FURTHER COMBINATORIAL PROPERTIES OF COHEN FORCING

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Abstract. The combinatorial properties of Cohen forcing imply the existence of a countably closed, $\aleph_2$-c.c. forcing notion $\mathbb{P}$ which adds a $\mathbb{C}(\omega_2)$-name $\mathbb{Q}$ for a σ-centered poset such that forcing with $\mathbb{Q}$ over $V^{\mathbb{P}\times\mathbb{C}(\omega_2)}$ adds a real not split by $V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega$ and preserves that all subfamilies of size ω_1 of the Cohen reals are unbounded.

1. Introduction

The results presented in this paper originate in the study of the combinatorial properties of the real line and in particular the bounding and the splitting numbers. A special case of the developed techniques appeared in [5]. Following standard notation for κ, λ regular cardinals, $[\kappa]^\lambda$ denotes the set of all subsets of λ of size κ, $\mathcal{P}(\lambda)$ is the power set of λ and ${}^\lambda\kappa$ is the collection of all functions from λ into κ. Throughout V denotes the ground model. If f, g are functions in ${}^\omega\omega$, then g dominates f, denoted $f \leq^* g$ if $\exists n \forall k \geq n(f(k) \leq g(k))$. A family $\mathcal{B} \subseteq {}^\omega\omega$ is unbounded, if $\forall f \in {}^\omega\omega \exists g \in \mathcal{B}(g \not\leq^* f)$. The bounding number $\mathfrak{b}$ is the minimal size of an unbounded family (see [9]). If $A, B \in [\omega]^\omega$ then A is split by B if both $A \cap B$ and $A \cap B^c$ are infinite. A family $S \subseteq [\omega]^\omega$ is splitting, if $\forall A \in [\omega]^\omega \exists B \in S$ such that B splits A. The splitting number $\mathfrak{s}$ is the minimal size of splitting family (see [9]). It is relatively consistent with the usual axioms of set theory, that $\mathfrak{s} < \mathfrak{b}$ as well as $\mathfrak{b} < \mathfrak{s}$. The consistency of $\mathfrak{s} < \mathfrak{b}$ holds in the Hechler model (see [2]) and the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$ is due to S. Shelah (see [7]). J. Brendle (see [3]) showed the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$, for κ regular uncountable cardinal and V. Fischer, J. Steprans (see [6]) showed the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

However the consistency of $\omega_1 < \mathfrak{b} < \mathfrak{b}^+ < \mathfrak{s}$ remains open. One way to approach this more general problem, is to obtain a ccc poset which preserves the unboundedness of a given unbounded family, adds a real not split by $V \cap [\omega]^\omega$ and iterate it with finite supports (note that in the desired generic extension $\aleph_3 < \mathfrak{c}$). There are two results which should be mentioned in this context. In 1988 [4], M. Canjar showed that if $\mathfrak{d} = \mathfrak{c}$, where $\mathfrak{d}$ is the dominating number, defined as the minimal size of a family $D \subseteq {}^\omega\omega$ such that $\forall f \in {}^\omega\omega \exists g \in D(f \leq^* g)$ and $\mathfrak{c}$ is the size of the continuum, then there is an ultrafilter U such that the relativized Mathias forcing $\mathbb{M}_U$, preserves the unboundedness of $V \cap {}^\omega\omega$ and certainly adds a real not split by the ground model infinite subsets of ω. This poset $\mathbb{M}_U$ however, can not be used to obtain a model in which $\mathfrak{b} < \mathfrak{c}$, since in order to obtain such a model, along the iteration one has to preserve the unboundedness of a chosen witness for $\mathfrak{b}$. That is in fact the main result of [6], where with a given unbounded directed family $\mathcal{H} \subseteq {}^\omega\omega$ of size $\mathfrak{c}$, one associates a σ-centered poset $Q_\mathcal{H}$ which preserves the unboundedness of $\mathcal{H}$ and adds a real not split by $V \cap [\omega]^\omega$. Consequently an appropriate iteration of $Q_\mathcal{H}$ gives the consistency of $\mathfrak{s} = \mathfrak{b}^+$ mentioned earlier. However the restriction $|\mathcal{H}| = \mathfrak{c}$, prevents the method of [6] from solving the more general consistency problem, since for this at certain stages of the iteration one has to preserve the unboundedness of a fixed family of size $< \mathfrak{c}$.

In the following we obtain a generic extension V_1, in which there is a σ-centered poset Q which preserves the unboundedness of a given family of size $< \mathfrak{c}$ and adds a real not split by $V_1 \cap [\omega]^\omega$. Thus the construction can be considered a first step towards obtaining the consistency of $\omega_1 < \mathfrak{b} < \mathfrak{b}^+ < \mathfrak{s}$.

2. Logarithmic measures and Cohen forcing

The notion of logarithmic measure is due to S. Shelah. In the presentation of logarithmic measures (Definitions 1, 2, 3) we follow [1].

