PRESERVATION OF UNBOUNDEDNESS AND THE CONSISTENCY OF $b < s$

VERA FISCHER

1. The Weakly Bounding Property

Recall the following definitions:

Definition 1. Let f and g be functions in ${}^{\omega}\omega$. We say that f is *dominated* by g iff there is some natural number n such that $f \leq_n g$, i.e. $(\forall i \geq n)(f(i) \leq g(i))$. Then $<^* = \cup \leq_n$ is called *the bounding relation* on ${}^{\omega}\omega$. If $\mathcal{F}$ is a family of functions in ${}^{\omega}\omega$ we say that $\mathcal{F}$ is *dominated* by the function g, and denote it by $\mathcal{F} <^* g$ iff $(\forall f \in \mathcal{F})(f <^* g)$. We say that $\mathcal{F}$ is *unbounded* (also *not dominated*) iff there is no function $g \in^{\omega} \omega$ which dominates it.

Definition 2. A forcing notion $\mathbb{P}$ is called *weakly bounding* iff for every $(V, \mathbb{P})$-generic filter G, the ground model reals are unbounded in $V[G]$. That is for every $f \in V[G] \cap^{\omega} \omega$ there is a ground model function g such that $\{n : g(n) \leq f(n)\}$ is infinite.

Theorem 1. If δ is a limit, and $\langle \mathbb{P}_i : i \leq \delta \rangle$ is a countable support iteration of proper forcing notions such that every initial stage of the iteration $\mathbb{P}_i$ is weakly bounding, then $\mathbb{P}_\delta$ is weakly bounding.

Proof. The proof is by induction on δ. Let $\dot{f}$ be a $\mathbb{P}$-name of a function, and p an arbitrary condition in $\mathbb{P}$. We will show that there is a ground model function g and an extension q of p such that $q \Vdash_\delta g \not\leq^* \dot{f}$. Note that this is equivalent to $q \Vdash \forall n \in \omega \exists k \geq n(\dot{f}(k) \leq g(k))$.

Consider a countable elementary submodel $\mathcal{M}$ of $H(\lambda)$, where $\lambda > 2^{|\mathbb{P}|}$, such that p, $\mathbb{P}_\delta$ and $\dot{f}$ are elements of $\mathcal{M}$. Since $\mathcal{M} \cap^{\omega} \omega$ is countable there is a function g which dominates all functions in $\mathcal{M}$. Similarly to the proof of the Properness Extension Lemma fix an increasing, unbounded sequence $\{\gamma_n\}_{n \in \omega}$ in $\mathcal{M} \cap \delta$. Inductively we will construct two sequences $\langle q_n : n \in \omega \rangle$ of $(\mathcal{M}, \mathbb{P}_{\gamma_n})$-generic conditions and $\langle \dot{p}_n : n \in \omega \rangle$ of $\mathbb{P}_{\gamma_n}$-names for conditions in $\mathcal{M} \cap \mathbb{P}_\delta$ such that:

(1) q_n is $(\mathcal{M}, \mathbb{P}_{\gamma_n})$-generic, and $q_n \restriction \gamma_{n-1} = q_{n-1}$.

Date: October 12, 2005.

(2) $\dot{p}_n$ is a $\mathbb{P}_{\gamma_n}$-name such that

$$q_n \Vdash_{\gamma_n} \quad (\dot{p}_n \in \mathcal{M} \cap \mathbb{P}_\delta) \wedge (\dot{p}_{n-1} \leq \dot{p}_n) \wedge (\dot{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n}) \wedge$$
$$(\dot{p}_n \Vdash_\delta \exists k \geq n(\dot{f}(k) \leq g(k)))$$

Begin with p_0 the given condition p and q_0 any $(\mathcal{M}, \mathbb{P}_{\gamma_0})$-generic condition extending $p_0 \restriction \gamma_n$. Suppose q_n and $\dot{p}_n$ have been defined and let G_{γ_n} be any $(V, \mathbb{P}_{\gamma_n})$-generic filter containing q_n. Then there is a condition p_n in $\mathcal{M} \cap \mathbb{P}_\delta$ such that $p_n = \dot{p}_n[G_{\gamma_n}]$. Let $r_0 = p_n$.

In $M[G_{\gamma_n}]$ we can construct inductively an increasing sequence $\langle r_n : n \in \omega\rangle$ of conditions in $\mathcal{M} \cap \mathbb{P}_\delta$ such that $r_n \restriction \gamma_n \in G_{\gamma_n}$ and

$$r_i \Vdash_\delta \dot{f}(i) = k \text{ for some } k\ .$$

Let f^* be the function thus interpreted. Note that since the sequence $\langle r_j : j \in \omega\rangle$ is increasing for every $j \in \omega$ we have $r_j \Vdash_\delta \dot{f} \restriction j = f^* \restriction j$. Since f^* belongs to $M[G_{\gamma_n}]$ and $\mathbb{P}_{\gamma_n}$ is weakly bounding there is a ground model function $h \in \mathcal{M} \cap^\omega \omega$ such that

$$M[G_{\gamma_n}] \vDash \{i : f^*(i) \leq h(i)\} \text{ is infinite}\ .$$

However h is a function from $\mathcal{M}$ and so is dominated by the function g. Thus there is some natural number k_0 such that for every $i \geq k_0$ we have $h(i) \leq g(i)$. But then there is an $i_0 \geq \max\{n+1, k_0\}$ such that $f^*(i_0) \leq h(i_0) \leq g(i_0)$. However for $j = i_0 + 1$ we have

$$r_j \Vdash_\delta \dot{f}(i_0) = f^*(i_0)\ .$$

Let $\dot{p}_{n+1}$ be a $\mathbb{P}_{\gamma_n}$-name for r_j. Then

$$q_n \Vdash_{\gamma_n} \quad (\dot{p}_{n+1} \in \mathcal{M} \cap \mathbb{P}_\delta) \wedge (\dot{p}_n \leq \dot{p}_{n+1}) \wedge (\dot{p}_{n+1} \restriction \gamma_n \in \dot{G}_{\gamma_n}) \wedge$$
$$(\dot{p}_{n+1} \Vdash_\delta \exists k \geq n+1(\dot{f}(k) \leq g(k)))$$

However by the Properness Extension Lemma applied to γ_n, γ_{n+1}, q_n and $\dot{p}_{n+1}$ there is an $(\mathcal{M}, \mathbb{P}_{\gamma_{n+1}})$-generic condition q_{n+1} such that

$$q_{n+1} \restriction \gamma_n = q_n$$

and

$$q_{n+1} \Vdash_{\gamma_{n+1}} \dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}\ .$$

