THE CONSISTENCY OF ARBITRARILY LARGE SPREAD BETWEEN $\mathfrak{u}$ AND $\mathfrak{d}$

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1. Preliminaries

In this talk we will consider the independence of $\mathfrak{u}$ and $\mathfrak{d}$, where

$$\mathfrak{u} = \min\{|\mathcal{G}| : \mathcal{G} \text{ generates an ultrafilter}\}$$

and

$$\mathfrak{d} = \min\{|D| : D \text{ is a dominating family}\}.$$

In particular we will obtain the consistency of arbitrarily large spread between $\mathfrak{u}$ and $\mathfrak{d}$.

Theorem 1 (GCH)**.** Let ν and δ be arbitrary regular uncountable cardinals. Then, there is a countable chain condition forcing extension in which $\mathfrak{u} = \nu$ and $\mathfrak{d} = \delta$.

As an application of the method used to obtain the result above, we will obtain the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$ where $\mathfrak{b}$ is the bounding number, $\mathfrak{s}$ is the splitting number and κ is an arbitrary regular cardinal.

Suppose $\nu \geq \delta$. Begin with a model of GCH and adjoin δ-many Cohen reals $\langle r_\alpha : \alpha \in \delta\rangle$ followed by ν-many Random reals $\langle s_\xi : \xi \in \nu\rangle$. That is, if V_δ is the model obtained after the first δ Cohen reals, the generic extension in which we are interested is obtained by finite support iteration of length ν of Random real forcing over V_δ. Since random forcing is ${}^\omega\omega$-bounding, the Cohen reals remain a dominating family in the final generic extension $V_{\delta,\nu}$. Furthermore for any family of reals of size smaller than δ there is a Cohen real which is unbounded by this family, and so $V_{\delta,\nu} \vDash \mathfrak{d} = \delta$. To verify that $\mathfrak{u} = \nu$, recall that if a is random real over some model M, then neither a, no $\omega - a$ contains infinite sets from M. Again since the ground model V satisfies GCH and the forcing notions with which we work have the countable chain condition, any set of reals $\mathcal{A}$ in $V_{\delta,\nu}$ of size smaller than ν is obtained at some initial stage of the random real forcing iteration $V_{\delta,\alpha}$ for some $\alpha < \nu$. But then neither s_α nor $\omega - s_\alpha$ contains an element of $\mathcal{A}$ and so $\mathcal{A}$ does not generate an ultrafilter. Therefore $\mathfrak{u} = \mathfrak{c} = \nu$.

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2. The consistency of $\mathfrak{d} = \delta < \mathfrak{s} = \nu$

In the following we assume that $\nu < \delta$. The model of $\mathfrak{u} = \nu < \mathfrak{d} = \delta$ will be obtained again as a countable chain condition forcing extension of aa model V of GCH. First adjoin δ many Cohen reals $\rangle r_\alpha : \alpha < \delta\langle$ to obtain a model $V(\delta, 0)$ (the model determined by $\langle r_\alpha : \alpha < \beta\rangle$ will be denoted $V(\beta, 0)$) and then for some appropriately chosen ultrafilters U_α $\alpha < \nu$ we will adjoin ν-name Mathias reals over $V(\delta, 0)$ to obtain the desired forcing extension $V(\delta, \nu)$. Again for $\xi < \nu$, $V(\delta, \xi)$ will denote the model obtained after adding the first ξ-name Mathias reals $\langle s_\eta : \eta < \xi\rangle$ over $V(\delta, 0)$.

For this purpose we will have to fix some terminology and consider some more basic properties of the required forcing notions.

Definition 1. Let U be an ultrafilter on ω. Then the Mathias forcing associated with U, $Q(U)$ consists of all pairs (a, A) where a is a finite subset of ω, $A \in U$. We say that (a, A) extends (b, B) (and denote this by $(a, A_{\leq}(b, B)))$ iff a end-extends b, $a\backslash b \subseteq B$ and $A \subseteq B$.

Note that $Q(U)$ is σ-centered and so has the countable chain condition for every ultrafilter u. Let G be $Q(U)$-generic. Then

$$s(G) = \cup\{a : \exists A \in U((a, A) \in Q(U))\}$$

is called the Mathias real adjoined by $Q(U)$. For every condition (a, A) in $Q(U)$ we have

$$(a, A) \Vdash (s(G) \subseteq^* A) \wedge (a \subseteq s(G)).$$

Thus (a, A) has the information of the generic real $s(G)$, that a is an initial segment $s(G)$ and that $s(G)\backslash a \subseteq^* A$.

