PRESERVATION OF ${}^{\omega}\omega$-BOUNDING PROPERTY

VERA FISCHER

1. Preliminaries

Recall the following definitions:

Definition 1. We say that the partial order P is a projection of the partial order Q and denote this by $P \lhd Q$, if there is an onto mapping $\pi : Q \to P$ which is order preserving and such that

$$\forall q \in Q \forall p \in P \text{ s.t. } \pi(q) \leq p \text{ there is } q' \in Q\ (q \leq_Q q') \wedge (\pi(q) = p).$$

Furthermore whenever $\pi(q) \leq p$ there is a condition q_1 in Q which is usually denoted $p + q$ such that $q \leq q_1$ and for every $r \in Q$ such that $p \leq \pi(r)$ and $q \leq r$ we have $q_1 \leq r$.

The notion of projection is closely related to the notion of two-step iteration. Suppose that $P \lhd Q$ and let G be a P-generic filter. Then in $V[G]$ define $Q/G = \{q \in Q : \pi(q) \in G\}$ with extension relation defined in the following way: for $q_1, q_2 \in Q/G$ let

$$q_1 \leq_{Q/G} q_2 \text{ iff } \exists g \in G \text{ s.t. } q_1 \leq_Q g + q_2.$$

Since the partial order Q/G is defined in a P-generic extension we can fix a P-name for it, say $\dot{Q}$. Now in the ground model we can consider the two step iteration $P * \dot{Q}$. Then the original partial order Q is densely embedded in $P * \dot{Q}$ and so we can consider forcing with Q as two step iteration: forcing by P followed by forcing with the quotient poset Q/G where G is a P-generic filter (sometimes we denote the P-name for the quotient poset also Q/P). Note that if H is a Q-generic filter and $G = \pi"H$ then $H \subseteq Q/G$ is also a Q/G-generic filter. For more on quotient forcing see [3] and [2].

2. Preservation of the Bounding Property

In the following functions from ω to ω will be called reals and names for functions in ${}^{\omega}\omega$ will also be referred to as names for reals. Recall that $<^* = \cup_{n\in\omega} \leq_n$ is the bounding relation (also called the dominating relation) on the reals, where we say that $f \leq_n g$ if for every $k \geq$

Date: December 7, 2005.

$n(f(k) \leq g(k))$. Furthermore if $f \leq_0 g$ we say that f is absolutely dominated by g.

Definition 2. We say that the family $D \subseteq {}^\omega\omega$ is dominating if for every real f there is some d in D such that $f <^* g$. The dominating number d is defined to be the minimal size of a dominating family.

In this talk we will consider a class of forcing notion which have the property that they do not increase the dominating number.

Definition 3. A forcing poset $\mathbb{P}$ is said to be ${}^\omega\omega$-bounding if for every generic filter G the ground model reals form a dominating family in the generic extension. That is for every $\mathbb{P}$-name $\dot{f}$ of a real and every condition $p \in \mathbb{P}$ there is an extension $q \geq p$ and a ground model function g such that $q \Vdash \dot{f} <^* g$. Note that we can require $q \Vdash \dot{f} \leq_0 g$.

Definition 4. Let $\mathbb{P}$ be a forcing poset and $\dot{f}$ a $\mathbb{P}$-name for a real. An increasing sequence $\bar{p} = \langle p_i : i \in \omega \rangle$ of conditions in $\mathbb{P}$ is said to interpret $\dot{f}$ as $f^* \in {}^\omega\omega$ if for every $i \in \omega$ $p_i \Vdash \dot{f} \restriction i = f^* \restriction i$. We denote the function f^* by $\text{intp}(\bar{p}, \dot{f})$. The sequence $\bar{p}$ is said to respect the function g if $\text{intp}(\bar{p}, \dot{f}) \leq_0 g$.