With this inductive construction of the sequences $\langle q_n : n \in \omega\rangle$ and $\langle \dot{p}_n : n \in \omega\rangle$ is completed. But then just as in the Properness Extension Lemma we obtain that $q = \cup_{n \in \omega} q_n$ is an extension of p such that

$$q \Vdash_\delta \dot{p}_n \in \dot{G}_\delta \text{ for every } n \in \omega\ .$$

So, if G is $(V, \mathbb{P}_\delta)$-generic and $q \in G$, then

$$V[G] \vDash \forall n \in \omega \exists k \geq n(\dot{f}(k) \leq g(k))\ ,$$

i.e. $q \Vdash_\delta g \not\leq^* \dot{f}$. □

Remark. Note that in the previous theorem we required that each initial stage $\mathbb{P}_i$ of the iteration is weakly bounding, rather than each iterand. The reason is that a finite iteration of weakly bounding posets is not necessarily weakly bounding. For example if $\mathbb{P}$ is the forcing notion for adding ω_1 Cohen reals, and $\dot{Q}$ is a $\mathbb{P}$-name for the Hechler forcing associated to the collection of all ground model reals, then for any $(V, \mathbb{P}*\dot{Q})$ generic filter G, the ground model reals are not unbounded in $V[G]$, yet $\dot{Q}[G_0]$ is weakly bounding in $V[G_0]$ for $G_0 = G \cap \mathbb{P}$. However there is a stronger condition, the almost ${}^{\omega}\omega$-bounding property which will remedy this situation.

2. The Almost Bounding Property

Definition 3. The partial order $\mathbb{P}$ is called *almost ${}^{\omega}\omega$-bounding* if for every $\mathbb{P}$-name $\dot{f}$, of a function in ${}^{\omega}\omega$ and every condition $p \in \mathbb{P}$ there is a ground model function g in ${}^{\omega}\omega$ such that for every infinite subset A of ω there is an extension q_A of q such that

$$q_A \Vdash \forall n \exists k \geq n \text{ s.t. } k \in A \text{ and } \dot{f}(k) \leq g(k)\ .$$

Lemma 1. *If $\mathbb{P}$ is a weakly bounding forcing notion and $\dot{Q}$ is a $\mathbb{P}$-name of an almost bounding forcing notion, then $\mathbb{P} * \dot{Q}$ is weakly bounding.*

Proof. Consider arbitrary $\mathbb{P} * \dot{Q}$-name of a real $\dot{f}$ and condition $(p, \dot{q})$ in $\mathbb{P} * \dot{Q}$. Let G be a $(V, \mathbb{P} * \dot{Q})$-generic filter containing $(p, \dot{q})$ and $G_0 = G \cap \mathbb{P}$. Then $\dot{q}[G_0]$ is a condition in $\dot{Q}[G_0]$ and furthermore $\dot{Q}[G_0]$ is an almost bounding poset in $V[G_0]$. Recall from the proof of Lemma 2 on the preservation of properness under CS iteration, that there is a $\mathbb{P}$-name f^*, such that for every $\mathbb{P}$-generic filter H_1, $f^*[H_1]$ is a $Q[H_1]$-name of a real and furthermore for every $Q[H_1]$generic filter H_2, $\dot{f}[H_1 * H_2] = f^*[H_1][H_2]$. Then in particular $f^*[G_0]$ is a $Q[G_0]$-name for a function in ${}^{\omega}\omega$ and so by the definition of the almost bounding property, there is a function g in $V[G_0]$ such that for every $A \in [\omega]^{\omega}$ there is an extension q_A of $\dot{q}[G_0]$ which forces that there are infinitely many $i \in A$ for which $g(i) \leq f^*(i)$. However since g is a function in $V[G_0]$ and $\mathbb{P}$ is weakly bounding there is a function h in V such that the set $A = \{i : g(i) \leq h(i)\}$ is infinite. If the second generic extension G_1 contains q_A, then

$$V[G_0 * G_1] \vDash \exists^{\infty} i \in A(\dot{f}(i) \leq h(i))$$

and so $\mathbb{P} * \dot{Q}$ is weakly bounding. □

Therefore by Theorem 1 we obtain

Theorem 2. The countable support iteration of proper almost ${}^\omega\omega$-bounding posets is weakly bounding.

Other preservation theorems, which will be used in the consistency result to be presented later are:

Theorem 3. Assume CH. Let $\langle \mathbb{P}_i : i \leq \delta \rangle$ where $\delta < \omega_2$, be a countable support iteration of proper forcing posets of size $\aleph_1$. Then the CH holds in $V^{\mathbb{P}_\delta}$.

Theorem 4. Assume CH. Let $\langle \mathbb{P}_i : i \leq \delta \rangle$ where $\delta \leq \omega_2$, be a countable support iteration of proper forcing posets of size $\aleph_1$. Then $\mathbb{P}_\delta$ satisfies the $\aleph_2$-chain condition.

Note that by the previous theorems if we assume the CH in the ground model and if $\langle \mathbb{P}_i : i \leq \omega_2 \rangle$ is a countable support iteration of proper forcing notions of size $\aleph_1$, then forcing with $\mathbb{P}_{\omega_2}$ does not collapse cardinals: ω_1 is not collapsed since $\mathbb{P}_{\omega_2}$ is proper, and cardinals greater or equal ω_2 are not collapsed by the $\aleph_2$-chain condition.

We are ready to proceed with the consistency of the bounding number smaller than the splitting number.

3. The Partial Order Q

Recall the following definitions:

Definition 4. A family $B \subseteq^\omega \omega$ is said to be *unbounded* if for every $f \in^\omega \omega$ there is a function $g \in B$ such that $g \not\leq^* f$, i.e. there are infinitely many i such that $f(i) \leq g(i)$. Then

$$b = \min\{|B| : B \subseteq^\omega \omega \text{ and } B \text{ is unbounded}\}$$

is called *the bounding number.*

Definition 5. A family $S \subseteq [\omega]^\omega$ is called *splitting* if for any infinite subset A of ω there is a set $B \in S$ such that $A \cap B$ and $A \cap B^c$ are infinite. Then

$$s = \min\{|S| : S \subseteq [\omega]^\omega \text{ and } S \text{ is splitting}\}$$

is called *the splitting number.*

In the remaining sections we will establish the following result:

Theorem 5. Assume CH. Then there is a generic extension in which cardinals are not collapsed, $2^{\aleph_0} = \aleph_2$, $b = \omega_1$ and $s = \omega_2$.

