STRONGLY UNFOLDABLE, SPLITTING AND BOUNDING

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Abstract. Assuming GCH, we show that generalized eventually narrow sequences on a strongly inaccessible cardinal κ are preserved under a one step iteration of the Hechler forcing for adding a dominating κ-real. Moreover, we show that if κ is strongly unfoldable, $2^\kappa = \kappa^+$ and λ is a regular cardinal such that $\kappa^+ < \lambda$, then there is a set generic extension in which $\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \mathfrak{c}(\kappa) = \lambda$.

1. Introduction

The topic "cardinal characteristics of the continuum" is a broad subject, which has been studied in many research articles and surveys like [3] or [15]. Combinatorial cardinal invariants and their generalizations to larger cardinals, some of which build the subject of this article, give an insight to the combinatorial and topological properties of the real line and the higher Baire spaces.

The splitting, bounding and dominating numbers, denoted by $\mathfrak{s}, \mathfrak{b}$ and $\mathfrak{d}$, are due to David D. Booth, Fritz F. Rothberger and Miroslav Katětov respectively. While $\mathfrak{s}, \mathfrak{b} \leq \mathfrak{d}$, the characteristics $\mathfrak{s}$ and $\mathfrak{b}$ are independent. Introducing the notion of an eventually narrow sequence and showing the preservation of such sequences under finite support iterations of Hechler forcings for adjoining a dominating real, Baumgartner and Dordal showed that consistently $\aleph_1 = \mathfrak{s} < \mathfrak{b}$ holds (see [1]). The consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$ is due to S. Shelah (see [13]) and is in fact the first appearance of the method of creature forcing. Studying the existence of ultrafilters $\mathcal{U}$, which have the property that for a given unbounded family $\mathcal{H} \subseteq {}^\omega\omega$, the relativized Mathias poset $\mathbb{M}(\mathcal{U})$ preserves the family $\mathcal{H}$ unbounded (appearing more recently in the literature as $\mathcal{H}$-Canjar filters, see [9]), the second author jointly with J. Steprāns generalized the result to an arbitrary regular uncountable κ, i.e. showed the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ (see [7]). The more general inequality of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$ was obtained only after certain developments of the method of matrix iteration, namely the appearance of a method of preserving maximal almost disjoint families along matrix iterations introduced by the second author and J. Brendle in [2].

In strong contrast to the countable case, the generalized bounding and splitting numbers are not independent. Indeed, Raghavan and Shelah showed that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for each regular uncountable κ (see [12] and Definitions 1, 3). Moreover, by a result of Motoyoshi (see [14, p. 34]), $\mathfrak{s}(\kappa) \geq \kappa$ if and only if κ is strongly inaccessible. Later T. Suzuki showed that under the same assumption

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$\mathfrak{s}(\kappa) \geq \kappa^+$ if and only if κ is weakly compact (see e.g. [14]). An easy diagonalization argument shows that $\kappa^+ \leq \mathfrak{b}(\kappa)$ and so unless κ is weakly compact, $\mathfrak{s}(\kappa) < \kappa^+ \leq \mathfrak{b}(\kappa)$.

In this article we further address the behaviour of $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ for κ regular uncountable. In particular, we introduce the concept of a generalized eventually narrow sequence, or κ-eventually narrow sequence (see Definition 2) and show that if κ is strongly inaccessible, then generalized eventually narrow sequences on κ are preserved by κ-Hechler forcing (see Theorem 13):

Theorem. Assume GCH, κ is strongly inaccessible and $\mathrm{cof}(\lambda) > \kappa$. If $\tau = \langle a_\xi : \xi < \lambda\rangle$ is a κ-eventually narrow sequence in V, then τ remains eventually narrow in $V^{\mathbb{D}(\kappa)}$.

It becomes a natural question, if the same result holds for iterations of κ-Hechler forcing. However, an attempt to generalize the preservation arguments of Baumgartner and Dordal from [1] to such iterations leads to significant difficulties. In fact, in order to obtain our main result, we take a slightly different approach (see Section 4):

Theorem. Assume κ is strongly unfoldable, $2^\kappa = \kappa^+$ and λ is a regular cardinal such that $\kappa^+ < \lambda$. Then there is a set generic extension in which $\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \mathfrak{c}(\kappa) = \lambda$.

Controlling $\mathfrak{s}(\kappa)$ strictly above κ^+ simultaneously with $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ and 2^κ remains an interesting open question. For a model of $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{c}(= \mathfrak{a})$ see [5], while a model of $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ can be found in [8].

2. Preliminaries

We recall some preliminaries and definitions.

Definition 1. Let κ be regular. Let a and b be elements in $[\kappa]^\kappa = \{x \in \mathcal{P}(\kappa)\colon |x| = \kappa\}$.

(1) We write $a \subseteq^* b$, if $|a \setminus b| < \kappa$ holds.
(2) The set a *splits* the set b if $|b \setminus a| = |a \cap b| = \kappa$.
(3) A family $\mathcal{S} \subseteq [\kappa]^\kappa$ is *splitting* if for each $b \in [\kappa]^\kappa$ there is an element $a \in \mathcal{S}$ such that a splits b.
(4) Finally, $\mathfrak{s}(\kappa)$ denotes the *generalized splitting number*:
$$\mathfrak{s}(\kappa) = \min\{|\mathcal{S}|\colon \mathcal{S} \subseteq [\kappa]^\kappa \text{ is splitting}\}.$$

Definition 2.