Theorem 1. Let $\mathbb{P}$ be an ${}^\omega\omega$-bounding poset, $\dot{f}$ a $\mathbb{P}$-name for a real, $\bar{p} = \langle p_i : i \in \omega \rangle$ an increasing sequence of conditions which interprets $\dot{f}$. Let $\mathcal{M}$ be a countable elementary submodel of H_κ for some sufficiently large κ such that $\mathbb{P}, \dot{f}, \bar{p} \in \mathcal{M}$. Furthermore let $g \in {}^\omega\omega$ be a real which dominates the reals of $\mathcal{M}$ and such that the sequence $\bar{p}$ respects g. Then there is a condition $s \in \mathcal{M} \cap \mathbb{P}$ such that $s \Vdash \dot{f} \leq_0 g$.

Proof. Since the forcing notion $\mathbb{P}$ is ${}^\omega\omega$-bounding,

$$H_\kappa \vDash \forall i \in \omega \exists p'_i \geq p_i \exists h_i \in {}^\omega\omega (p'_i \Vdash \dot{f} \leq_0 h_i).$$

However $\mathcal{M}$ is a countable elementary submodel of H_κ and so we can fix a sequence $\langle p'_i : i \in \omega \rangle$ of conditions in $\mathcal{M} \cap \mathbb{P}$ and a family $\langle h_i : i \in \omega \rangle$ of reals in $\mathcal{M} \cap {}^\omega\omega$ such that $\forall i \in \omega (p'_i \geq p_i) \wedge (p'_i \Vdash \dot{f} \leq_0 h_i)$. Since p'_i is an extension of p_i, and p_i forces that $\dot{f} \restriction i = f^* \restriction i$ where $f^* = \text{intp}(\bar{p}, \dot{f})$ we can assume that $h_i \restriction i = f^* \restriction i$. Thus consider the function

$$u(m) = \max\{h_i(m) : i \leq m\} \text{ for every } m \in \omega.$$

Note that $u \in \mathcal{M} \cap {}^\omega\omega$ and so in particular $u <^* g$. Say $u \leq_l g$ for some $l \in \omega$. We claim that p'_l is the desired condition. Notice that $h_l \leq_0 g$: if $k < l$ then $h_l(k) = f^*(k)$ by construction and since $f^*(k) \leq_0 g(k)$ we obtain $h_l(k) \leq g(k)$; if $l \leq k$ then $h_l(k) \leq u(k)$ by definition of u and $u(k) \leq g(k)$ since $u \leq_l g$. However $p'_l \Vdash \dot{f} \leq_0 h_l$ and so $h_l \leq_0 g$ implies that $p'_l \Vdash \dot{f} \leq_0 g$. □

Definition 5. Let $P \lhd Q$ with projection π, $\dot{f}$ a Q-name for a real and $\bar{r} = \langle r_i : i \in \omega \rangle$ a Q_2-increasing sequence which interprets $\dot{f}$. Let G be a P-generic filter. Inductively define a sequence $\bar{s} = \langle s_i : i \in \omega \rangle$ as follows:

(1) if $\pi(r_i) \in G$ let $s_i = r_i$,
(2) if $\pi(r_i) \notin G$ let s_{i-1} be the first condition in Q (under some fixed well-order on Q) which extends s_{i-1} and $\pi(s_i) \in G$.

The sequence $\bar{s}$ is contained in Q/G and is called the derived sequence. Since $\bar{s}$ is obtained in a P-generic extension it has a P-name which we denote by $\dot{\delta}_P(\bar{r}, \dot{f})$. If G is a P generic filter the evaluation of this name is also sometimes denoted by $\delta_G(\bar{r}, \dot{f})$.

Lemma 1. *Let $Q_1 \lhd Q_2$ where Q_1 an ${}^{\omega}\omega$-bounding forcing notion. Let $\dot{f}$ be a Q_2-name, $\bar{r}$ a Q_2-increasing sequence which interprets $\dot{f}$, $p \in Q_2$ such that $\bar{r}$ is above p in the Q_2-ordering. Let $\mathcal{M}$ be a countable elementary submodel of H_k such that $Q_1, Q_2, \dot{f}, \bar{r}, p \in \mathcal{M}$. Furthermore let g be a function which dominates the reals of $\mathcal{M}$ and such that $\bar{r}$ respects g. Then there is a condition $s \in Q_1 \cap \mathcal{M}$ such that $\pi(p) \leq s$*

$$s \Vdash \textit{intp}(\dot{\delta}_{Q_1}(\bar{r}, \dot{f}), \dot{f}) \leq_0 g \textit{ and } s \Vdash p \leq_{Q_2} \dot{\delta}_{Q_1}(\bar{r}, \dot{f})(0).$$