By the remarks from the previous section under the CH, any countable support iteration of length ω_2 of proper forcing notions of size $\aleph_1$ does not collapse cardinals. Therefore if in addition we require the forcing posets to be almost ${}^\omega\omega$-bounding, by Theorem 2 the resulting iteration will be weakly bounding and so in every generic extension the ground model reals will be an unbounded family of size ω_1. However in order the splitting number to be ω_2 we have to require something more: that at each successor stage of the iteration we add an infinite subset of ω, which is not split by the ground model reals. Therefore it is sufficient to obtain the following:

Theorem 6. Assume CH. There is a proper, almost ${}^\omega\omega$-bounding poset Q of size $\aleph_1$ such that in every (V, Q)-generic extension there is an infinite subset of ω which is not split by any ground model real.

In order to define the partial order, which will demonstrate Theorem 6 we need the notion of logarithmic measure.

Definition 6. Let S be a subset of ω and $h : \mathcal{P}_\omega(S) \to \omega$, where $\mathcal{P}_\omega(S)$ is the family of all finite subsets of ω. The function h is called a logarithmic measure, if for every $A \in \mathcal{P}_\omega(S)$ and for every A_0, A_1 such that $A = A_0 \cup A_1$ if $h(A) \geq l + 1$ for some $l \geq 1$, then $h(A_0) \geq l$ or $h(A_1) \geq l$. If S is a finite set, then $h(S)$ is called the level of S.

Corollary 1. *If h is a logarithmic measure and $h(A_0 \cup \cdots \cup A_{n-1}) \geq l+1$ then for some j, $0 \leq j \leq n-1$ $h(A_j) \geq l - j$.*

Furthermore we will work with logarithmic measures induced by positive sets, which will be essential in order to obtain the almost bounding property (see section 6).

Definition 7. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family. Then P induces a logarithmic measure h on $[\omega]^{<\omega}$ defined inductively on $|s|$ for $s \in [\omega]^{<\omega}$ in the following way:

(1) $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$
(2) $h(e) > 0$ iff $e \in P$
(3) for $l \geq 1$, $h(e) \geq l + 1$ iff $|e| > 1$ and whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq l$ or $h(e_1) \geq l$.

Then $h(e) = l$ iff l is the maximal natural number for which these hold.

Corollary 2. *If h is a logarithmic measure induced by positive sets and $h(e) \geq l$, then for every a such that $e \subseteq a$, $h(a) \geq l$.*

Example 1. Let P be the family of all sets containing at least two points and h the logarithmic measure induced by P on $[\omega]^\omega$. Then for every $x \in P$, $h(x) = i$ where i is the minimal natural number such that $|x| \leq 2^i$.

Now we can define the partial order Q, which satisfies Theorem 6.

Definition 8. Let Q be the set of all pairs (u, T) where u is a finite subset of ω and $T = \langle t_i : i \in \omega \rangle$ (here $t_i = (s_i, h_i)$, $s_i = \text{int}(t_i)$ is a finite subsets of ω and h_i is a given logarithmic measure on s_i) is a sequence of logarithmic measures such that

(1) $\max(u) < \min s_0$
(2) $\max s_i < \min s_{i+1}$
(3) $h_i(s_i) < h_{i+1}(s_{i+1})$.

The finite part u is called *the stem* of the condition $p = (u, T)$, and $T = \langle t_i : i \in \omega \rangle$ *the pure part* of p. Also $\text{int}(T) = \cup\{s_i : s \in \omega\}$. In case that $u = \emptyset$ we say that $(\emptyset, T)$ is a pure condition and usually denote it simply by T.

We say that (u_1, T_1) is extended by (u_2, T_2), where $T_l = \langle t^l_i : i \in \omega \rangle$ for $l = 1, 2$, and denote it by

$$(u_1, T_1) \leq (u_2, T_2)$$

iff the following conditions hold:

(1) u_2 is an end-extension of u_1 and $u_2 \backslash u_1 \subseteq \text{int}(T_1)$
(2) $\text{int}(T_2) \subseteq \text{int}(T_1)$ and furthermore there is an infinite sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max u_2 < \min \text{int}(t_j)$ for $j = \min B_0$, $\max(B_i) < \min(B_{i+1})$ and $s^2_i \subseteq \cup\{s^1_j : j \in B_i\}$.
(3) for every h^2_i positive subset e of s^2_i there is some $j \in B_i$ such that $e \cap s^1_j$ is h^1_j-positive.

In case that $u_1 = u_2$ we say the (u_2, T_2) is a pure extension of (u_1, T_1).

4. The Splitting Number

The reason that in every generic extension via Q there is a real which is not split by the ground model subsets of ω is the same as for Mathias forcing. We will need the following lemma.

Lemma 2. *Suppose T is a pure condition and A is an infinite subset of ω. Then there is a pure extension T' of T such that int(T') is contained in A or in A^c.*

Proof. Let $T = \langle t_i : i \in \omega \rangle$ where $t_i = (s_i, h_i)$. For every i define $r_i = (s_i \cap A, h_i \restriction s_i \cap A)$ or $r_i = (s_i \cap A^c, h_i \restriction s_i \cap A^c)$ depending on whether $h_i(s_i \cap A) \geq h_i(s_i) - 1$ or $h_i(s_i \cap A^c) \geq h_i(s_i) - 1$. Then there is an infinite index set I such that $\forall i \in I$ $\text{int}(r_i) \subset A$ or alternatively $\forall i \in I$ $\text{int}(r_i) \subset A^c$. Then the pure condition $T' = \langle r_i : i \in I \rangle$ is well defined (i.e. the measures r_i are strictly increasing), extends T and $\text{int}(T')$ is contained in A or in A^c. $\square$

Lemma 3. *Let G be a Q-generic filter. Then the real*

$$U_G = \bigcup\{u : \exists T(u, T) \in G\}$$

is not split by any ground model subset of ω.

