PRESERVATION OF PROPERNESS UNDER COUNTABLE SUPPORT ITERATION

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1. Preliminaries on Generic Conditions

If $\leq$ is a preorder on a set P and $p_0 \leq p_1$, we say that p_1 is an extension of p_0. Recall that a preorder is separative if and only if whenever p_1 is not an extension of p_0 there is an extension of p_1 which is incompatible with p_0. We say that $\mathbb{P} = (P, \leq)$ is a forcing notion (also forcing poset) if $\leq$ is a separative preorder with minimal element $0_{\mathbb{P}}$. Note that if $\mathbb{P}$ is separative and $p_1 \Vdash \check{p}_0 \in \dot{G}$ then $p_0 \leq p_1$ (here $\dot{G}$ is the canonical name of the $\mathbb{P}$-generic set). Also often in forcing formulas we write a instead of $\check{a}$ for an element a of the ground model V.

Definition 1. Let $\mathbb{P}$ be a forcing notion, $\lambda > 2^{|\mathbb{P}|}$ and $\mathcal{M}$ countable elementary submodel of $H(\lambda)$ with $\mathbb{P} \in \mathcal{M}$. We say that $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic iff for every dense subset D of $\mathbb{P}$ which belongs to $\mathcal{M}$ the set $D \cap \mathcal{M}$ is predense above q.

Definition 2. The forcing notion $\mathbb{P}$ is called proper iff $\forall \lambda > 2^{|\mathbb{P}|}$ and every countable elementary submodel $\mathcal{M}$ of $H(\lambda)$ such that $\mathbb{P} \in \mathcal{M}$, every condition $p \in \mathbb{P} \cap \mathcal{M}$ has an $(M, \mathbb{P})$-generic extension.

We will use the following characterizations of $(M, \mathbb{P})$-generic conditions.

Lemma 1. *Let $\mathbb{P}$ be a forcing notion, $\lambda > 2^{|\mathbb{P}|}$ and $\mathcal{M}$ a countable elementary submodel of $H(\lambda)$ such that $\mathbb{P} \in \mathcal{M}$. Let $q \in \mathbb{P}$. Then the following conditions are equivalent:*

(1) *q is $(\mathcal{M}, \mathbb{P})$-generic.*
(2) *for every dense $D \subseteq \mathbb{P}$ which belongs to $\mathcal{M}$, $q \Vdash D \cap \mathcal{M} \cap \dot{G} \neq \emptyset$.*
(3) *$q \Vdash \mathcal{M}[\dot{G}] \cap Ord = \mathcal{M} \cap Ord$*
(4) *$q \Vdash \mathcal{M}[\dot{G}] \cap V = \mathcal{M} \cap V$.*

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Proof. The equivalence of (1) and (2) is straightforward from the definition of $(\mathcal{M},\mathbb{P})$-generic conditions. Thus we proceed with the equivalence of (2) and (3).

Suppose $\dot\tau\in\mathcal{M}$ is a name of an ordinal. We have to show that $q\Vdash\dot\tau\in\mathcal{M}$. Let $D=\{p\in\mathbb{P}: p\Vdash\dot\tau=\check\alpha \text{ for some ordinal } \alpha\}$. Then D is a dense subset of $\mathbb{P}$ and since D is definable from τ, $\mathbb{P}$ the set D is also an element of $\mathcal{M}$. Let f be a function defined on D such that $(\forall d\in D)(f(d)=\alpha \text{ iff } d\Vdash\dot\tau=\check\alpha)$. Then the function f is definable from D and so f also belongs to the elementary submodel $\mathcal{M}$. By our assumption, i.e. part (2), $q\Vdash D\cap\mathcal{M}\cap\dot G\neq\emptyset$. Consider any $(V,\mathbb{P})$-generic filter G which contains q. Then

$$V[G]\vDash\exists d(d\in D\cap\mathcal{M}\cap G)\ .$$

Since d is an element of the generic filter, $V[G]\vDash(\dot\tau[G]=\alpha)$ where $d\Vdash\dot\tau=\check\alpha$. But $d\in\mathcal{M}$ and so $f(d)=\alpha\in\mathcal{M}$. Therefore $V[G]\vDash(\dot\tau[G]\in\mathcal{M})$ and since G was arbitrary generic with $q\in G$, $q\Vdash\dot\tau\in\mathcal{M}$.