(1) A sequence $\langle a_\xi \colon \xi < \lambda\rangle$, where each a_ξ is in $[\kappa]^\kappa$, is *κ-eventually splitting* if $\forall a \in [\kappa]^\kappa$ $\exists \xi < \lambda$ $\forall \eta > \xi$ a_η splits a.
(2) A sequence $\langle a_\xi \colon \xi < \lambda\rangle$, where each a_ξ is in $[\kappa]^\kappa$, is *κ-eventually narrow* if $\forall a \in [\kappa]^\kappa$ $\exists \xi < \lambda$ $\forall \eta > \xi$ $a \not\subseteq^* a_\eta$.

If κ is clear from the context, we write just "eventually narrow" instead of "κ-eventually narrow". Note that $\tau = \langle a_\xi \colon \xi < \lambda\rangle$ is κ-eventually splitting iff the sequence $\tau' = \langle b_\xi \colon \xi < \lambda\rangle$, defined as $b_{2\xi} = a_\xi$ and $b_{2\xi+1} = \kappa \setminus a_\xi$, is κ-eventually narrow.

Definition 3. Let κ be regular and let f and g be functions from κ to κ, i.e. $f, g \in {}^\kappa\kappa$.

(1) We say that g *eventually dominates* f, denoted by $f \leq^* g$, if $\exists \alpha < \kappa\, \forall \beta > \alpha\; f(\beta) \leq g(\beta)$.
(2) A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *dominating* if $\forall g \in {}^\kappa\kappa\; \exists f \in \mathcal{F}\; [g \leq^* f]$.
(3) A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *unbounded* if $\forall g \in {}^\kappa\kappa\; \exists f \in \mathcal{F}\; [f \not\leq^* g]$.
(4) $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ denote the *generalized bounding* and *dominating numbers* respectively:
$$\mathfrak{b}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa \text{ is unbounded}\},$$
$$\mathfrak{d}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa \text{ is dominating}\}.$$
(5) Finally, $\mathfrak{c}(\kappa) = 2^\kappa$.

In [4], it is shown that $\kappa^+ \leq \mathfrak{b}(\kappa) = \mathrm{cof}(\mathfrak{b}(\kappa)) \leq \mathrm{cof}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq \mathfrak{c}(\kappa)$ holds.

Definition 4. Let κ be regular uncountable.
(1) If $\mathcal{A} \subseteq \mathcal{P}(\kappa)$, then $\mathcal{A}$ has the *strong intersection property (SIP)* if $\forall \mathcal{A}' \in [\mathcal{A}]^{<\kappa}\; [|\bigcap \mathcal{A}'| = \kappa]$.
(2) A subset $X \subseteq \kappa$ is called a *pseudo-intersection of* $\mathcal{A}$ if $X \subseteq^* A$ for any $A \in \mathcal{A}$.

Definition 5.
(1) The *generalized pseudo-intersection number* $\mathfrak{p}(\kappa)$ is the minimal size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ with the SIP but no pseudo-intersection.
(2) The invariant $\mathfrak{p}_{cl}(\kappa)$ is the minimal size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ of clubs (closed and unbounded sets) in κ having no pseudo-intersection.

A proof of $\mathfrak{p}_{cl}(\kappa) = \mathfrak{b}(\kappa)$ for a regular uncountable κ is given in [6, Observation 4.2.]. A consistency proof of $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa)$, which uses the identity $\mathfrak{p}_{cl}(\kappa) = \mathfrak{b}(\kappa)$, is given in [6, Section 4]. The relevant poset for this proof and for our proof in section 4 is the following:

Definition 6. Let $\mathcal{C}$ denote the collection of all clubs in κ. The conditions in the forcing poset $\mathbb{M}(\mathcal{C})$ are pairs (a, C), where $a \in [\kappa]^{<\kappa}$ and $C \in \mathcal{C}$. The order is given by $(a', C') \leq (a, C)$ if $C' \subseteq C$ and $a' \setminus a \subseteq C$.

It is easy to see that, if κ is regular uncountable and $\kappa^{<\kappa} = \kappa$, then $\mathbb{M}(\mathcal{C})$ is κ-closed and has the κ^+-c.c., which is mainly due to the regularity of κ. If $G \subseteq \mathbb{M}(\mathcal{C})$ is generic over the ground model, then the real $c = \bigcup\{a : \exists (a, C) \in G\}$ is a pseudo-intersection of all clubs in the ground model. Clearly, conditions with the same first coordinate, referred to as stem, are compatible.

If κ is regular then let $\kappa^{<\kappa}\!\uparrow = \{s \in \kappa^{<\kappa} \colon s \text{ is strictly increasing}\}$ and let ${}^\kappa\kappa\!\uparrow = \{f \in {}^\kappa\kappa \colon f$ is strictly increasing$\}$.