Proof. Suppose by way of contradiction that there is a ground model subset A of ω such that $U_G \cap A$ and $U_G \cap A^c$ are infinite. Let $D_A = \{(u, T) \in Q : \text{int}(T) \subset (A) \text{ or } \text{int}(T) \subseteq A^c\}$. Then by Lemma 2 the set D_A is a dense subset of Q and so $G \cap D_A$ is nonempty. However if (u_0, T_0) belongs to this intersection then by the definition of D_A $\text{int}(T_0)$ is contained in A or in A^c. But (u_0, T_0) also belongs to G. It is not difficult to see from the definition of the extension relation on Q that $U_G \subseteq^* \text{int}(T)$ for every condition $p = (u, T)$ which belongs to G. Therefore $U_G \subseteq^* \text{int}(T_0)$ and so U_G is almost contained in A or in A^c. This is a contradiction since it implies that the intersection of U_G with A^c or A respectively, is finite. □

Lemma 4. *If $\langle \mathbb{P}_i : i \leq \delta \rangle$ is a countable support iteration of length δ, where $cf(\delta) > \omega$, then any real is added at some initial stage δ_0 of the iteration such that $\delta_0 < \delta$.*

Proof. Let $\dot{f}$ be a $\mathbb{P}_\delta$-name of a real and p an arbitrary condition in $\mathbb{P}$. We can assume that

$$\dot{f} = \bigcup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in A_i, i \in \omega, j^i_p \in \omega\}$$

where for each i, A_i is a maximal antichain in $\mathbb{P}$. Consider any countable elementary submodel $\mathcal{M}$ of $H(\lambda)$, λ is sufficiently large, such that $\mathbb{P}$, $\dot{f}$, p, A_i for every i belong to $\mathcal{M}$. If q is an $(\mathcal{M}, \mathbb{P})$-generic condition extending p and G a $(V, \mathbb{P})$-generic filter containing q, then for every i we have $A_i \cap G = \mathcal{M} \cap A_i \cap G$. That is for

$$\mathcal{M} \cap \dot{f} = \bigcup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{M} \cap A_i, i \in \omega, j^i_p \in \omega\}$$

and $i \in \omega$ we have $q \Vdash_\delta \dot{f}(i) = (\mathcal{M} \cap \dot{f})(i)$. Since $\mathcal{M}$ is a countable model, the intersection $\mathcal{M} \cap A_i$ is also countable and so if $\alpha_i = \sup\{\alpha_p : p \in \mathcal{M} \cap A_i\}$ where for every $p \in \mathcal{M} \cap A_i$ we define $\alpha_p = \sup \text{suppt}(p)$, then $\delta_0 = \sup\{\alpha_i : i \in \omega\}$ is an ordinal of countable cofinality which is smaller than δ. Then every condition p in $A_i \cap \mathcal{M}$ has support in δ_0. Therefore we can consider $\mathcal{M} \cap \dot{f}$ as a $\mathbb{P}_{\delta_0}$-name of a real such that $q \Vdash_\delta \dot{f} = \mathcal{M} \cap \dot{f}$. □

Theorem 7. If $\langle \mathbb{P}_i : i \leq \omega_2 \rangle$ is a countable support iteration of proper forcing notions, then any set of reals of cardinality ω_1 is added at some proper initial stage if the iteration.

Proof. Let A be an arbitrary family of size $\aleph_1$ of reals in $V^{\mathbb{P}_{\omega_2}}$. Consider any $(V, \mathbb{P})$-generic filter G. Then for every $\dot{f} \in A$ there is an ordinal α_f of countable cofinality such that $\dot{f}[G] = \dot{f}[G_{\alpha_f}]$. But then $A \subseteq V[G_\alpha]$ where $\alpha = \sup\{\alpha_f : \dot{f} \in A\}$. Since A is of size $\aleph_1$, $cf(\alpha) \leq \omega_1$. Therefore $\alpha < \omega_2$ and $A \subseteq V[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$. □

Note that by the previous theorem if we iterate the forcing notion Q ω_2-times with countable support, than any family A of ω_1-reals in the generic extension is not splitting. Really if G is $\mathbb{P}_{\omega_2}$-generic, where $\langle \mathbb{P}_i : i \leq \omega_2 \rangle$ is the iteration of Q, then by Theorem 7 there is some $\delta_0 < \omega_2$, such that $A \subseteq V[G_{\delta_0}]$ where $G_{\delta_0} = \mathbb{P}_{\delta_0} \cap G$. By Lemma 3 in $V[G_{\delta_0+1}]$ there is a real which is not split by A.

5. Axiom A Implies Properness

Definition 9. A forcing poset $\mathbb{P} = (P, \leq)$ is said to satisfy Axiom A, iff the following conditions hold:

1. There is a sequence of separative preorders on P $\{\leq_n\}_{n\in\omega}$, where $\leq_0 = \leq$, such that $\leq_m \subseteq \leq_n$ for every $m \leq n$. That is, whenever $m \leq n$ and p, q are conditions in P such that $p \leq_m q$, then $p \leq_n q$.
2. If $\{p_n\}_{n\in\omega}$ is a sequence of conditions in P such that $p_n \leq_{n+1} p_{n+1}$ for every n, then there is a condition p such that $p_n \leq_n p$ for every n. The sequence $\{p_n\}_{n\in\omega}$ is called a *fusion sequence* and p is called the *fusion* of the sequence.
3. For every $D \subseteq \mathbb{P}$ which is dense, and every condition p, for every $n \in \omega$ there is a condition p' such that $p \leq_n p'$ and a countable subset D_0 of D which is predense above p'.

Lemma 5. *If the forcing notion $\mathbb{P}$ satisfies axiom A, then $\mathbb{P}$ is proper.*

Proof. Let $\mathcal{D}$ be the family of all dense subsets of $\mathbb{P}$, and $\mathcal{D}'$ the family of all countable subsets of $\mathbb{P}$. Since the partial order $\mathbb{P}$ satisfies Axiom A, there is a function

$$\sigma : \omega \times \mathbb{P} \times \mathcal{D} \to \mathbb{P} \times \mathcal{D}'$$

such that $\sigma(n, p, D) = (p', D')$ iff $p \leq_n p'$ and D' is a countable subset of D which is predense above p'.

Let $\mathcal{M}$ be a countable elementary submodel of $H(\lambda)$, λ sufficiently large, such that $\mathbb{P}$, σ belong to $\mathcal{M}$. We will show that every condition in $\mathbb{P} \cap \mathcal{M}$ has an $(\mathcal{M}, \mathbb{P})$-generic extension. Fix an enumeration $\langle D_n : n \in \omega \rangle$ of the dense subsets of $\mathbb{P}$ which belong to $\mathcal{M}$ and let $p = p_0$ be a given condition in $\mathcal{M} \cap \mathbb{P}$. Since σ is an element of $\mathcal{M}$, also $\sigma(1, p_0, D_1) = (p_1, D_1')$ belongs to $\mathcal{M}$. But then p_1, and D_1' are elements of $\mathcal{M}$ themselves. Proceed inductively to define a fusion sequence $\langle p_n : n \in \omega\rangle$ of conditions in $\mathcal{M} \cap \mathbb{P}$ and a sequence $\langle D'_n : n \in \omega\rangle$ of countable subsets of $\mathbb{P}$, such that for every $n \in \omega$ $D'_n \in \mathcal{M}$, $D'_n \subseteq D_n$ and D'_n is predense above p_n. Let q be the fusion of $\{p_n\}_{n\in\omega}$ and D an arbitrary dense subset of $\mathbb{P}$ which belongs to $\mathcal{M}$. Then $D = D_m$ for some m. Since $p_m \leq_m q$, and D'_m is predense above p_m, D'_m is also predense above q. But D'_m is countable, and since it belongs to $\mathcal{M}$ it is a subset of $\mathcal{M}$. Therefore $D'_m \subseteq \mathcal{M} \cap D_m = \mathcal{M} \cap D$, which implies that $\mathcal{M} \cap \mathcal{D}$ is predense above q. □

In the remainder of this and next section we will show that the forcing notion Q satisfies Axiom A. For this consider the following preorders defined on Q: Let $\leq_0$ be just the order of Q.