Definition 2. Let f be a name for a function in ${}^\omega\omega$. We say that f is *normalized* if there is a countable family of maximal antichains W_n, $n \in \omega$ and functions $f_n : W_n \to \omega$ such that for every $p \in W_n$ we have

$$f_n(p) = m \text{ iff } p \Vdash f(n) = m.$$

We denote this by $f = ((W_n, f_n) : n \in \omega)$.

In the following we will assume that all names for reals are normalized.

The desired model will bo obtained as a countable chain condition extension over a model V of GCH by adding δ-name Cohen reals to obtain a model $V(\delta, 0)$ followed by the finite support iteration of Mathias forcing for appropriately chosen ultrafilters. The family $\langle r_\alpha : \alpha < \delta\rangle$ will be witness to $\mathfrak{d} = \delta$. The only requirement that we will insist on the ultrafilters U_α to have is that it contains all Mathias reals obtained at a previous stage of the iteration. That is $\langle s_\xi : \xi < \alpha\rangle \subseteq U_\alpha$. Therefore the α'th Mathias real s_α is almost contained in the preceding ones. But then the sequence $\langle s_\xi : \xi < \nu\rangle$ in $V(\delta, \nu)$ together with its intersections with cofinite subsets of ω generates an ultrafilter in $V(\delta, \nu)$. Therefore $\mathfrak{u} \leq \nu$. We will show that no family of size smaller than ν in $V(\delta, \nu)$ generates an ultrafilter.

Really. Consider any family $\mathcal{G} \subseteq V(\delta, \nu) \cap [\omega]^\omega$ of cardinality smaller than ν. Since we work with forcing notions having the countable chain condition over a model of GCH there is an initial stage of the Mathias iteration over $V(\delta, 0)$, namely $V(\delta, \alpha)$ for some $\alpha < \nu$ such that $\mathcal{G}$ is contained in $V(\delta, \alpha)$. Let s_α be the α'th Mathias real and let

$$X = \{n : |s_\alpha \cap n| \text{ is even}\}.$$

Then $X \in V(\delta, \alpha + 1)$ and we will see that no infinite subset of $V(\delta, \alpha)$ is contained in X or in $\omega - X$. Suppose not. Then there is an infinite subset Y of ω and a condition $(a, A) \in Q(U_\alpha)$ which forces that Y is a subset of X or a subset of $\omega - X$. Let $m = \min A$ and let y be any condition in A which is greater than m. Then certainly $(a, A - y)$ and $(a \cup \{m\}, A - y)$ are extensions of (a, A). However

$$(a, A - y) \Vdash_{Q(U_\alpha)} s_\alpha \cap y = a$$

and

$$(a \cup \{m\}, A - y) \Vdash_{Q(U_\alpha)} s_\alpha \cap y = a \cup \{m\}.$$

Therefore one of these extensions forces that $y \in X$ and the other one $y \notin X$ which is impossible.

Therefore $\mathcal{G}$ does not generate an ultrafilter and since $\mathcal{G}$ was arbitrary of size smaller than ν, we obtain that $\mathfrak{u} = \nu$.

To preserve then δ-many Cohen reals unbounded it is essential that we choose the ultrafilters U_α very carefully, since for example if U_α is selective then it adds a dominating real. The following Lemma will allow us to achieve this.

Lemma 1. *Let $M \subseteq M'$ be models of ZFC^* (sufficiently large portion of ZFC)Let $U \in M$ be an ultrafilter in ω and $g \in M' \cap {}^\omega\omega$ a real which is not dominated by the reals of M. Then*

1. *(1) $\exists U'$ ultrafilter in M' such that $U \subseteq U'$*
2. *(2) every maximal anitchain of $Q(U)$ in M is a maximal antichain for $Q(U')$*
3. *(3) for every $Q(U)$-name for a real f we have $\mathbb{1} \Vdash g \not\leq^* f$.*

Proof. We will analyze what it means there not to be an ultrafilter extending U with the desired properties. We will say that an infinite subset A of ω is *forbidden* by a finite set a and a maximal antichain L

of $Q(U)$ in M, if (a, A) is incompatible with all elements of L. That is there is no finite subset e of A such that $a \cup e$ is the finite part of a common extension of (a, A) and a member of L.