Definition 7. The *κ-Hechler poset* is defined as the set $\mathbb{D}(\kappa) = \{(s, f) \colon s \in \kappa^{<\kappa}\!\uparrow, f \in {}^\kappa\kappa\!\uparrow\}$ with extension relation given by $(t, g) \leq_{\mathbb{D}(\kappa)} (s, f)$ iff
$$s \subseteq t \wedge \forall \alpha \in \kappa\; [g(\alpha) \geq f(\alpha)] \wedge \forall \alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)\; [t(\alpha) \geq f(\alpha)].$$
$\mathbb{D}$ denotes $\mathbb{D}(\omega)$. For a condition $p \in \mathbb{D}(\kappa)$, p_0 (resp. p_1) denotes its first (resp. second) coordinate.

Remark 8. It is well-known that $\mathbb{D}(\kappa)$ is κ^+-c.c. and κ-closed, provided κ is regular and $\kappa^{<\kappa} = \kappa$.

Although the exact definition of a strongly unfoldable cardinal is not strictly necessary to understand the results of this article, we state it for the sake of completeness. First, if κ is strongly inaccessible, then a κ-model denotes a transitive structure M of size κ, such that $M \vDash ZFC - P$, $\kappa \in M$ and $M^{<\kappa} \subseteq M$, i.e. M is closed under sequences of size less than κ.

Definition 9 ([16], see [11])**.**

(1) Let λ be an ordinal. A cardinal κ is λ-strongly unfoldable iff
   (a) κ is strongly inaccessible
   (b) for every κ-model M there is an elementary embedding $j : M \to N$ with critical point κ such that $\lambda < j(\kappa)$ and $V_\kappa \subseteq N$.

(2) A cardinal κ is called strongly unfoldable if it is θ-strongly unfoldable for every ordinal θ.

A strongly unfoldable cardinal is in particular weakly compact. In Section 4 we will also use T. A. Johnstone's theorem concerning the indestructiblity of strongly unfoldable cardinals.

Theorem 10 ([11])**.** *Let κ be strongly unfoldable. Then there is a set forcing extension where the strong unfoldability of κ is indestructible by forcing notions of any size which are $<\kappa$-closed and have the κ^+-c.c..*

3. Eventually narrow sequences and Hechler forcing at the uncountable

Throughout the section, let κ be regular uncountable and $\kappa^{<\kappa} = \kappa$. Next, we show that κ-eventually narrow sequences are preserved in the one-step forcing extension via $\mathbb{D}(\kappa)$.

Definition 11. Let D be open dense in $\mathbb{D}(\kappa)$, i.e. $\forall p \in \mathbb{D}(\kappa)\ \exists q \in D\ [q \leq p]$ and whenever $p \in D$ and $q \leq p$ then $q \in D$. Define a sequence of subsets of $\kappa^{<\kappa}\uparrow$, referred to as a *sequence of derivatives*, as follows:

(1) $D_0 = \{s \in \kappa^{<\kappa}\uparrow \mid \exists p \in D\ [p_0 = s]\}$,
(2) $D_{\alpha+1} = \{s \in \kappa^{<\kappa}\uparrow \mid$
   (a) $s \in D_\alpha$, or
   (b) $\exists\gamma \in \kappa\, [\gamma > \mathrm{dom}(s) \wedge \exists\{t_\delta : \delta < \kappa\} \subseteq D_\alpha \forall\beta < \kappa\, [s \subseteq t_\beta \wedge \mathrm{dom}(t_\beta) = \gamma \wedge t_\beta(\mathrm{dom}(s)) > \beta]]\}$, and
(3) $D_\alpha = \bigcup\{D_\beta \mid \beta < \alpha\}$ if α is a limit ordinal.

In item (2b), first γ fixes a length. Then for every $\beta \in \kappa$, an element t_β is found such that each one's domain is γ. Further, each sequence in $\{t_\delta : \delta < \kappa\}$ is an end-extension of the sequence s and $\{t_\delta(\mathrm{dom}(s)) : \delta < \kappa\}$ is unbounded in κ.

Due to (2a) and (3) this sequence is increasing, i.e. $D_\alpha \subseteq D_{\alpha+1}$. Consequently this increasing sequence of derivatives has to stabilize at some index below κ^+, as $\kappa^{<\kappa} = \kappa$. That is, there exists $\gamma < \kappa^+$ such that $D_\gamma = D_{\gamma+1}$.

Lemma 12. Assume GCH and κ is strongly inaccessible. Let γ be the least such that $D_\gamma = D_{\gamma+1}$. Then $D_\gamma = \kappa^{<\kappa}\uparrow$.

Proof. Suppose not and let $s \in \kappa^{<\kappa}\uparrow \setminus D_\gamma$. For the purposes of this proof, we will use the following notion:

Definition. A sequence $t \in D_\gamma$ is said to be a *minimal extension of* s if $s \subseteq t$ and whenever $\gamma \in \kappa$ is such that $\mathrm{dom}(s) \leq \gamma < \mathrm{dom}(t)$ then $t \restriction \gamma \notin D_\gamma$.

First we claim that for a given length, there are less than κ many minimal extensions with this length:

Claim. For every $\gamma \in \kappa$ with $\mathrm{dom}(s) \leq \gamma$ we have $|T_\gamma| < \kappa$, where
$T_\gamma = \{t \in \kappa^{<\kappa} \uparrow : t \text{ is a minimal extension of } s \text{ with } \mathrm{dom}(t) = \gamma\}$.