For any two conditions (u_1, T_1) and (u_2, T_2) we say that

$$(u_1, T_1) \leq_1 (u_2, T_2) \text{ iff } u_1 = u_2 \text{ and} (u_1, T_1) \leq_0 (u_2, T_2) \ .$$

Furthermore for every $i \geq 1$, if $T_l = \langle t^l_i : i \in \omega\rangle$ for $l = 1, 2$ we say that

$$(u_1, T_1) \leq_{i+1} (u_2, T_2) \text{ iff } t^j_1 = t^j_2 \ \forall j = 0, \ldots, i-1 \ .$$

That is the stem and the first i logarithmic measures are not changed in the extension.

Then if $\{p_n\}_{n\in\omega} = \{(u, T_n)\}_{n\in\omega}$ where $T_n = \langle t^n_j : j \in \omega\rangle$, the condition $p = (u, T)$ where $T = \langle t_j : j \in \omega\rangle$ for $t_j = t^{j+1}_j$ is a fusion of this sequence. Thus in order to verify Axiom A we still have to show that part (3) is satisfied. For this we will need the notion of a preprocessed condition which is considered in the next section.

6. Preprocessed Conditions

Definition 10. Suppose D is a dense open set. We say that the condition $p = (u, T)$ where $T = \langle t_i : i \in \omega\rangle$, is preprocessed for D and i if for every subset of i which end-extends u the condition $(v, \langle t_j : j \geq i\rangle)$ has a pure extension in D if and only if $(v, \langle t_j : j \geq i\rangle)$ belongs to D.

Lemma 6. *If D is a dense open set and $i \in \omega$ if (u, T) is preprocessed for D and i, then any extension of (u, T) is also preprocessed for D and i.*

Proof. Suppose (w, R) extends (u, T) and let $v \subset i$ such that $(v, \langle r_j : j \geq i\rangle)$ has a pure extension in D. Since R extends T, by definition of the extension relation on Q we obtain that $\langle r_j : j \geq i)\rangle$ is an extension of $\langle t_j : j \geq i\rangle$. Therefore $(v, \langle t_j : j \geq i\rangle$ has a pure extension in D and since (u, T) is preprocessed for D and i the condition $(v, \langle t_j : j \geq i\rangle$

belongs to D. But D is open and since $(v, \langle r_j : j \geq i\rangle) \geq (v, \langle t_j : j \geq i\rangle)$ we obtain that $(v, \langle r_j : j \geq i\rangle)$ belongs to D itself. □

Lemma 7. *Every condition (u, T) has an $\leq_{i+1}$ extension which is preprocessed for D and i.*

Proof. Let $T = \langle t_j : j \in \omega\rangle$. Fix an enumeration of all subsets of i: $v_1, \dots, v_k$. Consider $(v_1, \langle t_j : j \geq i\rangle)$. If $(v_1, \langle t_j : j \geq i\rangle)$ has a pure extension in D, denote it $(v_1, \langle t_j^1 : j \geq i\rangle)$. If there is no such pure extension, let $t_j^1 = t_j$ for every $j \geq i$. In the next step consider similarly $(v_2, \langle t_j^1 : j \geq i\rangle)$. If it has a pure extension in D, denote it $(v_2, \langle t_j^2 : j \geq i\rangle$. If there is no such pure extension, then for every $j \geq i$ let $t_j^2 = t_j^1$. At the k-th step we will obtain a condition $(v_k, \langle t_j^k : j \geq i\rangle)$. Then $(u, \langle t_j^k : j \in \omega\rangle)$ where for every $j < i$, $t_j^k = t_j$ is an $\leq_{i+1}$ extension of (u, T) which is preprocessed for D and i.

Really suppose $(v, \langle t_j^k : j \geq i\rangle)$ has a pure extension in D where $v \subset i$. Then $v = v_m$ for some m, $1 \leq m \leq k$. Then at step m, we must have had that $(v_m, \langle t_j^{m-1} : j \geq i\rangle$ has a pure extension in D, and so we have fixed such a pure extension $(v_m, \langle t_j^m : j \geq i\rangle) \in D$. However since $m - 1 < k$, we have

$$\langle t_j^m : j \geq i\rangle \leq \langle t_j^k : j \geq i\rangle.$$

But D is open and so $(v_m, \langle t_j^k : j \geq i\rangle)$ is an element of D itself. □

Lemma 8. *Let D be a dense open set. Then any condition has a pure extension which is preprocessed for D and every natural number i.*

Proof. Let $p = (u, T)$ be an arbitrary condition. Then be Lemma 7 we can construct inductively a fusion sequence $\{p_i\}_{i\in\omega}$ such that $p_0 = p$ and p_{i+1} is an $\leq_{i+1}$ extension of p_i which is preprocessed for D and i. Then if q is the fusion of the sequence for every $i \in \omega$ we have that $p_{i+1} \leq_{i+1} q$. This implies that $p_{i+1} \leq q$ and so by Lemma 6 q is preprocessed for D and i. □

Remark. Whenever p is a condition which is preprocessed for a given dense open set and every natural number n, we will simply say that p is preprocessed for D.

We are ready to show that the forcing notion Q satisfies Axiom A, part (3). Let D be a dense open set and p an arbitrary condition. By Lemma 8 there is a pure extension $q = (u, T)$ for $T = \langle t_j : j \in \omega\rangle$ which is preprocessed for D and every natural number. Recall that q is obtained as a fusion of a sequence and so in particular $p \leq_n q$ for every n. Furthermore the set

$$D_0 = \{(v, \langle t_j : j \geq i\rangle) \in D : v \subseteq i, i \in \omega, v \text{ end-extends } u\}$$

is a countable subset of D which is predense above q. Really let (w, R) be an arbitrary extension of q. Then since D is dense (w, R) has an extension $(w \cup w', R')$ in D. However $R' \geq R \geq \langle t_j : j \geq k_w \rangle$, where $k_w = \min\{j : \max w < \min \mathrm{int} t_j\}$. Therefore $(w \cup w', \langle t_j : j \geq k_w \rangle)$ has a pure extension in D and since q is preprocessed for D the condition $(w \cup w', \langle t_j : j \geq k_w \rangle)$ belongs to D. Thus in particular $(w \cup w', \langle t_j : j \geq k_w \rangle)$ belongs to D_0 and is compatible with (w, R) (with common extension $(w \cup w', R')$).