Proof. Let G_1 be a Q_1-generic filter and $\delta = \delta_{G_1}(\bar{r}, \dot{f})$ the derived sequence. Let h^* be the interpretation of the derived sequence of $\dot{f}/G_1$ and $\dot{h}$ the Q_1-name of this real. Let $\bar{p} = \langle p_i : i \in \omega \rangle$ where $p_i = \pi(r_i)$ for $\bar{r} = \langle r_i : i \in \omega \rangle$. Then $p_i \Vdash \pi(r_i) \in \dot{G}_1$ and so $p_i \Vdash \dot{\delta}(i) = r_i$. Then $p_i \Vdash \dot{h} \restriction i = f^* \restriction i$ where $f^* = \mathrm{intp}(\dot{f}, \bar{r})$. Therefore

$$\mathrm{intp}(\bar{p}, \dot{h}) = \mathrm{intp}(\bar{r}, \dot{f})$$

and so $\mathrm{intp}(\bar{p}, \dot{h}) \leq_0 g$. By Theorem 1 there is $s \in Q_1 \cap \mathcal{M}$ such that $s \Vdash \dot{h} \leq_0 g$. That is

$$s \Vdash \mathrm{intp}(\dot{\delta}_{Q_1}(\bar{r}, \dot{f}), \dot{f}) \leq_0 g.$$

Furthermore $s \geq p_0 = \pi(r_0)$ and so $s \Vdash \pi(r_0) \in \dot{G}_1$ which implies that the first element of the derived sequence is r_0 and so is above p in the Q_2-ordering. Note that this implies that the entire derived sequence is above p in the Q_2-ordering. □

Lemma 2. *If $P \lhd Q$ and Q is proper, then P is proper.*

Proof. Let $p \in P \cap \mathcal{M}$ for $\mathcal{M}$ countable elementary submodel of H_κ for κ sufficiently large with $P, Q \in \mathcal{M}$. We have to show that there is $p' \geq p$ which is $(\mathcal{M}, P)$-generic. Identify p with $p + 0_q$. Since Q is proper there is $(\mathcal{M}, Q)$-generic condition q which extends p. Then $p \leq \pi(q)$

and it is sufficient to show that $\pi(q)$ is $(\mathcal{M}, P)$-generic. Let D be a dense subset of P which belongs to $\mathcal{M}$. Then $D' = \{q \in Q : \pi(q) \in D\}$ is a dense subset of Q which belongs to $\mathcal{M}$. Let G be a P-generic filter containing $\pi(q)$. There is a Q-generic filter H which contains q and such that $\pi"H = G$. Since q is $(\mathcal{M}, Q)$-generic there is some $x \in D' \cap \mathcal{M} \cap H$. But then $\pi(x) \in D \cap M \cap G$ and so in particular $D \cap \mathcal{M} \cap G$ is nonempty. Since D was arbitrary this proves that $\pi(q)$ is an $(\mathcal{M}, P)$-generic condition. □

Lemma 3. *Let P be a proper, ${}^\omega\omega$-bounding poset, $\mathcal{M}$ countable elementary submodel of H_κ and g a real which dominates $\mathcal{M} \cap {}^\omega\omega$. Let q be $(\mathcal{M}, P)$-generic condition and G a P-generic filter containing q. Then the function g dominates $\mathcal{M}[G] \cap {}^\omega\omega$.*

Proof. Let $\dot{f} \in \mathcal{M} \cap V^P$ be a name for a real. Since P is ${}^\omega\omega$-bounding

$$H_\kappa[G] \vDash \exists h \in {}^\omega\omega \cap V(\dot{f}[G] <^* h).$$

However $\mathcal{M}[G]$ is an elementary submodel $H_\kappa[G]$ and so

$$\mathcal{M}[G] \vDash \exists h \in {}^\omega\omega \cap (\mathcal{M}[G] \cap V)(\dot{f}[G] <^* h).$$