We will say that A is *forbidden* by a finite set a and a $Q(U)$-name f for a function in ${}^\omega\omega$ if for every $n \in \omega$ the condition (a, A) is not compatible with any condition $p \in Q(U)$ such that $p \Vdash f(n) < g(n)$. That is, if $f = ((W_n, f_n) : n \in \omega)$ is a normalized name and (a, A) is compatible with some $p \in W_n$ then $g(n) \leq f_n(p)$.

By Zorn's Lemma it is sufficient to show that no infinite set $Z \in U$ is covered by finitely many forbidden sets in M'. Suppose to the contrary that there is a set $Z \in U$ such that Z is the disjoint union of $A_1, \dots, A_k$, $B_1, \dots, B_k$ such that for every $i \leq k$, A_i is forbidden by a finite set a_i and a maximal antichain L_i in $Q(U)$, and B_i is forbidden by a finite set b_i and a $Q(U)$-name for a real f. Let n_0 be an integer greater than a_i, b_i for every $i \leq k$. We can assume that $Z \subseteq \omega - n_0$.

Claim. For every $n \in \omega$ there is $h(n) > n$ such that whenever $Z \cap [n, h(n))$ is partitioned into $2k$-pieces at least one of them, say P, has the following two properties:

(1) $\forall i \leq k$, there is a finite subset e of P such that $a_i \cup e$ is permitted by a member of L_i,
(2) $\forall i \leq k$, there is a finite subset e of P such that $a_i \cup e$ is permitted by some $p \in W_n$ for which $f_n(p_n) < h(n)$.

Proof. Suppose there is $n \in \omega$ for which this is not true. Then by Koenig's Lemma there is a partition of Z into $2k$ pieces none of which has the above two properties no matter how large $h(n)$ is. However U is an ultrafilter and so at least one of those pieces, say P belongs to U. Let $i \leq k$. Then there is a finite subset e of P such that $a_i \cup e$ is compatible with an element of L_i and so P satisfies condition (i) above. Similarly there is a finite subset e of P such that $b_i \cup e$ is permitted by a condition $p \in W_n$. However (ii) holds as long as we choose $h(n)$ sufficiently large, which is a contradiction since P should not satisfy both of conditions (i) and (ii). $\square$

Consider any $n > n_0$ and partition $Z \cap [n, h(n)]$ into $2k$ pieces, namely $A_i = Z \cap [n, h(n))$, $B_i = Z \cap [n, h(n))$. By the above claim at least one of them, say P has properties (i) and (ii).

If $P = A_i \cap [n, h(n))$ then there is a finite subset e of A_i permitted by an element of L_i, which is a contradiction since A_i is forbidden by a_i and L_i. Thus it must be the case that $P = B_i \cap [n, h(n))$ for some $i \leq k$ and so there is a finite subset e of B_i such that $b_i \cup e$ is permitted by some element p of W_n for which $f_n(p) < h(n)$. Since B_i is forbidden by f it must be the case that $g(n) \leq h(n)$. However this holds for every $n > n_0$ and so $g \leq^* h$. Note that $h \in M$ which is the desired contradiction. □

Corollary 1. *Let G' be $Q(U')$-generic filter over M'. Then*

1. *$G = G' \cap Q(U)$ is $Q(U)$-generic over M,*
2. *if $s(G')$ is the real added by $Q(U')$ and $s(G)$ is the real added by $Q(U)$ then $s(G) = s(G')$,*
3. *for every $Q(U)$-name for a real f, the evaluations of f with respect to G and G' coincide.*

Proof. Note that if $(a, A) \in G'$ for some $Q(U')$-generic filter over M', then $(a, \omega - a)$ is also in G' and so $(a, \omega - a) \in G' \cap Q(U)$. □

Thus we can proceed with the actual construction of the Mathias extension over $V(\delta, 0)$. On the ground model $V(0,0)$ choose an arbitrary ultrafilter $U(0,0)$. Since r_1 is Cohen over $V(0,0)$, r_1 is unbounded by the reals on $V(0,0)$ we can apply the Main Lemma to obtain an ultarfilter $U(1,0)$ which extends the given one and has the properties from the main Lemma. Furthermore, by transfinite induction of length ν we can obtain a sequence $U(\alpha, 0)$ of ultrafilters in $V(\alpha, 0)$ with the following properties. For every $\alpha \leq \delta$