Definition 1. Let $s \subseteq \omega$ and let $h : [s]^{<\omega} \to \omega$, where $[s]^{<\omega}$ is the family of finite subsets of s. Then h is *a logarithmic measure* if $\forall A \in [s]^{<\omega}$, $\forall A_0, A_1$ such that $A = A_0 \cup A_1$, $h(A_i) \geq h(A) - 1$ for $i = 0$ or $i = 1$ unless $h(A) = 0$. Whenever s is a finite set and h a logarithmic measure on s, the pair $x = (s, h)$ is called *a finite logarithmic measure*. The value $h(s) = \|x\|$ is called *the level of* x, the underlying set of integers s is denoted $\mathrm{int}(x)$. Whenever h is a finite logarithmic measure on x and $e \subseteq x$ is such that $h(e) > 0$, we will say that e is *h-positive*.

If h is a logarithmic measure and $h(A_0 \cup \cdots \cup A_{n-1}) \geq \ell + 1$ then $h(A_j) \geq \ell - j$ for some j, $0 \leq j \leq n - 1$.

Definition 2. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family which does not contain singletons. Then P induces a logarithmic measure h on $[\omega]^{<\omega}$ defined inductively on $|s|$ for $s \in [\omega]^{<\omega}$ as follows:

1. $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$
2. $h(e) > 0$ iff $e \in P$
3. for $\ell \geq 1$, $h(e) \geq \ell + 1$ iff whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \ell$ if ℓ is maximal for which $h(e) \geq \ell$. The elements of P are called *positive sets* and h is said to be *induced by* P.

If h is an induced logarithmic measure and $h(e) \geq \ell$, then for every a such that $e \subseteq a$, $h(a) \geq \ell$. A known example of induced logarithmic measure is the standard measure (see Shelah, [8]). That is the measure h induced by $P = \{a \subseteq \omega : |a| < \omega \text{ and } |a| \geq 2\}$. Note that $\forall x \in P$, $h(x) = \min\{i : |x| \leq 2^i\}$. Let LM be the set of finite logarithmic measures and for $n \in \omega$ let $L_n = \{x \in LM : \|x\| \geq n, \min \mathrm{int}(x) \geq n\}$. By [LM] denote the set of all families of finite logarithmic measures X such that $\forall n \in \omega (X \cap L_n \neq \emptyset)$. For $X \in$ [LM] let $\mathrm{int}(X) = \cup\{\mathrm{int}(t) : t \in X\}$ be the underlying set of integers.

Claim. If $\mathcal{E} \subseteq$ [LM] is a centered, then there is $U \subseteq$ [LM] which is centered and such that for every $X \in$ [LM] either $X \in U$ or $\exists Y \in U (X \cap Y \notin$ [LM]$)$.

Definition 3. Let Q be the partial order of all $(u, X) \in [\omega]^{<\omega} \times [\mathrm{LM}]$ such that $\forall x \in X(\max u < \min \mathrm{int}(x))$. If $u = \emptyset$ we say that $(\emptyset, X)$ is a *pure condition* and denote it by X. Then (u_2, X_2) extends (u_1, X_1), denoted $(u_2, X_2) \leq (u_1, X_1)$, if u_2 is an end-extension of u_1, $u_2 \backslash u_1 \subseteq \mathrm{int}(X_1)$, $\mathrm{int}(X_2) \subseteq \mathrm{int}(X_1)$, $\forall x \in X_2 \exists B_x \in [X_1]^{<\omega}$ such that $\mathrm{int}(x) \subseteq \cup\{\mathrm{int}(y) : y \in B_x\}$, $\forall y \in B_x(u_2 \cap \mathrm{int}(y) = \emptyset)$ and $\forall e \subseteq \mathrm{int}(x)$ which is x-positive $\exists y \in B_x(e \cap \mathrm{int}(y)$ is y-positive$)$.

Definition 4. If $\mathcal{F}$ is a family of pure conditions, then $Q(\mathcal{F})$ is the suborder of Q consisting of all $(u, X) \in Q$ such that $\exists Y \in \mathcal{F}(Y \leq X)$.

If C is a centered family of pure conditions, then $Q(C)$ is σ-centered. Conditions of $Q(C)$ are compatible as conditions in $Q(C)$ if and only if they are compatible as conditions in Q.

Unless specified otherwise Γ denotes a countable subset of ω_2. Also $\mathbb{C}(\Gamma)$ is the forcing notion of all partial functions $p : \Gamma \times \omega \to \omega$ with finite domain and extension relation $p \leq q$ if $q \subseteq p$. Thus $\mathbb{C}(\Gamma)$ is the forcing notion for adding Γ Cohen reals, e.g. $\mathbb{C}(\{0\}) = \mathbb{C}$ is just Cohen forcing, $\mathbb{C}_n = \mathbb{C}(n)$ is the forcing for adding n Cohen reals, etc. If $p \in \mathbb{C}(\Gamma)$, then $\mathbb{C}(\Gamma)^+(p) = \{q \in \mathbb{C}(\Gamma) : q \leq p\}$. A family $\Gamma' = \{\Gamma_j\}_{j \in n} \subseteq \mathcal{P}(\lambda)$ for some ordinal λ, where $n \in \omega$ and $\forall j \in n - 1 \sup \Gamma_j < \min \Gamma_{j+1}$ is called a *finite ordered partition of* $\Gamma = \cup_{j \in n} \Gamma_j$. Note that if Γ is a countable set of ordinals, then Γ has only countably many finite ordered partitions. $\mathcal{FP}(\Gamma)$ denotes the set of all finite ordered partitions of Γ. For $k, n \in \omega$ let ${}^{\leq n}k = \cup_{j=0}^{n-1} {}^{\{0,\dots,j\}}k$.