Let D be a dense subset of $\mathbb{P}$, such that $D\in\mathcal{M}$. In $H(\lambda)$ there is an onto mapping f, defined on $|D|$ and taking values in D. Since $\mathcal{M}$ is elementary submodel of $H(\lambda)$ there is such an f in $\mathcal{M}$. Let $\dot\tau=\min\{i: f(i)\in\dot G_{\mathbb{P}}\}$. Then since D is a dense subset of $\mathbb{P}$, $\dot\tau$ is a name of an ordinal. Furthermore $\dot\tau$ is definable from f, $\mathbb{P}$ and so $\dot\tau$ is an element of $\mathcal{M}$. By assumption $q\Vdash\dot\tau\in\mathcal{M}$. Thus fix any $(V,\mathbb{P})$-generic filter G containing q. Then $V[G]\vDash(\dot\tau[G]\in\mathcal{M})$. But $\dot\tau[G]=\min\{i: f(i)\in G\}$ and so

$$V[G]\vDash(\exists i\in\mathcal{M})(f(i)\in D\cap G)$$

(take $i=\dot\tau[G]$). However since $i\in\mathcal{M}$, also $f(i)\in\mathcal{M}$ and so $V[G]\vDash D\cap G\cap M\neq\emptyset$. But G was arbitrary and so $q\Vdash D\cap G\cap\mathcal{M}\neq\emptyset$.

The equivalence of (2) and (4) is done in a similar way. □

Lemma 2. *Let $\mathbb{P}$ be a forcing notion, $\dot Q$ a $\mathbb{P}$-name of a forcing notion (i.e. $0_{\mathbb{P}}\Vdash\dot Q$ is a forcing notion), λ sufficiently large cardinal and $\mathcal{M}$ countable elementary submodel of $H(\lambda)$ s.t. $\mathbb{P}*\dot Q\in\mathcal{M}$. Then if p_0 is an $(\mathcal{M},\mathbb{P})$-generic condition and $p_0\Vdash$ "$\dot q_0$ is $(\mathcal{M}[\dot G],\dot Q[\dot G])-$ generic" then*

$$(p_0,\dot q_0)\ \textit{is}\ (\mathcal{M},\mathbb{P}*\dot Q)-\textit{generic}\ .$$

Proof. We will show that $(p_0,\dot q_0)$ is $(\mathcal{M},\mathbb{P}*\dot Q)$-generic by using part (3) of Lemma 1. Let G be any $(\mathcal{M},\mathbb{P}*\dot Q)$-generic filter containing $(p_0,\dot q_0)$. Then $G_0=G\cap\mathbb{P}$ is $(V,\mathbb{P})$-generic and $p_0\in G_0$. Since p_0 is $(\mathcal{M},\mathbb{P})$-generic by part (3) of Lemma 1

$$\mathcal{M}[G_0]\cap\text{Ord}=\mathcal{M}\cap\text{Ord}\ .$$

Similarly, if $G_1 = G/G_0 = \{\dot{q}[G_0] : (\exists p)(p,\dot{q}) \in G\}$ then G_1 is $(V[G_0],\dot{Q}[G_0])$-generic and since p_0 belongs to the generic filter G_0, $\dot{q}_0[G_0]$ is $(\mathcal{M}[G_0],\dot{Q}[G_0])$-generic. Again by Lemma 1 part (3)

$$(\mathcal{M}[G_0])[G_1] \cap \mathrm{Ord} = \mathcal{M}[G_0] \cap \mathrm{Ord}\ .$$

So it is left to check that $\mathcal{M}[G] \subseteq \mathcal{M}[G_0][G_1]$. However for every $\mathbb{P} * \dot{Q}$-name $\dot{\tau}$ there is a $\mathbb{P}$-name $\dot{\tau}_*$ definable from $\dot{\tau}$ such that for every $\mathbb{P}$-generic filter H_1, $\dot{\tau}_*[H_1]$ is a $\dot{Q}[H_1]$-name, such that for every $(V[H_1],\dot{Q}[H_1])$-generic filter H_2, $\dot{\tau}[H_1 * H_2] = \dot{\tau}_*[H_1][H_2]$.