1. $\forall \beta < \alpha$, $U(\beta, 0) \subseteq U(\alpha,)$
2. $\forall \beta < \alpha$ every maximal antichain of $Q(U_\beta)$ from $V(\beta, 0)$ remains maximal in $V(\alpha, 0)$
3. for every $Q(U_\alpha)$-name f for a real in $V(\alpha, 0)$ we have

$$\Vdash_{Q(U(\alpha+1,0))} r_\alpha \not\leq^* f.$$

At successor stages choose $U(\alpha+1, 0)$ applying the Main Lemma. At stages λ of uncountable cofinality define $U(\lambda, 0) = \cup_{\alpha<\lambda} U(\alpha, 0)$ and at stages λ of countable cofinality essentially repeat the proof of the Main Lemma to obtain an ultrafilter $U(\lambda, 0)$ extending $\cup_{\alpha<\lambda} U(\alpha, 0)$ such that every maximal antichain of $Q(U(\alpha, 0))$ from $V(\alpha, 0)$ remains a maximal antichain of $Q(U(\lambda, 0))$. Let $U_0 = \cup_{\alpha<\delta} U(\alpha, 0)$. Then is s_0 is the Mathias real adjoined by $Q(U_0)$, be the Corollary above s_0 is generic over $V(\alpha, 0)$ for every $\alpha < \delta$ and so

$$V(\delta, 0)[s_0] \Vdash \forall \alpha \in \delta (r_\alpha \not\leq^* s_0).$$

Now for every $\alpha < \delta$ let $V(\alpha, 1) = V(\alpha, 0)[s_0]$. We can repeat the same process to obtain a sequence of ultrafilters $U(\alpha, 1)$ in $V(\alpha, 1)$ which satisfy the analogous properties of $V(\alpha, 0)$ just in the same way. Certainly we can repeat the same process any finite number of times n which results in adjoining a finite sequence $\langle s_i : i < n \rangle$ of finitely many

Mathias reals over $V(\delta,0)$ and the model $V(\delta,n)$. Again the sequence $\langle s_i : i < n\rangle$ is generic over $V(\alpha,0)$ for every $\alpha<\delta$ and so in particular we have obtained an extension $V(\alpha,n)$.

All of the above could have been defined as a finite support iteration of length n of appropriate forcing notions over $V(\delta,0)$. For this we will fix the following notation: $T(\delta,n)$ where $T(\delta,n+1)=T(\delta,n)*Q(U_n)$. Since this can be done for every $n\in\omega$ we can define the finite support iteration $T(\delta,\omega)$ of $\langle T(\delta,n):n\in\omega\rangle$ which adds the sequence $\langle s_n : n\in\omega\rangle$ of Mathias reals to $V(\delta,0)$. Before we can continue the inductive construction we have to verify that $T(\alpha,\omega)$ which is the finite support iteration of $\langle T(\alpha,n):n\in\omega\rangle$ does not add a real dominating r_α.

Lemma 2. *Let $\alpha<\delta$ and let $D\in V(\alpha,\omega)$ be a dense subset of $T(\alpha,\omega)$. Then D is a pre-dense subset of $T(\delta,\omega)$.*

Proof. Consider arbitrary condition $p\in T(\delta,\omega)$. By definition of finite support iteration there is $k\in\omega$ such that $p\in T(\delta,k)$. Recall also that $T(\delta,\omega)=T(\delta,k)*R$ for some forcing notion R over $V(\delta,k)$. Similarly $T(\alpha,\omega)=T(\alpha,k)*R'$. The set

$$\bar{D}=\{r\in T(\alpha,k):\exists q'\in R'((r,q')\in D)\}$$

is dense in $T(\alpha,k)$ and so by inductive hypothesis (our assumption on the construction of $T(\alpha,n)$) $\bar{D}$ is pre-dense in $T(\delta,k)$. Therefore there is some $r\in\bar{D}$ such that r is compatible with p. But then for some $q'\in R'$, $(r,q')\in D$ and certainly (r,q') is compatible with p. $\square$