But q is $(\mathcal{M}, P)$-generic and so $q \Vdash \mathcal{M}[\dot{G}] \cap V = \mathcal{M} \cap V$. Therefore

$$\mathcal{M}[G] \vDash \exists h \in {}^\omega\omega \cap (\mathcal{M} \cap V)(\dot{f}[G] <^* h).$$

Fix any such h. But then h belongs to $\mathcal{M}$ and so h is dominated by g. This implies that $(\dot{f}[G] <^* g)^{V[G]}$. □

Lemma 4. *If $P \lhd Q$ and Q is ${}^\omega\omega$-bounding, then P is ${}^\omega\omega$-bounding.*

Proof. Suppose P is not ${}^\omega\omega$-bounding. Then there is a P-generic filter G such that the ground model reals do not form a dominating family in $V[G] \cap {}^\omega\omega$. That is there is a P-name $\dot{f}$ for a real such that $\dot{f}[G]$ is not bounded by any ground model real. Thus if H is Q-generic filter with $\pi"H = G$, the real $\dot{f}[H]$ (which is equal to $\dot{f}[G]$) is not dominated by any ground model real, which is a contradiction to Q being ${}^\omega\omega$-bounding. □

Lemma 5. *Let $Q_0 \lhd Q_1 \lhd Q_2$ where Q_1 is proper and ${}^\omega\omega$-bounding. Let $\dot{f}$ be a Q_2-name for a real, $\mathcal{M}$ countable elementary submodel of H_κ for some sufficiently large κ such that $Q_0, Q_1, Q_2, \dot{f} \in \mathcal{M}$. Furthermore let*

1. *q_0 be $(\mathcal{M}, P)$-generic condition, $g \in {}^\omega\omega$ such that ${}^\omega\omega \cap \mathcal{M} <^* g$*
2. *$\dot{p} \in V^{Q_0}$ such that $q_0 \Vdash \dot{p} \in Q_2/G_0 \cap \mathcal{M}$*
3. *q_0 forces that in $M[G_0]$ there is a Q_2-increasing sequence $\bar{r} = \langle r_i : i \in \omega\rangle$ of conditions in Q_2/G_0 which is above $\dot{p}[G_0]$ in Q_2-ordering, interprets $\dot{f}$ and respects g.*

Then there is $(\mathcal{M}, Q_1)$*-generic condition* q_1 *such that* $\pi_{1,0}(q_1) = q_0$*,* $q_1 \Vdash \pi_{2,1}(\dot{p}) \in \dot{G}_1$ *and furthermore* q_1 *forces that in* $M[G_1]$ *there is a* Q_2*-increasing sequence* $\bar{r} = \langle r_i : i \in \omega\rangle$ *of conditions in* Q_2/G_1 *which is above* $\dot{p}$ *in* Q_2*-ordering, interprets* $\dot{f}$ *and respects* g*.*

Proof. Note that by Lemma 2 the forcing notion Q_0 is proper and by Lemma 4 also ${}^\omega\omega$-bounding. Let G_0 be (V, Q_0)-generic with $q_0 \in G_0$. Then in $V[G_0]$ we can evaluate $\dot{p}[G_0]$. Furthermore by assumption (3) in $\mathcal{M}[G_0] \cap Q_2/G_0$ there is a Q_2-increasing sequence $\bar{r}$, which is above $\dot{p}[G_0]$, interprets $\dot{f}$ and respects g. Since q_0 is $(\mathcal{M}, Q_0)$-generic by Lemma 3 $\mathcal{M}[G_0] \cap {}^\omega\omega$ is dominated by g. But then all the assumptions of Lemma 1 hold in $V[G_0]$ for the partial orders Q_1/G_0 and Q_2/G_0. That is $Q_1/G_0 \lhd Q_2/G_0$, $\dot{f}/G_0$ is Q_2/G_0-name for a real, $\dot{p}[G_0] \in Q_2/G_0 \cap \mathcal{M}[G_0]$, the reals of $\mathcal{M}[G_0] \cap {}^\omega\omega$ are dominated by g and all of $\dot{f}/G_0$, Q_1/G_0, Q_2/G_0, $\bar{r}$, $p = p[G_0]$ belong to $\mathcal{M}[G_0]$. Therefore there is $s \in Q_1/G_0 \cap \mathcal{M}[G_0]$ such that

$$s \Vdash_{Q_1/G_0} \mathrm{intp}(\delta_{Q_1/G_0}(\bar{r}, \dot{f}/G_0), \dot{f}/G_0) \leq_0 g \text{ and } s \Vdash_{Q_1/G_0} \dot{p} \leq_{Q_2/G_0} \dot{\delta}(0).$$

