu_{G(s_0(0),\langle s_0,x_0\rangle)}$, where $\langle\cdot,\cdot\rangle$ denotes some (fixed) reasonable recursive way of coding a pair $(s,x) \in p_c(2^\omega)^{\leq\omega} \times 2^\omega$ as a single element of 2^ω. Note that $G(s_0(0),\langle s_0,x_0\rangle) \in L_{\delta+\omega}$, since G is recursive.

It is clear that

$$\hat{A} = \{\nu_\alpha : \alpha < \omega_1\}$$

is a maximal orthogonal set of measures. Thus it remains only to see that $\hat{A}$ is Π^1_1.

We first define a relation $Q \subseteq p_c(2^\omega)^{\leq\omega} \times 2^\omega$, similar to P in Proposition 3.3. We let $Q(s,x)$ if and only if

(a) M_x is wellfounded and transitive, $M_x \models \sigma_0$, and for some $m \in \omega$ we have $\pi_x(m) = s$.
(b) $\{\mu_{s(n)} : n < \mathrm{lh}(s)\}$ is a set of orthogonal continuous measures.
(c) For all $n < \mathrm{lh}(s)$ it holds that $s(n)$ is the $<_L$-smallest code for a continuous measure orthogonal to all $s(k)$ for which $s(k) <_L s(n)$.
(d) $(\forall k > 0) s(k) <_L s(0)$.

That the relation Q is Π^1_1 follows as in the proof of Proposition 3.3.

Now define a relation $\hat{P} \subseteq p_c(2^\omega) \times 2^\omega$ by letting $\hat{P}(f,x)$ if and only if

1. M_x is wellfounded and transitive, $M_x \models \sigma_0$, and for some $m \in \omega$ we have $\pi_x(m) = f$.
2. $f \in \mathrm{dom}(R)$, $r(f) = \langle s, w\rangle$ for some $(s,w) \in p_c(2^\omega)^{\leq\omega} \times 2^\omega$, and $Q(s,w)$.
3. $(\forall s' \in p_c(2^\omega)^{\leq\omega})[\{s(n) : n \in \mathrm{lh}(s)\} = \{s'(n) : n \in \mathrm{lh}(s')\} \wedge s(0) = s'(0)] \longrightarrow \neg(s' <_L s)$
4. If $M_w \simeq L_{\delta'}$ then $w \in L_{\delta'+\omega}$, and w is $<_L$ least such that this holds and for some $m \in \omega$, $\pi_w(m) = s$.
5. $f = G(s(0), \langle s, w\rangle)$.

Conditions (1) and (2) are Π^1_1 and (5) is Π^0_1. If (1) and (2) hold then (3) is equivalent to

$$(\forall s' \in p_c(2^\omega)^{\leq\omega})((\forall l)(\exists l')s(l) = s'(l') \wedge (\forall l)(\exists l')s'(l) = s(l') \wedge s(0) = s'(0)) \\ \longrightarrow ((\forall n_0)(\forall n_1)(\pi_x(n_0) = s \wedge \pi_x(n_1) = s' \longrightarrow \neg n_1 <^x_{\varphi_0} n_0)).$$

which is a Π^1_1 predicate.

To verify that (4) is a Π^1_1 condition, define as in [6, p. 170] the restriction $M_x \restriction k$, for $x \in 2^\omega$ and $k \in \omega$, to be the $\mathcal{L}_\epsilon$ structure

$$M_x \restriction k = (\{n : n \in_x k\}, \in_x).$$

Assuming that (1)–(3) hold (4) is equivalent to the conjunction of the following two conditions:

$$(\forall k)((M_x \restriction k \models \sigma_0 \wedge (\exists l) l \in_x k \wedge M_x \restriction l \simeq M_w) \longrightarrow \\ ((\exists n_0, n_1) n_0 \in_x k \wedge n_1 \in_x k \wedge \pi_x(n_0) = s \wedge \pi_x(n_1) = w))$$

and

$$(\forall w')(\forall k)((M_{w'} \simeq M_x \restriction k \wedge M_{w'} \models \sigma_0 \wedge w' <_L w) \longrightarrow$$
$$(\forall n_0, n_1)((n_0 \in_x k \wedge n_1 \in_x k) \longrightarrow \pi_x(n_0) \neq s \wedge \pi_x(n_1) \neq w')),$$

where $\simeq$ denotes isomorphism between $\mathcal{L}_\epsilon$ structures. Since $\simeq$, which is Σ^1_1, only occurs on the left-hand side of the above implications, and since $<_L$ may be replaced by using $<^x_{\varphi_0}$ as before, this shows that (4) may be replaced by a Π^1_1 predicate, which proves that $\hat{P}$ is a Π^1_1 relation.