Proof of the Claim. Suppose not and let $\gamma \in \kappa$ be such that $|T_\gamma| \geq \kappa$. Then $|T_\gamma| = \kappa$, as $T_\gamma \subseteq \kappa^{<\kappa} \uparrow$, $|\kappa^{<\kappa} \uparrow| = \kappa$. For each $t \in T_\gamma$ let $\rho(t) = \sup\{t(\alpha) : \alpha < \gamma\}$ and let $\rho = \sup\{\rho(t) : t \in T_\gamma\}$. If $\rho < \kappa$, then $|T_\gamma| < \kappa$. This is due to the inaccessibility of κ. Thus, $\rho = \kappa$. Now, if for each $\alpha < \gamma$, $\mu_\alpha = \sup\{t(\alpha) : t \in T_\gamma\} < \kappa$, then $\rho = \sup\{\mu_\alpha : \alpha < \gamma\} < \kappa$, which is a contradiction. Therefore, there is $\alpha < \gamma$ such that $\mu_\alpha = \kappa$. Pick α least such that $\mu_\alpha = \kappa$. Then in particular, $|\{t \restriction \alpha : t \in T_\gamma\}| < \kappa$ and so we can find $u \in \kappa^{<\kappa} \uparrow$ and $T' \subseteq T_\gamma$ of cardinality κ such that for each $t \in T'$, $t \restriction \alpha = u$ and $\{t(\alpha) : t \in T'\}$ is unbounded in κ. Note that $\mathrm{dom}(s) \subseteq \mathrm{dom}(u)$. Then u is an element of $D_{\gamma+1} = D_\gamma$, contradicting the minimality of the elements in T'. $\square$

We continue with the proof of the theorem. As there are $<\kappa$-many minimal extensions for a fixed length β, we can define a function which goes above all minimal extensions in T_β. For any $\gamma + 1 \in \kappa$ let $\rho_{\gamma+1} = \sup\{t(\gamma) : t \in T_{\gamma+1}\}$. By regularity and the above claim $\rho_{\gamma+1} < \kappa$ for any $\gamma + 1 \in \kappa$. Therefore, for any γ such that $\mathrm{dom}(s) \leq \gamma < \kappa$ define the function $f \in {}^\kappa\kappa \uparrow$ such that $f(\gamma) > \rho_{\gamma+1}$. Thus, f dominates at point γ the values of all minimal extensions t, whose length is $\gamma+1$. If $\gamma \in \kappa$ is a limit and $\alpha < \gamma$, then let $\rho'_\alpha = \sup\{t(\alpha) : t \in T_\gamma\}$. Define inductively $h \in {}^\kappa\kappa \uparrow$ such that for each limit $\gamma \in \kappa$ and $\alpha \in \gamma$, we have $h(\alpha) > \rho'_\alpha$ if $h(\alpha)$ is not defined yet.

Let $f' \in {}^\kappa\kappa \uparrow$ be above the pointwise maximum of f and h. Consider the condition $(s, f') \in \mathbb{D}(\kappa)$. By the density of D we can find a condition $(t, g) \leq (s, f')$ such that $(t, g) \in D$. So the element $t \in \kappa^{<\kappa} \uparrow$ is in D_0 and $s \subseteq t$ (by the extension relation). Then, some initial segment t' of t must be a minimal extension of s (the existence of such a minimal extension t' follows from the fact that $\in$ well-orders κ). As $t' \neq s$ (because this would imply $s \in D_\gamma$), $\mathrm{dom}(t') > \mathrm{dom}(s)$. If $\mathrm{dom}(t') = \alpha + 1$ is a successor, then $t(\alpha) = t'(\alpha) < f(\alpha) \leq f'(\alpha)$ which is a contradiction to $(t, g) \leq (s, f')$. Suppose $\alpha = \mathrm{dom}(t')$ is a limit. Take $\beta \in \alpha$. Then $t(\beta) = t'(\beta) < h(\beta) \leq f'(\beta)$ which is a contradiction to $(t, g) \leq (s, f')$. $\square$

Theorem 13. (GCH) *Assume that κ is strongly inaccessible and λ is an ordinal such that* $\mathrm{cof}(\lambda) > \kappa$. *Then any κ-eventually narrow sequence $\tau = \langle a_\xi : \xi < \lambda\rangle$ remains κ-eventually narrow in $V^{\mathbb{D}(\kappa)}$.*

Proof. Suppose by way of contradiction that τ is not eventually narrow in the generic extension. So fix a condition $p' \in \mathbb{D}(\kappa)$ and a name $\dot{a}$ for a subset of κ of size κ such that

$$p' \Vdash_{\mathbb{D}(\kappa)} \forall \xi < \lambda \; \exists \eta > \xi \; [\dot{a} \subseteq^* a_\eta].$$

Let θ be a regular cardinal such that $\mathbb{D}(\kappa) \in H(\theta) = \{x \in WF : |\mathrm{trcl}(x)| < \theta\}$. Let $\mathcal{N}$ be an elementary substructure of $H(\theta)$ of size κ such that $\mathbb{D}(\kappa) \in \mathcal{N}$, $\dot{a} \in \mathcal{N}$ and $p' \in \mathcal{N}$. Since τ is eventually narrow, for every $c \in [\kappa]^\kappa \cap \mathcal{N}$ there is a $\xi < \lambda$ such that for all $\eta > \xi$ we have $c \not\subseteq^* a_\eta$. However $|\mathcal{N}| = \kappa$, τ is of length λ and $\mathrm{cof}(\lambda) > \kappa$, so this will yield κ-many ordinals ξ smaller than λ. Hence there is an ordinal $\xi' < \lambda$ such that $\forall c \in \mathcal{N} \cap [\kappa]^\kappa\ \forall \eta' \geq \xi'\ [c \not\subseteq^* a_{\eta'}]$.

