READING COURSE: SET THEORY
WS 2017

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This is a master level course in set theory. We will consider some of the central set theoretic, combinatorial properties of the real line, as well as their connections to some of the topological and measure theoretic properties of the reals. **The lectures are taking place Fridays from 9:00 to 10:30 at the Seminar room of the KGRC (room 101).** The final exam will be oral.

Detailed information on the material covered during the semester, including relevant references to the literature, will be regularly given here.

Lecture 1, 06.10.: The notion of a cardinal characteristics of the continuum was introduced. In particular, we discussed two types of characteristics *combinatorial*, as well as *categoriacal and measure theoretic* ones.

The first reading task is **Chapter 8** *of the book of L. Halbesien, Combinatorial set theory (see* [4]*).*

Excellent expository articles regarding the cardinal charactersitcs of the continuum are A. Blass, *Combinatorial Cardinal Characteristics of the Continuum* (see [2]) and T. Bartoszynski, *Invariants of measure and category* (see [1]). Both of those articles should be available on-line, but if you cannot access them, please contact me to get a copy. In addition, for those students in the class, who are familiar with the classical cardinal characteristics of the continuum, we briefly introduced the recently defined *rearrangement number* (see [3]), which will be discussed in detail later in the semester.

Lecture 2, 13.10: The characteristics $\mathfrak{p}, \mathfrak{b}, \mathfrak{d}, \mathfrak{s}, \mathfrak{r}$ and $\mathfrak{a}$ were introduced. We established many of the existing ZFC relations between those (see [4]), and briefly discussed the lack of ZFC dependence between each of $\{\mathfrak{b}, \mathfrak{s}\}$, $\{\mathfrak{s}, \mathfrak{a}\}$, and $\{\mathfrak{a}, \mathfrak{d}\}$.

As some of our colleagues showed interest and wanted to find out more about these independence results, here are some good references (note that as this is rather advanced material, it will not be required at the final exam):

- The consistency of $\mathfrak{s} < \mathfrak{b}$: [J. Baumgartner; P. Dordal, *Adjoining dominating functions.* J. Symbolic Logic 50 (1985), no. 1, 94101.] As $\mathfrak{b} \leq \mathfrak{a}$, this is also a model of $\mathfrak{s} < \mathfrak{a}$.
- The consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$: [S. Shelah, *On cardinal invariants of the continuum.* Axiomatic set theory (Boulder, Colo., 1983), 183207,

Contemp. Math., 31, Amer. Math. Soc., Providence, RI, 1984.] This is the first appearance of a forcing technique, known as *creature forcing*, which found broad applications, among others, in the study of the cardinal characteristics associated with measure and category.
- The consistency of $\mathfrak{a} < \mathfrak{d}$ holds in the Cohen model. For those that have taken "Axiomatic Set Theory I", please review the proof: on one side there is a Cohen indestructible m.a.d. family, and on the other hand Cohen forcing adds an unbounded real.
- The consistency of $\aleph_2 \leq \mathfrak{d} = \mathfrak{b} < \mathfrak{a}$ is due to S. Shelah and holds in his template model. An axiomatic treatment of the construction can be found in [J. Brendle, *Mad families and iteration theory.* Logic and algebra, 131, Contemp. Math., 302, Amer. Math. Soc., Providence, RI, 2002.]. The fact that in the final extension there are no m.a.d. families of size λ, where $\mathfrak{b} \leq \lambda < \mathfrak{c}$ is due to an isomorphism-of-names argument. The same idea stays behind the proof, that in the Cohen model, every maximal almost disjoint family is of size $\aleph_1$ or $\mathfrak{c}$ (we will discuss these properties of the Cohen extension in detail in the next few lectures).

On the 20th of October, we will continue with the discussion of **Chapter 8** *of the book of L. Halbesien, Combinatorial set theory.*

Lecture 3, 20.10.: We studied the independence number and showed that there is always a maximal independent family of size $\mathfrak{c}$, that $\mathfrak{r} \leq \mathfrak{i}$ and also that $\mathfrak{d} \leq \mathfrak{i}$. Among the combinatorial cardinal characteristics, $\mathfrak{i}$ is one of those for which upper bounds apart from $\mathfrak{c}$ are not known.

Next time, we will prove Ramsey's theorem (see [4, Theorem 2.1]*) and consider* $\mathfrak{h}$*,* $\mathfrak{par}$ *and* $\mathfrak{hom}$*.* In addition, time permiting, for those familiar with forcing we will outline a proof of the consistency of Spec(m.a.d. families) = $\{\aleph_1, \mathfrak{c}\}$ where $\mathfrak{c} > \aleph_1$.