It is clear from the definition of $\hat{P}$ that

$$\hat{A} = \{\mu_f \in P_c(2^\omega) : (\exists x)\hat{P}(f, x)\}.$$

To see that $\hat{A}$ is Π^1_1, suppose that $\mu_f \in \hat{A}$. Then $\hat{P}(f, x)$ for some x, and so $f \in \mathrm{dom}(R)$ and $r(f) = \langle s, w\rangle$ where $M_w \simeq L_{\delta'}$. By Lemma 3.4 there is $w' \in \Delta^1_1(w)$ such that $M_{w'} \simeq L_{\delta'+\omega}$ and by condition (4) above $w \in L_{\delta'+\omega}$. But then $\hat{P}(f, w')$ holds. Thus we have shown that

$$(\exists x)\hat{P}(f, x) \iff (\exists x \in \Delta^1_1(f))\hat{P}(f, x),$$

which proves that $\hat{A}$ is Π^1_1. ⊣

§4. **Final remarks.** In this final section we consider two natural questions: What is the cardinality of a maximal orthogonal family of measures? And is it consistent that there is no co-analytic maximal orthogonal family of measures?

Both questions can be answered using the product measure construction of Kechris and Sofronidis described in §2. Any maximal orthogonal family in $P(2^\omega)$ must have size $\mathfrak{c}$ since there are $\mathfrak{c}$ many point measures. But even if we only consider non-atomic measures we reach the same conclusion:

PROPOSITION 4.1. *Let $A \subseteq P_c(2^\omega)$, $|A| < \mathfrak{c}$, be a set of pairwise orthogonal measures. Then A is not maximal orthogonal in $P_c(2^\omega)$. In fact, there is a product measure which is orthogonal to all elements of A.*

PROOF. Suppose $A = \{\nu_\alpha : \alpha < \kappa\}$, where $\kappa < \mathfrak{c}$, is an orthogonal family. Since E_I (as defined in §2) has meagre classes, it follows by Mycielski's Theorem (see e.g. [2]) that there are perfectly many E_I classes, and so we can find a sequence $(\mu_\alpha : \alpha < \mathfrak{c})$ of orthogonal product measures. For each ν_α there can be at most countably many $\beta < \mathfrak{c}$ such that

$$\mu_\beta \not\perp \nu_\alpha$$

since $\ll$ is ccc below ν_α (see e.g. the proof of [12, Theorem 3.1]). Since $\kappa < \mathfrak{c}$ it follows that there must be some $\beta < \mathfrak{c}$ such that μ_β is orthogonal to all elements of A. $\dashv$

PROPOSITION 4.2. *If there is a Cohen real over L then there is no Π^1_1 maximal orthogonal set of measures.*

PROOF. We will use the following result of Judah and Shelah [4]: If there is a Cohen real over L then every Δ^1_2 set of reals is Baire measurable.

Suppose $A \subseteq P(2^\omega)$ is a Π^1_1 maximal orthogonal set of measures. Then define a relation $Q \subseteq 2^\omega \times P(2^\omega)^\omega$ by

$$Q(x,(\nu_n)) \iff (\forall n)(\nu_n \in A \wedge \nu_n \not\perp \mu^x) \wedge (\forall \mu)(\mu \not\perp \mu^x \longrightarrow (\exists n)\nu_n \not\perp \mu)$$

Then for each $x \in 2^\omega$ the section Q_x is non-empty since A is maximal. Using Π^1_1 uniformization, we obtain a function $f : 2^\omega \to P(2^\omega)^\omega$ having a Π^1_1 graph and such that

$$(\forall x)Q(x, f(x)).$$

Now if $U \subseteq P(2^\omega)^\omega$ is a basic open set then

$$\begin{aligned} x \in f^{-1}(U) &\iff (\exists (\nu_n) \in P(2^\omega)^\omega) f(x) = (\nu_n) \wedge (\nu_n) \in U \\ &\iff (\forall (\nu_n) \in P(2^\omega)^\omega) f(x) \neq (\nu_n) \vee (\nu_n) \in U. \end{aligned}$$