Lemma 3. *No real in $V(\alpha,\omega)$ dominates r_α.*

Proof. For every $\alpha\le\delta$ and $\xi\le\omega$ let $V(\alpha,\xi)$ be obtained as a finite support iteration over V of a forcing notion $P(\alpha,\xi)$. As we described the iteration $P(\alpha,\xi)$ consists of the finite support iteration of length α of Cohen forcing followed by a finite support iteration of Mathias forcing of length ξ. Thus suppose there is a $P(\alpha,\omega)$-name for a real f and a condition $p\in P(\delta,\omega)$ such that

$$p\Vdash r_\alpha\le^* f.$$

There is some $k\in\omega$ such that $p\in P(\delta,k)$. Let $G(\delta,k)$ be a $P(\delta,k)$-generic filter containing p. Similarly let $G(\alpha,k)$ be the restriction of $G(\delta,k)$ to $P(\alpha,k)$. By the observations from above, $G(\alpha,k)$ is $P(\alpha,k)$-generic filter. In $V(\alpha,k)$ define $g\in^\omega\omega$ as follows:

$$g(n)=\min\{m:\exists q\in W'_n(f'_n(q)=m)\}.$$

Then g is a function defined in $V(\alpha,k)$. Let H be a $R(\delta,k)$-generic filter over $V[G(\alpha,k)]$ containing some q such that $q\Vdash g(n)=f'(n)$.

Let $H' = h \cap R'(\alpha, k)$. Again by the observation from above, H' is $R'(\alpha, k)$-generic over $V[G(\alpha, k)]$. But then

$$f_{G(\delta,k)*H}(n) = f_{G(\alpha,k)*H'}(n) = f'_{H'}(n) = g(n).$$

However $p \in G(\delta, k)$ and so $r_\alpha(n) \leq g(n)$. But this can be done for every n which implies that r_α is dominated by g. It remains to observe that $g \in V(\alpha, k) \cap {}^\omega\omega$ which contradicts the construction of the model. □

Since no real in $V(\alpha, \omega)$ dominates r_α we can repeat the construction and obtain ultrafilter U_ω and an associated forcing notion Q_ω. The same process can be certainly repeated ν-many times.

To verify that $V(\delta, \nu) \vDash \mathfrak{d} = \nu$ it remains to see that every set of reals in $V(\delta, \nu)$ of size smaller than ν is contained in $V(\alpha, \nu)$ for some $\alpha < \delta$.

Lemma 4. *Let $\xi \leq \nu$.*

(1) Every $P(\delta, \xi)$-condition is $P(\alpha, \xi)$-condition for some $\alpha < \delta$.
(2) Every $P(\delta, \xi)$-name for a real f, is $P(\alpha, \xi)$-name for a real for some $\alpha < \delta$.

Proof. It is sufficient to show part (i) since part (ii) follows from it. If $\xi = 0$ then this is just a property of the finite support iteration of Cohen forcing. If ξ is a limit, then the same argument holds. If $\xi = \alpha + 1$ then $p = (t, q)$ where $t \in P(\delta, \alpha)$ and $q \in Q(U_\alpha)$. By inductive hypothesis $p \in P(\eta_1, \alpha)$ for some $\eta < \delta$. Note that $q = (a, A)$ is a $P(\delta, \alpha)$-name for a real and so again by the inductive hypothesis there is some $\eta_2 < \delta$ such that q is $P(\eta_2, \alpha)$-name for a real. If $\eta = \max\{\eta_1, \eta_2\}$ then p is a condition in $P(\eta, \alpha)$. Since $\eta < \delta$ the inductive proof is complete. □

It remains to observe that if $\mathcal{G}$ is set of reals of size smaller than ν in $V(\delta, \nu)$ then there is $\alpha < \nu$ such that G is contained in $V(\alpha, \nu)$. But then r_α is unbounded by $\mathcal{G}$ and so $\mathcal{G}$ is not a dominating family. Therefore $V(\delta, \nu) \vDash \mathfrak{d} = \mathfrak{c} = \nu$.

3. The consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$

Note that the same model can be used to obtain the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$. Just begin by adding ω_1 Cohen reals followed be a finite support iteration of length κ of Mathias forcing for ultrafilters chosen just as in the proof of the Main Lemma from the previous section. Any set of reals in $V(\omega_1, \kappa)$ of size smaller than κ is obtained at some initial stage of the iteration $V(\omega_1, \alpha)$. We claim that s_α is not split by any infinite subset of ω from $V(\omega_1, \alpha)$.

Let X be an arbitrary infinite set. Then there is some $\eta < \alpha$ such that $s_\eta \subseteq^* X$ or $s_\eta \subseteq \omega - X$. But $s_\alpha \subseteq^* s_\eta$ and so $s_\alpha \subseteq^* X$ or $s_\alpha \subseteq^* \omega - X$. Therefore s_α is not split by X and so $V(\omega_1, \kappa) \vDash (\mathfrak{s} = \kappa)$.

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