Let $\dot{s}$ be a Q_0-name for s. Then in particular $q_0 \Vdash \pi_{1,0}(\dot{s}) \in \dot{G}_0$ and so by the Properness Extension Lemma there is $(\mathcal{M}, Q_1)$-generic condition q_1 such that $q_1 \Vdash \dot{s} \in \dot{G}_1$ and $\pi_{1,0}(q_1) = q_0$. Let G_1 be a (V, Q_1)-generic filter containing g_1 and let $G_0 = \pi_{1,0}"G_1$. Note that $G_1 \subset Q_1/G_0$ is also a Q_1/G_0-generic filter. However $s = \dot{s}[G_0] \in G_1$ and so $V[G_1]$ satisfies everything that s forces: the derived sequence $\bar{p} = \langle p_n : n \in \omega\rangle$ is Q_2/G_0-increasing, contained in $Q_2/G_1 \cap \mathcal{M}[G_1]$ and is above $p = \dot{p}[G_0]$ in the Q_2/G_0-ordering. We will define inductively a sequence $\langle g_n + p_n : n \in \omega\rangle$ which is contained in $\mathcal{M}[G_1] \cap Q_2/G_1$, which is Q_2-increasing and is above $p = \dot{p}[G_0]$ in the Q_2-ordering, interprets $\dot{f}$ and respects g.

Since $p_n \Vdash_{Q_2/G_0} \dot{f}/G_0 \restriction n = e_n$ for some finite function e_n, there is $g'_n \in G_0$ such that $g'_n + p_n \Vdash \dot{f} \restriction n = e_n$. Since $\mathcal{M}[G_1] \prec H_\kappa[G_1]$ for every $i \in \omega$ we can fix a condition $g'_n \in \mathcal{M}[G_1] \cap G_0$ with the above properties. Consider the following inductive construction. Since $p \leq_{Q_2/G_0} p_1$ there is a condition $g \in G_0$ such that $p \leq_{Q_2} g + p_1$ and again since $\mathcal{M}[G_1] \prec H_\kappa[G_1]$ we can obtain such a condition g in $\mathcal{M}[G_1]$. Then for g_0 a common extension of g, g'_0 in $\mathcal{M}[G_1] \cap G_0$ the condition $g_0 + p_0$ extends p in Q_2-ordering and forces (in Q_2-ordering) that $\dot{f} \restriction 0 = e_0$. Proceed inductively. Suppose g_n has been defined. Then let g_{n+1} be any common extension of g'_{n+1}, g_n and g which belongs to $\mathcal{M}[G_1] \cap G_0$, where g is a condition in $\mathcal{M}[G_1] \cap G_0$ with $p_n \leq_{Q_2} g + p_{n+1}$. Then $g_n + p_n \leq_{Q_2} g_{n+1} + p_{n+1}$ and $g_{n+1} + p_{n+1} \Vdash \dot{f} \restriction n + 1 = e_{n+1}$. □

Theorem 2. Let $\langle \mathbb{P}_i : i \leq \delta \rangle$ be a countable support iteration of proper, ${}^\omega\omega$-bounding posets. Then $\mathbb{P}_\delta$ is proper and ${}^\omega\omega$-bounding.

Proof. The proof is by induction on δ. For δ successor the result is straightforward. So, we can assume that δ is a limit. Furthermore we can assume that $P_0 = \{0\}$ is the trivial poset. Suppose that $\dot{f}$ is a $\mathbb{P}_\delta$-name for a real and let $p \in \mathbb{P}$ be arbitrary condition in $\mathbb{P}$. We have to show that there is a condition $q \geq p$ such that for some ground model function g $q \Vdash \dot{f} \leq_0 g$.