Since $p' \Vdash \forall \xi < \lambda\ \exists \eta > \xi\ |\dot{a} \setminus a_\eta| < \kappa$, in particular p' forces the existence of an η_0 greater than ξ' such that $a \setminus a_{\eta_0}$ is of size less than κ in the extension. Let $p = (s, f) \leq p'$ be such that there are ordinals $\alpha_0 \in \kappa$ and $\eta_0 > \xi'$ such that $p \Vdash \forall j \geq \alpha_0\ [j \in \dot{a} \to j \in a_{\eta_0}]$.

Let $\dot{h}$ be a $\mathbb{D}(\kappa)$-name such that $\Vdash$ "$\dot{h}$ enumerates $\dot{a}$". So in particular $\Vdash \forall \zeta < \kappa\ [\dot{h}(\zeta) \geq \zeta]$. To define $\dot{h}$ we only used $\dot{a}$ which was in $\mathcal{N}$, so $\dot{h} \in \mathcal{N}$ holds as well. We claim the following:

Claim. Let $t \in \kappa^{<\kappa} \uparrow$ be such that $(t, f) \leq (s, f)$ and let $\zeta \geq \alpha_0$. Then $Z_t(\zeta) \neq \emptyset$, where

$$Z_t(\zeta) = \{j \colon \forall g \in {}^\kappa\kappa \uparrow \exists r \leq_{\mathbb{D}(\kappa)} (t, g)\ [r \Vdash \dot{h}(\zeta) = j]\}.$$

Proof. Fix $\zeta \geq \alpha_0$ and let $D = \{u \in \mathbb{D}(\kappa) \colon \exists j \in \kappa\ [u \Vdash \dot{h}(\zeta) = j]\}$. Then D is dense, open and we can form the sequence of derivatives $\langle D_\alpha \rangle_{\alpha \leq \gamma}$ where γ is the least ordinal with $D_\gamma = D_{\gamma+1} = \kappa^{<\kappa} \uparrow$. We will prove the claim by induction on the rank $\alpha \leq \gamma$ for all sequences t as above.

If $t \in D_0$, we have that there exists a condition $u \in D$ such that $u_0 = t$, where u_0 is the first coordinate of u, and $\exists j\ [u \Vdash \dot{h}(\zeta) = j]$. Let g be in ${}^\kappa\kappa \uparrow$. Then (t, g) and u $(= (t, u_1))$ are compatible with common extension r. Thus, $r \Vdash \dot{h}(\zeta) = j$ and so $Z_t(\zeta) \neq \emptyset$. For limit ordinals α the claim is true by the induction hypothesis, since $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$.

Finally let $t \in D_{\alpha+1} \setminus D_\alpha$ be such that $(t, f) \leq (s, f)$. By definition of $D_{\alpha+1}$ there is a sequence $\langle t_\beta \colon \beta \in \kappa \rangle$ of elements of D_α such that for each $\beta \in \kappa$, $\mathrm{dom}(t_\beta) = \gamma$ (for some $\gamma \in \kappa$) and $t_\beta(\mathrm{dom}(t)) > \beta$. Since such a sequence exists in $H(\theta)$ and the latter was an existential statement and $\mathcal{N} \preccurlyeq H(\theta)$, by the Tarski-Vaught-Criterion, we can find a witness in $\mathcal{N}$. So assume $\langle t_\beta \colon \beta \in \kappa \rangle \in \mathcal{N}$.

At this point we distinguish between two cases. Case 1: There is a $j \in \kappa$ such that $j \in Z_{t_\beta}(\zeta)$ for κ-many ordinals β. Let $g \in {}^\kappa\kappa$. We have that "$\exists r \leq (t_\beta, g)\ [r \Vdash \dot{h}(\zeta) = j]$" holds for κ-many t_β's, but not all of these pairs (t_β, g) extend (t, g). However since we have κ-many such t_β's and $\forall \beta < \kappa\ [t_\beta(\mathrm{dom}(t)) > \beta]$ we can find one (actually infinitely many) pair $(t_\beta, g)$with $(t_\beta, g) \leq (t, g)$ and consequently one (infinitely many) $r \leq (t, g)$ such that $j \in Z_{t_\beta}(\zeta)$; hence $j \in Z_t(\zeta) \neq \emptyset$. (To find a (t_β, g) as desired we choose β such that $\beta > g(\gamma)$. Then for such a β and any α with $\mathrm{dom}(t) \leq \alpha < \gamma$ we have $t_\beta(\alpha) > \beta > g(\gamma) > g(\alpha)$).