Thus if $\dot{\tau}$ is an $\mathbb{P} * \dot{Q}$-name of an ordinal which belongs to $\mathcal{M}$, then the corresponding name $\dot{\tau}_*$ also is in $\mathcal{M}$ and

$$\dot{\tau}[G] = \dot{\tau}[G_0 * G_1] = \dot{\tau}_*[G_0][G_1] \in \mathcal{M}[G_0][G_1]\ .$$

□

2. Properness Extension Lemma

Lemma 3. *Let $\mathbb{P}$ be a proper forcing notion, $\dot{Q}$ a $\mathbb{P}$-name of a proper forcing notion, i.e. $0_{\mathbb{P}} \Vdash$ "$\dot{Q}$ is proper". Let λ be sufficiently large cardinal and $\mathcal{M}$ countable elementary submodel of $H(\lambda)$ s.t. $\mathbb{P}*\dot{Q} \in \mathcal{M}$. If $\dot{r}$ is a $\mathbb{P}$-name and q_0 is an $(\mathcal{M},\mathbb{P})$-generic condition such that*

$$q_0 \Vdash \dot{r} \in \mathcal{M} \cap \mathbb{P} * \dot{Q}\ \wedge\ \pi(\dot{r}) \in \dot{G}_0$$

where $\dot{G}_0$ is the canonical name of the $\mathbb{P}$-generic filter and π is a projection from $\mathbb{P} * \dot{Q}$ onto the first coordinate, then there is a $\mathbb{P}$-name $\dot{q}_1$ such that $(q_0,\dot{q}_1)$ is $(\mathcal{M},\mathbb{P} * \dot{Q})$-generic and

$$(q_0,\dot{q}_1) \Vdash_{\mathbb{P}*\dot{Q}} \dot{r} \in \dot{G}$$

where $\dot{G}$ is the canonical name of the $\mathbb{P} * \dot{Q}$-generic filter.

Proof. Consider any $(V,\mathbb{P})$-generic filter G_0 which contains q_0 and let $r = (r_0,\dot{r}_1)$ be an element of $\mathcal{M} \cap \mathbb{P} * \dot{Q}$ such that $\dot{r}[G_0] = r$. Note that $\dot{r}_1$ is also an element of $\mathcal{M}$ and so $\dot{r}_1[G_0]$ belongs to $\dot{Q}[G_0] \cap \mathcal{M}[G_0]$. But $\dot{Q}[G_0]$ is proper in $V[G_0]$ and so

$$V[G_0] \models \exists x(x \text{ extends } \dot{r}_1[G_0] \wedge x \text{ is } (\mathcal{M}[G_0],\dot{Q}[G_0]) - \text{generic }).$$

Since G_0 was arbitrary generic containing q_0

$$q_0 \Vdash_{\mathbb{P}} \exists x(x \text{ extends the second coordinate of } \dot{r} \wedge\ x \text{ is } (\mathcal{M}[\dot{G}_0],\dot{Q}[\dot{G}_0]) - \text{generic})\ .$$

Then by existential completeness there is a $\mathbb{P}$-name $\dot{q}_1$ such that

$$q_0 \Vdash_{\mathbb{P}} \dot{q}_1 \text{ extends the second coordinate of } \dot{r} \wedge \dot{q}_1 \text{ is } (\mathcal{M}[\dot{G}_0],\dot{Q}[\dot{G}_0]) - \text{generic}\ .$$

Therefore by Lemma 2 $(q_0, \dot{q}_1)$ is $(\mathcal{M}, \mathbb{P} * \dot{Q})$-generic. We still have to show that

$$(q_0, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{Q}} \dot{r} \in \dot{G} \ .$$