Lecture 4, 27.10.: We completed our discussion of Chapter 8 of [4].

Lecture 5, 3.11.: We defined Martin's axiom, showed that MA(κ) implies that $\mathfrak{p} > \kappa$, proved Solovay's Lemma, defined add($\mathcal{M}$) and proved that MA(κ) implies that add($\mathcal{M}$) $> \kappa$. The material can be found in the old book of K. Kunen, "Set theory" (Chapter 2 of [6]), alternatively see Lemma III.1.25. of [5] for a proof of $\mathfrak{p} \leq$ add($\mathcal{M}$).

Next time we will start our discussion of Chapter 9 of [4]*. Please, read the first two sections of the chapter.*

A free reading assignement is the proof by P. Matet of the Canonical Ramsey Theorem, see [7]. Time permitting, we will also discuss the proof in class.

Lecture 6, 10.11.: We saw an overview of a proof that MA(κ) implies that add($\mathcal{N}$) $> \kappa$, as we all as an overview of Matet's proof of the Canonical Ramsey Theorem (the former can be found in [6], the latter in [7]). We introduced the ideal $\mathcal{R}_0$ of completely Ramsey null sets and proved that it is a σ-ideal (using a *fusion* argument). The proofs that $\mathcal{R}_0$ is proper and contains all $\{x\}$ where $x \in [\omega]^\omega$ was left as an exercise. A proof of the existence of a coloring $\pi : [\omega]^\omega \to 2$ without a π-homogeneous set can be found on page 18 of [4].

Lecture 7, 17.11.: We proved that $\text{add}(\mathcal{R}_0) = \text{cov}(\mathcal{R}_0) = \mathfrak{h}$, introduced the Ellentuck topology, showed that every open set is completely Ramsey, and that a set is nowhere dense if and only if it is completely Ramsey-null.

Lecture 8, 24.11.: We proved that a set X has the Baire property in the Ellentuck topology iff it is completely Ramsey. Then, we moved, by slightly changing the topic and discussed the Delta System Lemma.

Lecture 9, 01.12.: We started our discussion of trees and in particular constructed an Aronszajn tree from a coherent sequence. In addition, we showed (the proof will be completed next time) that there is a special Aronszajn tree. The material can be found in [5, Section III.5].

Lecture 10, 15.12.: We finished the proof of the existence of a special Aronszajn tree, and constructed a Sulsin line from a Suslin tree.

Lecture 11, 12.01.: We showed that if there is Suslin line, then there is a Sulin tree; if an ω_1-tree is ever branching and it does not have uncountable antichains, then it is Suslin. In addition, we stated the diamond principle and set to prove that it implies the existence of a Suslin tree.

As a reading assignment, please read on closed unbounded sets and stationary sets - Section III.6 from [5]*, pages 217 -220 (without Lemma III.6.15), as well as the proof of Theorem III.7.3. of* [5]*.*

As a **free** *reading assignement look at the Theorem III.7.13 from* [5]*, proving that in the Constructible Universe the diamond principle holds, as well as Section II.6 (up to an including Theorem II.6.20.).*

Lecture 12, 19.01: clubs, stationary sets, the existence of a diamond sequence implies the existence of a Suslin tree; We concluded with a brief description of the following two topics: the Constructible Universe and its well-order, as well as the generalized Suslin operation.

That was the **last lecture** for the semester. On **26.01.** at 9am will be the first exam session. Another possibility to take the exam is **01.03.** at 9am. If none of those are possible for you, please do send me an Email to arrange for another appointment.

References

[1] T. Bartoszynski *Invariants of measure and category.* Handbook of set theory. Vols. 1, 2, 3, 491555, Springer, Dordrecht, 2010.

[2] A. Blass *Combinatorial cardinal characteristics of the continuum.* Handbook of set theory. Vols. 1, 2, 3, 395489, Springer, Dordrecht, 2010.

[3] A. Blass, J. Brendle, W. Brian, J. D. Hamkins, M. Hardy, P. B. Larson *The rearangment number* arXiv:1612.07830

[4] L. Halbesien *Combinatorial set theory. With a gentle introduction to forcing.* Springer Monographs in Mathematics. Springer, London, 2012. xvi+453 pp. ISBN: 978-1-4471-2172-5; 978-1-4471-2173-2

[5] K. Kunen *Set theory.* Studies in Logic (London), 34. College Publications, London, 2011. viii+401 pp. ISBN: 978-1-84890-050-9

[6] K. Kunen *Set theory. An introduction to independence proofs.* Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam, 1980. xvi+313 pp. ISBN: 0-444-85401-0

[7] P. Matet, *An easier proof of the canonical Ramsey theorem.* Colloq. Math. 145 (2016), no. 2, 187191.

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