Case 2: Fix by the induction hypothesis one $j_\beta \in Z_{t_\beta}(\zeta)$ (e.g. choose the minimal one) and consider the set $J = \{j_\beta \colon \beta \in \kappa\}$. This set is of size κ, because otherwise it would have an upper bound in κ, so $\exists \alpha_0 < \kappa\ \forall \alpha, \beta \geq \alpha_0\ [j_\alpha = j_\beta]$. But then we would have a j which is in all $Z_{t_\beta}(\zeta)$'s for $\beta \geq \alpha_0$, so we would have a j which is in κ-many $Z_{t_\beta}(\zeta)$'s, which is in fact Case 1. So $|J| = \kappa$, but J consists of j_β's which are elements of $Z_{t_\beta}(\zeta)$ and these were defined using $\dot{h}$ which was in $\mathcal{N}$ and the sequence $\langle t_\beta \colon \beta \in \kappa \rangle$ which was also in $\mathcal{N}$, so we may take $J \in \mathcal{N}$. This further means that $|J \setminus a_{\eta_0}| = \kappa$. So choose β' large enough such that $j_{\beta'} \geq \alpha_0$, $\beta' \geq f(\gamma)$ and $j_{\beta'} \notin a_{\eta_0}$. Then for this particular β' we have $(t_{\beta'}, f) \leq (t, f) \leq p$. For the first extension relation note that we have $t_{\beta'}(\mathrm{dom}(t)) > \beta' \geq f(\gamma)$; hence this extension really holds. On the other hand since $j_{\beta'} \in Z_{t_{\beta'}}(\zeta)$ there is, by the definition of $Z_{t_{\beta'}}(\zeta)$, some $r_{\beta'} \leq (t_{\beta'}, f)$ such that $r_{\beta'} \Vdash \dot{h}(\zeta) = j_{\beta'}$.

But then $j_{\beta'} \in a_{\eta_0}$, since $j_{\beta'} \geq \alpha_0$ and $r_{\beta'} \Vdash \forall j \geq \alpha_0\ [j \in \dot{a} \to j \in a_{\eta_0}]$ (p forced this). However $j_\beta \in a_{\eta_0}$ contradicts the assumption on β'. $\square$

By the claim $Z_s(\zeta) \neq \emptyset$ for $\zeta \geq \alpha_0$ since $p \leq p$. Choose $k_\zeta \in Z_s(\zeta)$ for each $\zeta \geq \alpha_0$ and consider the set $K = \{k_\zeta \colon \zeta \geq \alpha_0\}$. Since $\Vdash \dot{h}(\zeta) \geq \zeta$ we have $k_\zeta \geq \zeta$ for all ζ, hence K is of size κ. Since K is definable from s and other parameters of $Z_s(\zeta)$, we have $K \in \mathcal{N}$, so $K \setminus a_{\eta_0}$ has size κ. Now let $k_\zeta \in K \setminus a_{\eta_0}$ be chosen; so by definition $\exists r \leq p$ such that $r \Vdash \dot{h}(\zeta) = k_\zeta$ and again $r \Vdash \forall j \geq \alpha_0\ [j \in \dot{a} \to j \in a_{\eta_0}]$, and $k_\zeta \geq \zeta \geq \alpha_0$ so we have $k_\zeta \in a_{\eta_0}$ which is a contradiction. $\square$

Even though the above result might seem promising towards separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ above κ, an attempt to establish the preservation of κ-eventually narrow sequences along $< \kappa$-supported iterations of κ-Hechler forcing halts already at stage ω of the iteration. In general, natural attempts in generalising the preservation arguments for eventually narrow sequences along finite support iterations of Hechler forcing of Baumgartner and Dordal from [1], to $< \kappa$-supported iterations of κ-Hechler forcing fail at stages α of the iteration, where α is of cofinality strictly below κ. So, in Section 4 we use a different approach to separate $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ strictly above κ.

4. $Con(\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \lambda)$

In this section, we show that if κ is a strongly unfoldable cardinal, then $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ can indeed be separated above κ. Recall that if there are strongly unfoldable cardinals, then they exists in the constructible universe L.

Theorem 14. *Let κ be a strongly unfoldable cardinal. Suppose $2^\kappa = \kappa^+$. Let $\lambda > \kappa^+$ be a regular uncountable cardinal. Then there is a set forcing generic extension, in which*

$$\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \mathfrak{c}(\kappa) = \lambda.$$

Proof. Let V_0 be the ground model and let V be the $\mathbb{P}^*$-generic extension of V_0, where $\mathbb{P}^*$ is the lottery preparation of κ. Then, κ remains strongly unfoldable in any further generic extensions obtained by $< \kappa$-closed, κ^+-cc forcing notions (see [11, Main Theorem]). Note that $\mathbb{P}^*$ is κ-cc, of size κ, and so $2^\kappa = \kappa^+$ in V. As κ is strongly unfoldable (in particular strongly inaccessible) in V, $V \vDash \kappa^{<\kappa} = \kappa$ holds as well.