Consider any extension $(u_0, \dot{u}_1)$ of $(q_0, \dot{q}_1)$ such that for some condition $r = (r_0, \dot{r}_1)$ in $\mathcal{M} \cap \mathbb{P} * \dot{Q}$

$$(u_0, \dot{u}_1) \Vdash_{\mathbb{P}*\dot{Q}} \dot{r} = \check{r} \ .$$

Since u_0 is an extension of q_0 and $q_0 \Vdash \pi(\dot{r}) \in \dot{G}_0$, we have that $q_0 \Vdash \check{r}_0 \in \dot{G}_0$. But $\mathbb{P}$ is separative and so u_0 is an extension of r_0. Also $u_0 \Vdash \dot{r}_1 \leq \dot{q}_1$ and since $u_0 \Vdash \dot{q}_1 \leq \dot{u}_1$, it is the case that $u_0 \Vdash \dot{r}_1 \leq \dot{u}_1$. Therefore $(u_0, \dot{u}_1)$ is an extension of $(r_0, \dot{r}_1)$ and so

$$(u_0, \dot{u}_1) \Vdash_{\mathbb{P}*\dot{Q}} \dot{r} = \check{r} \in \dot{G} \ .$$

The set of conditions in $\mathbb{P}*\dot{Q}$ which evaluate $\dot{r}$ as a condition in $\mathcal{M}\cap\mathbb{P}*\dot{Q}$ is dense above $(q_0, \dot{q}_1)$ and so

$$(q_0, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{Q}} \dot{r} \in \dot{G} \ .$$

□

Lemma 4 (Properness Extension Lemma)**.** *Let $\langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle$ be a countable support iteration of proper forcing notions, λ sufficiently large cardinal and $\mathcal{M}$ countable elementary submodel of $H(\lambda)$ such that $\gamma, \mathbb{P}_\gamma$ belong to $\mathcal{M}$. If $\gamma_0 \in \gamma \cap \mathcal{M}$, q_0 is $(\mathcal{M}, \mathbb{P}_{\gamma_0})$-generic and $\dot{p}_0$ is a $\mathbb{P}_{\gamma_0}$-name such that*

$$q_0 \Vdash_{\mathbb{P}_{\gamma_0}} \dot{p}_0 \in \mathcal{M} \cap \mathbb{P}_\gamma \wedge \dot{p}_0 \restriction \gamma_0 \in \dot{G}_{\gamma_0}$$

where $\dot{G}_{\gamma_0}$ is the canonical $\mathbb{P}_{\gamma_0}$-name of the generic filter, there is an $(\mathcal{M}, \mathbb{P}_\gamma)$-generic condition q such that $q \restriction \gamma_0 = q_0$ and

$$q \Vdash_{\mathbb{P}_\gamma} \dot{p}_0 \in \dot{G}_\gamma$$

where $\dot{G}_\gamma$ is the canonical $\mathbb{P}_\gamma$ name of the generic filter.

Proof. The proof is by induction on γ. If γ is a successor, i.e. $\gamma = \delta + 1$ for some δ then if γ is in the elementary submodel $\mathcal{M}$, already δ is in $\mathcal{M}$ and so by inductive hypothesis applied to γ_0, δ and q_0, we could extend q_0 to an $(\mathcal{M}, \mathbb{P}_\delta)$-generic condition with the required properties. Thus the successor case is reduced to the two step iteration which was considered in Lemma 3.