Let $\mathcal{M}$ be a countable elementary submodel of H_κ for some sufficiently large κ which contains $\mathbb{P}_\delta, \dot{f}, p$. Inductively construct an increasing sequence $\bar{r} = \langle r_i : i \in \omega \rangle$ of conditions in $\mathbb{P}_\delta \cap \mathcal{M}$ which interprets $\dot{f}$. Let g be a function dominating the reals of $\mathcal{M}$ and such that $\bar{r}$ respects g.

Let $\{g_n\}_{n\in\omega}$ be a cofinal, increasing sequence in $\mathcal{M} \cap \delta$. Inductively we will construct sequences $\langle p_n : n \in \omega \rangle$, $\langle \dot{q}_n : n \in \omega \rangle$ such that

(1) $q_0 = 0$ and q_n is $(\mathcal{M}, \mathbb{P}_{\gamma_n})$-generic, such that $q_{\gamma_{n+1}} \restriction \gamma_n = q_{\gamma_n}$
(2) $p_0 = p$ and $\dot{p}_n$ is a $\mathbb{P}_{\gamma_n}$-name such that

$$q_{\gamma_n} \Vdash_{\gamma_n} \dot{p} \in \mathbb{P}_\delta \cap \mathcal{M} \wedge \dot{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n} \wedge \dot{p}_{n-1} \leq_\delta \dot{p}_n$$

(3) $q_n \Vdash_{\gamma_n} (\dot{p}_n \Vdash_\delta \dot{f} \restriction n \leq_0 g \restriction n)$
(4) q_{γ_n} forces that in $M[\dot{G}_{\gamma_n}]$ there is a $\mathbb{P}_\delta$-increasing sequence contained in $\mathbb{P}_\delta/\mathbb{P}_{\gamma_n}$, which is above $\dot{p}_n[\dot{G}_{\gamma_n}]$ in $\mathbb{P}_\delta$-ordering, interprets $\dot{f}$ and respects g.

Suppose we have succeeded in this inductive construction. Let $q = \cup_{n\in\omega} q_n$. Just as in the proof of the Properness Extension Lemma one obtains that $q \Vdash_\delta \dot{p}_n \in \dot{G}_\delta$ and so by (3) $q \Vdash_\delta \dot{f} \leq_0 g$.

For $n = 0$ the conditions $(1)-(4)$ hold. Suppose we have constructed q_n and $\dot{p}_n$. Let G be any $\mathbb{P}_{\gamma_n}$ generic filter containing q_n. Then by (4) in $\mathcal{M}[G_{\gamma_n}]$ there is a $\mathbb{P}_\delta$ increasing sequence $\bar{r}$ of conditions in $\mathbb{P}_\delta/G$ which is above $\dot{p}_n$ in $\mathbb{P}_\delta$-ordering, interprets $\dot{f}$ and respects g. Let $\dot{p}_{n+1}$ be the $\mathbb{P}_{\gamma_n}$-name for the $(n+1)$th element of $\bar{r}$. To obtain q_{n+1} apply Lemma 5 to $\mathbb{P}_{\gamma_n}, \mathbb{P}_{\gamma_{n+1}}$, $\mathbb{P}_\delta$, q_n and $\dot{p}_n$. □

The proof discussed above is very similar to the proofs of the preservation of properness and the preservation of the weakly bounding property under countable support iterations. For general preservation theorems see [3] and [4].

References

[1] U. Abraham *Proper Forcing,* for the Handbook of Set-Theory.

[2] J. Baumgartner *Iterated Forcing,* in: Surveys in Set Theory (A.R.D. Mathias, editor), London Mathematical Society Lecture Notes Series, No. 8, Cambridge University Press, Cambridge 1983.

[3] M. Godstern *Tools for your forcing constructions,* In Set Theory of the Reals, vol.6 of Israel Mathematical Conference Proceedings, 305-360

[4] S. Shelah *Proper and Improper Forcing,* Second Edition. Springer, 1998.

E-mail address: vfischer@mathstat.yorku.ca