Let $\mathbb{C}_{\kappa^+}$ denote the $< \kappa$-support product of κ^+-many copies of the Cohen forcing $2^{<\kappa}$. We first add κ^+-many Cohen subsets of κ, $\langle y_\alpha \colon \alpha < \kappa^+ \rangle$ by forcing with $\mathbb{C}_{\kappa^+}$ and then iteratively diagonalize the club filter for λ-many steps. So the poset that we are forcing with is $\mathbb{P} = \mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$, where $\dot{\mathbb{M}}(\mathcal{C})_\lambda$ is a $\mathbb{C}_{\kappa^+}$-name for the $< \kappa$-support iteration of $\mathbb{M}(\mathcal{C})$ (from Definition 6) of length λ. The following property and notation have been established in [6]:

(1) This forcing $\mathbb{P}$ has the κ^+-c.c., is κ-closed and forces that $\mathfrak{c}(\kappa) = \lambda$.
(2) The set of conditions in $\mathbb{M}(\mathcal{C})_\lambda$ of the form $(\bar{a}, q)$, where
    i) $\bar{a}$, called the sequence of the stems, is of the form $\langle a_\beta \colon \beta \in I \rangle$ for some $I \in [\lambda]^{<\kappa}$ and $a_\beta \in [\kappa]^{<\kappa}$ for each $\beta \in I$,
    ii) q is a function with domain I and for each $\beta \in I$, $q(\beta)$ is a $\mathbb{M}(\mathcal{C})_i$-name for a club,

    is a dense subset of $\mathbb{M}(\mathcal{C})_\lambda$.

(3) The set of conditions in $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$ of the form $(p, \bar{a}, \dot{q})$, where
  i) $p \in \mathbb{C}_{\kappa^+}$ and $\bar{a} \in V$,
  ii) $\dot{q}$ is a $\mathbb{C}_{\kappa^+}$-name for a sequence as in (2, ii).

  is dense in $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$.

For a subset x of κ, a nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ has the form $\bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$ where A_α is an antichain in $\mathbb{M}(\mathcal{C})_\lambda$ and for each $(\bar{a}, q) \in A_\alpha$ and $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a nice $\mathbb{M}(\mathcal{C})_\beta$-name. So nice $\mathbb{M}(\mathcal{C})_\beta$-names for elements in $\mathcal{P}(\kappa)$ are defined by induction on $\beta \in \lambda$.

Claim 15. For each nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ for a subset of κ, $|\mathrm{trcl}(\dot{x})| \leq \kappa$ holds.

Proof. This is seen by induction on the length: Suppose the claim holds for nice $\mathbb{M}(\mathcal{C})_\beta$-names for every $\beta < \gamma$ and $\dot{x}$ is a nice $\mathbb{M}(\mathcal{C})_\gamma$-name. Then by the definition of nice names, $\dot{x}$ is of the form $\bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$, where A_α is an antichain in $\mathbb{M}(\mathcal{C})_\gamma$ (thus of size $\leq \kappa$). For every $(\bar{a}, q) \in A_\alpha$, $|\mathrm{dom}(q)| < \kappa$ and for each $\beta \in \mathrm{dom}(q)$, $q(i)$ is a nice $\mathbb{M}(\mathcal{C})_\beta$-name, which was assumed to have transitive closure of size at most κ. □

In the generic extension by $\mathbb{P}$ also $\mathfrak{b}(\kappa) = \mathfrak{p}_{cl}(\kappa) = \lambda$ holds: Any family F of clubs in κ of size $<\lambda$ appears, by the regularity of λ, at an earlier stage of the iteration and the next iterand $\mathbb{M}(\mathcal{C})$ adds a pseudo-intersection of the clubs of this intermediate model, so in particular a pseudo-intersection of those in F. We can assume that $\mathbb{P}$ does not destroy the strongly unfoldability of κ in the generic extension. Hence $V^{\mathbb{P}} \vDash \mathfrak{s}(\kappa) \geq \kappa^+$. So it is sufficient to find a splitting family of size κ^+. We are going to show that the Cohen reals $\bar{y} = \langle y_\alpha \colon \alpha < \kappa^+ \rangle$ build up such a family.

Claim 16. $\bar{y} = \{y_\alpha \colon \alpha < \kappa^+\}$ is a splitting family in the forcing extension obtained via $\mathbb{M}(\mathcal{C})_\lambda$.

Proof. Let $\dot{x}$ be a nice $\mathbb{M}(\mathcal{C})_\lambda$-name for a κ-real living in $V^{\mathbb{C}_{\kappa^+}} = V[\bar{y}]$. By Claim 15 and $\mathrm{cof}(\kappa^+) = \kappa^+ > \kappa$, there is a $\gamma < \kappa^+$ such that $\dot{x} \in V[\langle y_\alpha \colon \alpha < \kappa, \alpha \neq \gamma \rangle]$. We show that the κ-Cohen real y_γ splits $\dot{x}$. W.l.o.g. $\dot{x} \in V$ and we are adding a single Cohen κ-real $y_\gamma = y$ over V (by letting $V = V[\langle y_\alpha \colon \alpha < \kappa^+, \alpha \neq \gamma \rangle]$ be the new ground model). Then $V[y] \vDash (\Vdash_{\mathbb{M}(\mathcal{C})_\lambda}$ "$\check{y}$ splits $\dot{x}$").