7. Logarithmic Measures Induced by Positive Sets

Lemma 9. *Let P be an upwards closed family of finite subsets of ω and h the induced logarithmic measure. Let $l \geq 1$. Then for every subset A of ω if A does not contain a set of measure $\geq l+1$, then there are A_0, A_1 such that $A = A_0 \cup A_1$ and none of A_0, A_1 contain a set of measure greater or equal l.*

Proof. Note that if A is a finite set, then the given condition is exactly part 3 of Definition 7. Thus assume A is infinite. For every natural number k, let $A_k = A \cap k$ and let T be the family of all functions $f : m \to \bigcup_{0 \leq k \leq m} A_k \times A_k$, where $m \in \omega$, such that for every k,

$$f(k) = (a_0^k, a_1^k) \in A_k \times A_k$$

where $a_0^k \cup a_1^k = A_k$, $h(a_0^k) \not\geq l$, $h(a_1^k) \not\leq l$ and for every $k : 1 \leq k \leq m$, $a_0^{k-1} \subseteq a_0^k$, $a_1^{k-1} \subseteq a_1^k$.

Then T together with the end-extension relation is a tree. Furthermore for every $m \in \omega$, the m-th level of T is nonempty. Really consider an arbitrary natural number m. Then $A \cap m = A_m$ is a finite set which is not of measure greater or equal $l+1$. By Definition 7, part (3), there are sets a_0^m, a_1^m such that $A_m = a_0^m \cup a_1^m$ and $h(a_0^m) \not\geq l$, $h(a_1^m) \not\geq l$. Let $a_0^{m-1} = A_m \cap a_0^m$ and $a_1^{m-1} = A_m \cap a_1^m$. Then by Corollary 2 the measure of each of a_0^{m-1}, a_1^{m-1} is not greater or equal to l and $A_{m-1} = A \cap (m-1) = a_0^{m-1} \cup a_1^{m-1}$. Therefore in m steps we can define finite sequences $\langle a_0^k : 0 \leq k \leq m \rangle$, $\langle a_1^k : 0 \leq k \leq m \rangle$ such that for every k, $A_k = a_0^k \cup a_1^k$, $h(a_0^k) \not\geq l$, $h(a_1^k) \not\geq l$ and $\forall k : 0 \leq k \leq m-1$ $a_0^k \subseteq a_0^{k+1}$, $a_1^k \subseteq a_1^{k+1}$. Therefore $f : m \to \bigcup_{0 \leq k \leq m} A_k \times A_k$ defined by $f(k) = (a_0^k, a_1^k)$ is a function in the m'th level of T.

Therefore by König's Lemma there is an infinite branch through T. Let $f : \omega \to \bigcup_{k \in \omega} A_k \times A_k$ where $f(k) = (a_0^k, a_1^k)$, $a_0^k \cup a_1^k = A_k$, etc., be such an infinite branch. Then if $A_0 = \bigcup_{k \in \omega} a_0^k$, $A_1 = \bigcup_{k \in \omega} a_1^k$ we have that $A = A_0 \cup A_1$ and none of the sets A_0, A_1 contain a set of measure greater or equal l. Consider arbitrary finite subset x of A_0.

Then $x \subseteq a_0^k$ for some $k \in \omega$. But $h(a_0^k) \ngeq l$ and so $h(x) \ngeq l$. The same argument applies to A_1. □

Lemma 10 (Sufficient Condition for High Values)**.** *Let P be an upwards closed family of finite subsets of ω and h the logarithmic measure induced by P. Then if for every $n \in \omega$ and every partition of ω into n-sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is some $j \leq n-1$ such that A_j contains a positive set, then for every natural number k, for every $n \in \omega$ and partition of ω into n-sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is some $j \leq n-1$ such that A_j contains a set of measure greater or equal k.*

Proof. The proof proceeds by induction on k. If $k = 1$ this is just the assumption of the Lemma. So suppose we have proved the claim for $k = l$ and furthermore that it is false for $k = l + 1$. Then there is some $n \in \omega$ and partition of ω into n-sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ such that none of $A_0, \ldots, A_{n-1}$ contain a set of measure greater or equal $l + 1$. By Lemma 9 for each $j \leq n-1$ there are sets A_j^0, A_j^1 none of which contains a set of measure greater or equal l and such that $A_j = A_j^0 \cup A_j^1$. Then

$$\omega = A_0^0 \cup A_0^1 \cdots \cup A_{n-1}^0 \cup A_{n-1}^1$$

is a partition of ω into $2n$ sets, none of which contains a set of measure $\geq l$. This contradicts the inductive hypothesis for $k = l$. □

8. The Bounding Number

Lemma 11. *Let D be a dense open set, $T = \langle t_j : j \in \omega \rangle$ a pure condition which is preprocessed for D. Let $v \in [\omega]^{<\omega}$. Then the family $\mathcal{P}_v(T)$ which consists of all finite subsets x of ω such that*

(1) $\exists l \in \omega$ s.t. $x \cap int(t_l)$ is t_l positive
(2) $\exists w \subseteq x$ s.t. $(v \cup w, T) \in D$.

induces a logarithmic measure $h = h_v(T)$ which takes arbitrary high values.

Proof. The family $\mathcal{P}_v(T)$ is nonempty and upwards closed. Consider the condition (v, T). Since D is dense there is an extension $(v \cup w, R)$ of (v, T) which belongs to D. By definition of the extension relation $w \subseteq \text{int}(T)$ and so for some $l \in \omega$ we have $w \subseteq \cup\{\text{int}(t_j) : j = 0, \ldots, l-1\}$. However $(v \cup w, R)$ is a pure extension of $(v \cup w, \langle t_j : j \geq l \rangle)$ and since T is preprocessed for D (and every natural number) the condition $(v \cup w, \langle t_j : j \geq l \rangle)$ belongs to D. Then $x = \cup\{\text{int}(t_j) : j = 0, \ldots, l-1\}$ is an element of $\mathcal{P}_v(T)$.