Thus $f^{-1}(U)$ is Δ^1_2, and so if there is a Cohen real over L then it has the property of Baire. It follows that f is a function with the Baire property. But then we may argue just as in Kechris and Sofronidis' proof that no analytic set of measure is maximal orthogonal and arrive at a contradiction: Indeed, f is an E_I-invariant assignment of countable subsets of $P(2^\omega)$, and E_I is a turbulent equivalence relation, and so we must have that

$$x \mapsto A(x) = \{f(x)(n) : n \in \omega\}$$

is constant on a comeagre set. This contradicts that

$$(\forall x, x' \in 2^\omega)\neg(x E_I x') \Longrightarrow \mu^x \perp \mu^{x'}$$

and the ccc-below property of $\ll$. $\dashv$

The natural relativization of Proposition 4.2 gives us that if for every $x \in 2^\omega$ there is a Cohen real over $L[x]$ then there is no co-analytic (i.e., boldface $\mathbf{\Pi}^1_1$) maximal set of orthogonal measures. We do not know what happens with the complexity of maximal orthogonal sets if we add other types of reals. In fact, we do not know the answer to the following:

QUESTION 4.3. If there is a Π^1_1 maximal orthogonal set of measures, are all reals constructible?

REFERENCES

[1] F. R. Drake, ***Set theory, an introduction to large cardinals***, North-Holland Publishing Company, 1974.

[2] S. Gao, ***Invariant descriptive set theory***, Taylor and Francis, 2008.

[3] S. Gao and Y. Zhang, *Definable sets of generators in maximal cofinitary groups*, ***Advances in Mathematics***, vol. 217 (2008), no. 2, pp. 814–832.

[4] H. Judah and S. Shelah, Δ^1_2*-sets of reals*, ***Annals of Pure and Applied Logic***, vol. 42 (1989), no. 3, p. 207223.

[5] S. Kakutani, *On equivalence of infinite product measures*, ***Annals of Mathematics***, vol. 49 (1948), pp. 214–226.

[6] A. Kanamori, ***The higher infinite***, Springer Verlag, 1997.

[7] B. Kastermans, *Cofinitary groups and other almost disjoint families*, ***Ph.D. thesis***, University of Michigan, 2006, http://www.bartk.nl/Files/Papers/thesis.pdf.

[8] ———, *The complexity of maximal cofinitary groups*, ***Proceedings of the American Mathematical Society***, vol. 137 (2009), no. 1, pp. 307–316.

[9] B. Kastermans, J. Steprans, and Y. Zhang, *Analytic and coanalytic families of almost disjoint functions*, ***The Journal Symbolic Logic***, vol. 73 (2008), no. 4, pp. 1158–1172.

[10] A. Kechris, *Measure and category in effective descriptive set theory*, ***Annals of Pure and Applied Logic***, vol. 5 (1973), pp. 337–384.

[11] ———, ***Classical descriptive set theory***, Graduate Texts in Mathematics, no. 156, Springer-Verlag, 1995.

[12] A. Kechris and N. E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators*, ***Ergodic Theory and Dynamamical Systems***, vol. 21 (2001), pp. 1459–1479.

[13] R. Mansfield and G. Weitkamp, ***Recursive aspects of descriptive set theory***, Oxford University Press, 1985.

[14] A. W. Miller, *Infinite combinatorics and definability*, ***Annals of Pure and Applied Logic***, vol. 41 (1989), pp. 179–203.

[15] Y. N. Moschovakis, ***Descriptive set theory***, North Holland, 1980.

[16] D. Preiss and J. Rataj, *Maximal sets of orthogonal measures are not analytic*, ***Proceedings of the American Mathematical Society***, vol. 93 (1985), pp. 471–476.

KURT GÖDEL RESEARCH CENTER, UNIVERSITY OF VIENNA, WÄHRINGER STRASSE 25, 1090 VIENNA, AUSTRIA
E-mail: vfischer@logic.univie.ac.at

KURT GÖDEL RESEARCH CENTER, UNIVERSITY OF VIENNA, WÄHRINGER STRASSE 25, 1090 VIENNA, AUSTRIA
E-mail: asger@logic.univie.ac.at