So suppose γ is a limit and the lemma is true for every ordinal smaller than γ. Let $\langle \gamma_n : n \in \omega \rangle$ be an increasing and unbounded sequence of ordinals in $\gamma \cap \mathcal{M}$ and let $\langle D_n : n \in \omega \rangle$ be a fixed enumeration of the dense subsets of $\mathbb{P}_\gamma$ which belong to $\mathcal{M}$. Inductively we will construct sequences $\langle q_n : n \in \omega \rangle$ and $\langle \dot{p}_n : n \in \omega \rangle$ (starting with $\dot{p}_0$ - the given $\mathbb{P}_{\gamma_0}$-name, and q_0 - the given $(\mathcal{M}, \mathbb{P}_{\gamma_0})$-generic condition) such that

(1) q_n is $(\mathcal{M}, \mathbb{P}_{\gamma_n})$-generic condition and $q_{n+1} \restriction \gamma_n = q_n$
(2) $\dot{p}_n$ is a $\mathbb{P}_{\gamma_n}$-name such that

$$q_n \Vdash_{\mathbb{P}_{\gamma_n}} (\dot{p}_n \in \mathcal{M} \cap \mathbb{P}_\gamma) \wedge (\dot{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n}) \wedge (\dot{p}_{n-1} \leq \dot{p}_n) \wedge (\dot{p}_n \in D_{n-1})$$

where $\dot{p}_n \in D_{n-1}$ is required only for $n \geq 1$ and $\dot{G}_{\gamma_n}$ is the canonical name for the $\mathbb{P}_{\gamma_n}$-generic filter. For notational simplicity we will write $\Vdash_{\gamma_n}$ instead of $\Vdash_{\mathbb{P}_{\gamma_n}}$.

Suppose q_n and $\dot{p}_n$ have been defined and consider any $(V, \mathbb{P}_{\gamma_n})$-generic filter G_{γ_n} containing q_n. Let p_n be an element of $\mathcal{M} \cap \mathbb{P}_\gamma$ such that $p_n = \dot{p}_n[G_{\gamma_n}]$. The set

$$D' = \{d \restriction \gamma_n : p_n \leq d \text{ and } d \in D_n\}$$

is dense above $p_n \restriction \gamma_n$ and since it is definable from γ_n, p_n and D_n all of which belong to $\mathcal{M}$, D' is itself an element of $\mathcal{M}$. Then $D = D' \cup \{p \in \mathbb{P}_{\gamma_n} : p \perp (p_n \restriction \gamma_n)\}$ is a dense subset of $\mathbb{P}_{\gamma_n}$ which belongs to $\mathcal{M}$ and since $q_n \in G_{\gamma_n}$ is $(\mathcal{M}, \mathbb{P}_{\gamma_n})$-generic the intersection $D \cap \mathcal{M} \cap G_{\gamma_n}$ is nonempty. However $p_n \restriction \gamma_n \in G_{\gamma_n}$ and so if $x \in D \cap \mathcal{M} \cap G_{\gamma_n}$ then x is compatible with $p_n \restriction \gamma_n$. Therefore $D' \cap \mathcal{M} \cap G_{\gamma_n} \neq \emptyset$. But then

$$H(\lambda)[G_{\gamma_n}] \models \exists x (x \in \mathbb{P}_\gamma \wedge x \in D_n \wedge p_n \leq x \wedge x \restriction \gamma_n \in \mathcal{M} \cap G_{\gamma_n}) .$$

Since $\mathcal{M}[G_{\gamma_n}]$ is an elementary submodel of $H(\lambda)[G_{\gamma_n}]$ there is such an x in $\mathcal{M}[G_{\gamma_n}]$. However $\mathcal{M}[G_{\gamma_n}] \cap \mathbb{P}_\gamma = \mathcal{M} \cap \mathbb{P}_\gamma$ since $\mathbb{P}_\gamma \subseteq V$ and $\mathcal{M}[G_{\gamma_n}] \cap V = \mathcal{M} \cap V$ (see Lemma 1). Therefore

$$V[G_{\gamma_n}] \models \exists x (x \in \mathcal{M} \cap \mathbb{P}_\gamma \wedge x \in D_n \wedge p_n \leq x \wedge x \restriction \gamma_n \in G_{\gamma_n}) .$$

By existential completeness there is a $\mathbb{P}_{\gamma_n}$-name $\dot{p}_{n+1}$ such that

$$q_n \Vdash_{\gamma_n} \dot{p}_{n+1} \in \mathcal{M} \cap \mathbb{P}_\gamma \wedge \dot{p}_{n+1} \in D_n \wedge \dot{p}_n \leq \dot{p}_{n+1} \wedge \dot{p}_{n+1} \restriction \gamma_n \in G_{\gamma_n} .$$