To show that h takes arbitrarily high values it is enough to show that for every n and partition of ω into n-sets $\omega = A_0 \cup \ldots \mathcal{A}_{n-1}$, there is $k \leq n-1$ such that A_k contains a positive set. Thus fix a natural number n and a partition of ω. For every $k : 0 \leq k \leq n-1$ and $j \in \omega$ let $s^k_j = s_j \cap A_k$ where $t_j = (s_j, h_j)$. Suppose that for every k there is a constant M_k such that $h_j(s^k_j) \leq M_k$, i.e. the constant M_k bounds the measures of $s_j \cap A_k$. Then let $M = \max_{k \leq n-1} M_k$. Since T is a pure condition the measures $h_j(s_j)$ take arbitrarily high values and so in particular there is an $i \in \omega$ such that $h_j(s_j) \geq M+n+1$. By Corollary 1 there is a $k : 0 \leq k \leq n-1$ such that $h_i(s^k_i) \geq (M+n)-k \geq M+1 > M_k$ (notice that $s_i = s^0_i \cup \dots s^{n-1}_i$) which is a contradiction to the definition of M_k. Therefore there is some k such that the measures $h_j(s^k_j)$ take arbitrarily high values and so there is a pure extension $R = \langle r_j : j \in \omega\rangle$ of T such that $\text{int}(R) \subseteq A_k$. Since D is dense, there is an extension $(v \cup w, R')$ of (v, R) which belongs to D. By definition of the extension relation on Q, $w \subseteq \cup\{\text{int}(r_j) : j = 0, \dots, l\}$ for some $l \in \omega$. However $(v \cup w, R') \geq (v \cup w, T)$ and since T is preprocessed for D, $(v \cup w, T) \in D$. Therefore

$$x = \bigcup\{\text{int}(t_j) : j = 0, \dots, l-1\}$$

is a positive set contained in A_k. □

Corollary 3. *Let D be a dense open set and $T = \langle t_j : j \in \omega\rangle$ a pure condition which is preprocessed for D. Let $v \in [\omega]^{<\omega}$. Then there is a pure extension $R = \langle r_j : j \in \omega\rangle$ such that for every $l \in \omega$ and every $s \subseteq$ int(r_l) which is r_l-positive, there is $w \subseteq s$ such that $(v \cup w, \langle t_j : j \geq l+1\rangle) \in D$.*

Proof. Let h be the logarithmic measure induced by $\mathcal{P}_v(T)$. Consider the following inductive construction. Let x_0 be any positive set. Then there is $B_0 \in [\omega]^{<\omega}$ such that $x_0 \subseteq \cup\{\text{int}(t_j) : j \in B_0\}$. Let $r_0 = (x_0, h \restriction x_0 + 1)$. Furthermore let $A_0 = \max\{\text{int}(t_j) : j = \max(B_0)\} + 1$, $A_1 = \omega \backslash A_0$ and $H_1 = \max\{h(x) : x \subseteq A_0\}$. Then by the sufficient condition for arbitrarily high values there is $x_1 \subseteq A_1$ such that $h(x_1) \geq H_1 + 1$. Furthermore there is a finite set B_1 such that $\max B_0 < \min B_1$ and such that $x_1 \subseteq \cup\{\text{int}(t_j) : j \in B_1\}$. Let $r_1 = (x_1, h \restriction x_1 + 1)$.Proceed inductively. Suppose $\langle r_0, \dots, r_{k-1}\rangle$, $\langle B_0, \dots, B_{k-1}\rangle$ have been defined so that

1. $r_j = (x_j, h \restriction x_j + 1)$, $x_j \subseteq \cup\{\text{int}(t_i) : i \in B_j\}$
2. $h(x_j) < h(x_{j+1})$ and $\max B_j < \min B_{j+1}$.

To obtain r_k let $A_0 = \max\{\text{int}(t_j) : j = \max(B_{k-1})\} + 1$, $A_1 = \omega \backslash A_0$, $H_k = \max\{h(x) : x \subseteq A_0\}$. Then by the sufficient condition for high values there is $x_k \subseteq A_k$ such that $h(x_k) \geq H_k + 1$. Furthermore there is a finite set B_k such that $\max B_{k-1} < \min B_k$ and $x_k \subseteq \cup\{\text{int}(t_j) : j \in B_k\}$. Let $r_k = (x_k, h \restriction x_k + 1)$.

Let $R = \langle r_j : j \in \omega\rangle$ be the so constructed condition. Suppose $e \subseteq \text{int}(r_j) = x_j$ is r_j-positive. That is $h(e) > 0$ and so $x \in \mathcal{P}_v(T)$. But then by part (2) of the Definition of $\mathcal{P}_v(T)$ there is an $l \in B_j$ such that $e \cap \text{int}(t_l)$ is t_l-positive. This implies that R is an extension of T.

Furthermore, consider any $l \in \omega$ and $s \subseteq \text{int}(r_l)$ which is r_l-positive. Then $s \in \mathcal{P}_v(T)$ and so there is $w \subseteq s$ such that $(v \cup w, T) \in D$. But $(v \cup w, \langle r_j : j \geq l+1\rangle)$ extends $(v \cup w, T)$ and since D is open the condition $(v \cup w, \langle r_j : j \geq l+1\rangle)$ belongs to D itself. □

Remark. Whenever R is a pure condition which satisfies Corollary 3 for some given dense open set D, and finite subset v of ω we will say that $\phi(v, R, D)$ holds. Note also that any further pure extension of R preserves this property.

Corollary 4. *Let D be a dense open set, T a pure condition which is preprocessed for D and $k \in \omega$. Then there is a pure extension R of T, $R = \langle r_j : j \in \omega\rangle$ such that $\forall v \subset k \forall l \forall s \subseteq int(r_l)$ which is r_l-positive, there is $w_v \subseteq s$ such that $(v \cup w, \langle r_j : j \geq l+1\rangle) \in D$.*

Proof. Let $v_1, \dots, v_n$ be an enumeration of all (proper) subsets of k. By Corollary 3 for each $j = 1, \dots, n$ there is a pure extension T_j of T_{j-1} (where T_0 is the given condition T) such that $\phi(v_j, T_j, D)$. Then $R = T_n$ has the required property. □

Remark. Whenever R is a pure condition which satisfies the property of the above statement for some natural number k and dense open set D we will say that $\phi(k, R, D)$ holds.