Suppose for a contradiction that $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \setminus \epsilon \subseteq \dot{y}$ or $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \cap \dot{y} \subseteq \epsilon$ for some $\epsilon \in \kappa$ and a condition $(p, \bar{a}, \dot{q}) \in \mathbb{C} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$. Suppose $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \setminus \epsilon \subseteq \dot{y}$. Let y be a $\mathbb{C}$-generic over the ground model V with p in the generic filter, i.e. $p \subseteq y$. Define $y' \in 2^\kappa$ by letting $y'(i) = p(i) = y(i)$ for $i \in \mathrm{dom}(p)$ and $y'(i) = 1 - y(i)$ otherwise. Then $V[y] = V[y'] =: W$, but possibly $q := \dot{q}[y] \neq \dot{q}[y'] = q'$. In W, $(\bar{a}, q)$ and $(\bar{a}, q')$ are compatible, because their stems are the same. Let $(\bar{a}, r) \in \mathbb{M}(\mathcal{C})_\lambda$ be their common extensions.

Now let $(\bar{b}, s) \leq (\bar{a}, r)$ and $\delta \in \kappa \setminus \bigcup\{\epsilon, \mathrm{dom}(p)\}$ be such that $(\bar{b}, s) \Vdash \delta \in \dot{x}$.

As $y' \cap y \subseteq \epsilon$ we have $\delta \notin y$ or $\delta \notin y'$. Suppose $\delta \notin y$, then whenever G is $\mathbb{M}(\mathcal{C})_\lambda$-generic over W containing $(\bar{b}, s)$, $W[G] \vDash \dot{x}[G] \setminus \epsilon \not\subseteq y$. This is a contradiction because $(p, \bar{a}, q)$ is in the corresponding $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$-generic over V. Similarly for $\delta \notin y'$.

So suppose $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \cap \dot{y} \subseteq \epsilon$. Then again as $y' \cap y \subseteq \epsilon$ we have $\delta \in y$ or $\delta \in y'$. Suppose $\delta \in y$, then whenever G is $\mathbb{M}(\mathcal{C})_\lambda$-generic over W containing $(\bar{b}, s)$, $W[G] \vDash \dot{x}[G] \cap y \not\subseteq \epsilon$. This is a contradiction because $(p, \bar{a}, q)$ is in the corresponding $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$-generic over V. Similarly for the case $\delta \in y'$. □

This completes the proof of the theorem. □

References

[1] J. E. Baumgartner, P. Dordal: *Adjoining Dominating Functions*, The Journal of Symbolic Logic, Vol. 50, No. 1, 1985, pp. 94-101.

[2] J. Brendle, V. Fischer: *Mad families, splitting families and large concinuum*, Journal of Symbolic Logic 76(1), 2011, pp. 198-208.

[3] Blass, Andreas: *Combinatorial Cardinal Characteristics of the Continuum*; in: M. Foreman, A. Kanamori (Ed.): Handbook of Set Theory, Springer, Heidelberg/London/New York, 2010, pp. 395-489.

[4] J. Cummings, S. Shelah: *Cardinal invariants above the continuum*, Annals of Pure and Applied Logic 75, 1995, pp. 251-268.

[5] V. Fischer, D. Mejia: *Splitting, bounding, and almost disjointness can be quite different*, Canadian Journal of Mathematics, Vol.69(3), 2017, pp. 502-531.

[6] V. Fischer, D. C. Montoya, J. Schilhan, D. T. Soukup: *Towers and gaps at uncountable cardinals*, accepted at Fundament Mathematicae 2021, arXiv:1906.00843.

[7] V. Fischer, J. Steprāns: *The consistency of* $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$, Fundamenta Mathematicae 201(3), 2008, pp. 283-293.

[8] M. Goldstern, J. Kellner, D. A. MejÃa, S. Shelah: *Preservation of splitting families and cardinal characteristics of the continuum*, arXiv preprint, 2020, arXiv:2007.13500.

[9] O. Guzmán, D. Kalajdzievski: *The ultrafilter and almost disjointness numbers*, Advances in Mathematics, 386 (3), 2021.

[10] Jech, Thomas: *Set Theory. The Third Millennium Edition*, Springer, Berlin/Heidelberg/New York, 2006.

[11] Johnstone, Thomas A.: *Strongly Unfoldable Cardinals Made Indestructible*, The Journal of Symbolic Logic, 2008, Vol.73(4), pp. 1215-1248.

[12] D. Raghavan, S. Shelah: *Two inequalities between cardinal invariants*, Fund. Math., vol. 237, 2017, no. 2, pp. 187-200.

[13] Shelah, Saharon:*On cardinal invariants of the continuum.*; in: D. A. Martin, J. Baumgartner, and S. Shelah (Ed.): Axiomatic Set Theory, Contemporary Mathematics, vol. 31. American Mathematical Society, Providence, 1984, pp. 183-207.

[14] Suzuki, Toshio: *About splitting numbers*, Proceedings of the Japan Academy, Series A, Mathematical Sciences, 1998, pp. 33-35.

[15] van Douwen, Eric K.: *The integers and topology*, in: Handbook of set-theoretic topology, North-Holland, Amsterdam/New York, 1984, pp. 111-167.

[16] Villaveces, Andrés: *Chains of End Elementary Extensions of Models of Set Theory*, The Journal of Symbolic Logic, 1 September 1998, Vol.63(3), pp. 1116-1136.

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