Now by the inductive hypothesis of the Lemma applied to γ_n, γ_{n+1}, q_n and $\dot{p}_{n+1}$ there is an $(\mathcal{M}, \mathbb{P}_{\gamma_{n+1}})$-generic condition q_{n+1} such that $q_{n+1} \restriction \gamma_n = q_n$ and

$$q_{n+1} \Vdash_{\gamma_{n+1}} \dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}$$

where $\dot{G}_{\gamma_{n+1}}$ is the canonical $\mathbb{P}_{\gamma_{n+1}}$-name of the generic filter.

With this the inductive construction of the sequences $\langle q_n : n \in \omega \rangle$ and $\langle \dot{p}_n : n \in \omega \rangle$ is complete. Let $q = \cup_{n \in \omega} q_n$. Then q extends every q_n. We will show that for every n

$$q \Vdash_\gamma \dot{p}_n \in \dot{G}_\gamma .$$

But then $q \Vdash_\gamma \dot{p}_n \in \dot{G}_\gamma \cap \mathcal{M} \cap D_{n-1}$ and since $\langle D_n : n \in \omega \rangle$ is an enumeration of all dense subsets of $\mathbb{P}_\gamma$ which belong to $\mathcal{M}$, this implies that q is $(\mathcal{M}, \mathbb{P}_\gamma)$-generic.

Fix an arbitrary n. By condition 2 of the inductive construction for every m which is greater or equal to n, $q \Vdash_\gamma \dot{p}_n \leq \dot{p}_m$. But q also forces that $\dot{p}_m \restriction \gamma_m \in \dot{G}_{\gamma_m}$ and so

$$q \Vdash_\gamma \dot{p}_n \restriction \gamma_m \in \dot{G}_{\gamma_m} \text{ for every } m \geq n \ .$$

Consider any extension q' of q such that $q' \Vdash_\gamma \dot{p}_n = \check{p}_n$ for some $p_n \in \mathcal{M} \cap \mathbb{P}_\gamma$. Then

$$q' \Vdash_\gamma \check{p}_n \restriction \gamma_m \in \dot{G}_{\gamma_m} \text{ for every } m \geq n \ .$$

But $\mathbb{P}_{\gamma_n}$ is separative and so $p_n \restriction \gamma_m \leq q'$ for every $m \in \omega$. Since the condition p_n belongs to the elementary submodel $\mathcal{M}$, its domain is contained in $\mathcal{M}$ and so in particular the sequence $\langle \gamma_n : n \in \omega \rangle$ is unbounded in the domain of p_n. Therefore q' extends p_n and so

$$q' \Vdash_\gamma \dot{p}_n = \check{p}_n \in \dot{G}_\gamma \ .$$

Since the set of conditions which decide $\dot{p}_n$ as a condition in $\mathcal{M} \cap \mathbb{P}_\gamma$ is dense above q (it is dense above q_n and q is an extension of q_n)

$$q \Vdash_\gamma \dot{p}_n \in \dot{G}_\gamma \ .$$

□

Theorem 1. Let γ be a limit ordinal and $\langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle$ a countable support iteration of proper forcing posets. Then $\mathbb{P}_\gamma$ is proper.

Proof. Let $\mathbb{P}' = \{0\}$ be the trivial poset. Then $V[\{0\}] = V$ (note that $\{0\}$ is also the generic set) and so every element of the universe can be identified with its $\mathbb{P}'$-name. Since the trivial poset is completely embedded in every poset, we can apply Lemma 4 with $\gamma_0 = 0$, γ - the length of the iteration, $q_0 = 0$ and p_0 a given condition in $\mathcal{M} \cap \mathbb{P}_\gamma$, for which we want to show the existence of an $(\mathcal{M}, \mathbb{P}_\gamma)$-generic extension, considered as a $\mathbb{P}'$-name. □

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