Lemma 12. *Let $\dot{f}$ be a Q-name for a function in ${}^\omega\omega$ and p arbitrary condition in Q. Then there is a pure extension $q = (u, R)$ of p, where $R = \langle r_i : i \in \omega la$ such that $\forall i \forall v \subset i \forall s \subseteq int(r_i)$ which is r_i-positive, there is $w_v \subseteq s$ such that $(v \cup w_v, \langle r_j : j \leq i+1\rangle) \Vdash \dot{f}(i) = \check{k}$ for some $k \in \omega$.*

Proof. Consider the following inductive construction. Let $p = (u, T)$ where $T = \langle t_i : i \in \omega\rangle$. For every $n \in \omega$ denote by D_n the dense open set of all conditions in Q which decide the value of $\dot{f}(n)$. Let $k_0 = \min \text{int}(t_0)$. Then by Lemma 8 we can assume that the pure condition T is preprocessed for D_0 and so by Corollary 4 there is a pure extension $T_1 = \langle t_i^1 : i \in \omega\rangle$ of T such that $\phi(k_0, T_1, D_0)$. Then if $p_1 = (u, T_1)$ we have $p_0 \leq_1 p_1$. To define p_2 consider $k_1 = \max \text{int}(t_0^1) + 1$. Again we can assume that $\langle t_i^1 : i \geq 1\rangle$ is preprocessed for D_1 (otherwise by Lemma 8 pass to such an extension). Then there is a pure extension $T_2 = \langle t_i^2 : i \geq 1\rangle$ of $\langle t_i^1 : i \geq 1\rangle$ such that $\phi(k_1, T_2, D_1)$. Let $p_2 = (u, \langle t_i^2 : i \in \omega\rangle)$ where $t_0^2 = t_0^1$, $k_2 = \max \text{int}(t_1^2) + 1$.

Proceed inductively. Suppose $p_0, \dots, p_n$ have been defined so that $p_j \leq_{j+1} p_{j+1}$ for every $j = 1, \dots, n-1$, where $p_j = (u, \langle t_i^j : i \in \omega\rangle)$ and $\phi(k_j, \langle t_i^{j+1} : i \geq j\rangle, D_j)$. Let $k_n = \max \operatorname{int}(t_{n-1}^n) + 1$. We can assume that $\langle t_i^n : i \geq n\rangle$ is preprocessed for D_n. Then by Corollary 4 there is a pure extension $T_{n+1} = \langle t_i^{n+1} : i \geq n\rangle$ of $\langle t_i^n : i \geq n\rangle$ such that $\phi(k_n, T_{n+1}, D_n)$. Let $p_{n+1} = (u, \langle t_i^{n+1} : i \in \omega\rangle)$ where $t_i^{n+1} = t_i^{i+1}$ for every $i = 0, \dots, n-1$. Then $p_n \leq_{n+1} p_{n+1}$.

Let $q = (u, \langle r_j : j \in \omega\rangle)$ be the fusion of the sequence. Let $i \in \omega$, $v \subset i$ and $s \subset \operatorname{int}(r_i)$ which is r_i-positive. However $r_i = t_i^{i+1}$ and so $s \subseteq \operatorname{int}(t_i^{i+1})$ is t_i^{i+1}-positive. Also $\phi(k_i, T_{i+1}, D_i)$ holds and so there is $w_v \subseteq s$ such that $(v \cup w_v, \langle t_j^{i+1} : j \geq i+1\rangle) \in D_i$. It remains to notice that $\langle r_j : j \geq i+1\rangle$ is extends $\langle t_j^{i+1} : j \geq i+1\rangle$ and since D_i is open, $(v \cup w_v, \langle r_j : j \geq i+1\rangle) \in D_i$. By definition of D_i that is

$$(v \cup w_v, \langle r_j : j \geq i+1\rangle) \Vdash \dot{f}(i) = \check{k}$$

for some natural number k. □

Theorem 8. The forcing notion Q is almost ${}^\omega\omega$-bounding.

Proof. Let $\dot{f}$ be arbitrary Q-name of a function and p a condition in Q. Let $q = (u, T)$, where $T = \langle t_i : i \in \omega\rangle$ be a pure extension of p which satisfies the Main Lemma. Then for every $i \in \omega$ define

$$g(i) = \max\{k : v \subseteq i, w \subseteq \operatorname{int}(t_i), (v \cup w, \langle t_j : j \geq i+1\rangle) \Vdash \dot{f}(i) = \check{k}\} .$$

Consider any $A \in [\omega]^{<\omega}$ and let $q_A = (u, \langle t_i : i \in A\rangle)$. We claim that

$$q_A \Vdash \forall n \exists k (k \geq n \wedge k \in A \wedge \dot{f}(k) \leq g(k)) .$$

Fix any $n_0 \in \omega$. Let (v, R) be an arbitrary extension of q_A. Then there is $i_0 \in A$ such that $i_0 < n_0$, $v \subseteq i_0$ and $s = \operatorname{int}(R) \cap \operatorname{int}(t_{i_0})$ is t_{i_0}-positive. Note that $i_0 \leq k_{i_0} = \max \operatorname{int}(t_{i_0 - 1}) + 1$ and so $v \subset k_{i_0}$. But then by Lemma 12 there is $w \subseteq s$ such that $(v \cup w, \langle t_j : j \geq i_0 + 1\rangle) \Vdash \dot{f}(i_0) = \check{k}$ and so in particular

$$(v \cup w, \langle t_j : j \geq i_0 + 1\rangle) \Vdash \dot{f}(i_0) \leq g(i_0) .$$

However $(v \cup w, R)$ extends $(v \cup w, \langle t_j : j \geq i_0 + 1\rangle)$ and so $(v \cup w, R) \Vdash \dot{f}(i_0) \leq g(i_0)$. Note also that $(v \cup w, R)$ extends (v, R). Then, since (v, R) was an arbitrary extension of q_A, the set of conditions which force "$\exists i_0$ s.t. $i_0 \geq n_0 \wedge i_0 \in A \wedge \dot{f}(i_0) \leq g(i_0)$" is dense above q_A. Therefore

$$q_A \Vdash \exists k (k \geq n_0 \wedge k \in A \wedge \dot{f}(k) \leq g(k)) .$$

The natural number n_0 was arbitrary and this completes the proof of the theorem. □

References

[1] U. Abraham *Proper Forcing,* for the Handbook of Set-Theory.
[2] M. Godstern *Tools for your forcing constructions,* In Set Theory of the Reals, vol.6 of Israel Mathematical Conference Proceedings, 305-360
[3] S. Shelah *Proper and Improper Forcing,* Second Edition. Springer, 1998.
[4] S. Shelah *On cardinal invariants of the continuum[207]* In (J.E. Baumgartner, D.A. Martin, S. Shelah eds.) Contemporary Mathematics (The Boulder 1983 conference) Vol. 31, Amer. Math. Soc. (1984), 184-207.

E-mail address: `vfischer@mathstat.yorku.ca`