Definition 5. Let $\Gamma' = \{\Gamma_j\}_{j \in n} \in \mathcal{FP}(\Gamma)$, $k \in \omega$. Then $\mathbb{M}_k(\Gamma')$ is the set of all matrices $P = (p_i^j)_{i \in k, j \in n}$ with k rows and n columns, where the (i, j)-th entry p_i^j is a condition in $\mathbb{C}(\Gamma_j)$. Note that $\mathbb{M}_1(\Gamma')$ and $\mathbb{C}(\Gamma)$ can be identified. A matrix $P = (p_i^j) \in \mathbb{M}_k(\Gamma')$ is below $p = (p^j) \in \mathbb{M}_1(\Gamma')$ if $\forall i, j(p_i^j \leq p^j)$. Let $\mathbb{M}_{k,p}(\Gamma') = \{P \in \mathbb{M}_k(\Gamma') : P$ is below $p\}$, $\mathbb{M}(\Gamma') = \cup_{k \in \omega} \mathbb{M}_k(\Gamma')$ and $\mathbb{M}(\Gamma) = \cup\{\mathbb{M}(\Gamma') : \Gamma' \in \mathcal{FP}(\Gamma)\}$.

Definition 6. Let $\Gamma' = \{\Gamma_j\}_{i \in \omega} \in \mathcal{FP}(\Gamma)$ and $t :^{\leq n} k \to \cup_{j=0}^{n-1} \mathbb{C}(\Gamma_j)$ such that $\forall j \in n \forall a \in {}^{j+1}k\ t(a) \in \mathbb{C}(\Gamma_j)$. Then t induces a tree $T = \{T(a)\}_{a \in {}^{\leq n}k}$ where $T(a) = (T(b), t(a))$ whenever $a = (b, i), i \in k$ and $T(a) \leq_T T(b)$ iff $a \restriction |b| = b$. Let $\mathcal{T}_k(\Gamma')$ be the set of all trees induced by some t as above, $\mathcal{T}(\Gamma') = \cup_{k \in \omega} \mathcal{T}_k(\Gamma')$ and $\mathcal{T}(\Gamma) = \{\mathcal{T}(\Gamma') : \Gamma' \in \mathcal{FP}(\Gamma)\}$.

We use the convention that trees are denoted by a capital letter, while the inducing function is denoted by the corresponding small letter, e.g. T is induced by t. For $T \in \mathcal{T}_k(\Gamma')$, $\max T$ is the set of all maximal nodes of T. Note that $\max T \subseteq \mathbb{C}(\cup \Gamma')$. If ϕ is a formula in the $\mathbb{C}(\Gamma)$-language of forcing, T a tree in $\mathcal{T}_k(\Gamma')$, $\Gamma' \in \mathcal{FP}(\Gamma)$ then $T \Vdash \phi$ if $\forall t \in \max T(t \Vdash \phi)$. To emphasize that Γ' is a partition of Γ, we write $\mathbb{M}_k(\Gamma, \Gamma')$, $\mathcal{T}_k(\Gamma, \Gamma')$, etc.

Definition 7. Let $\Gamma' = \{\Gamma_j\}_{j \in n} \in \mathcal{FP}(\Gamma)$, $P = (p_i^j) \in \mathbb{M}_k(\Gamma')$. Then $\mathrm{ext}(P)$ is the set of all $T \in \mathcal{T}_k(\Gamma')$ such that if T is induced by $t :^{\leq n} k \to \cup_{j=0}^{n-1} \mathbb{C}(\Gamma_j)$ then $\forall j \in n \forall a \in {}^{j+1}k(t(a) \leq p_i^j)$. The elements of $\mathrm{ext}(P)$ are called trees of extensions of P.

Definition 8. A $\mathbb{C}(\Gamma)$-name $\dot{X}$ for a pure condition is Γ' *symmetric*, $\Gamma' \in \mathcal{FP}(\Gamma)$, if $\forall k \in \omega \forall P \in \mathbb{M}_k(\Gamma') \forall M \in \omega \exists T \in \text{ext}(P) \exists x \in L_M (T \Vdash \text{“}\check{x} \leq \dot{X}\text{”})$. Also $\dot{X}$ is *symmetric* if $\forall \Gamma' \in \mathcal{FP}(\Gamma)$ $\dot{X}$ is Γ'-symmetric.

Definition 9. A $\mathbb{C}(\Gamma)$-name for a pure condition $\dot{X}$ is Γ' *symmetric below* $p \in \mathbb{C}(\Gamma)$, where $\Gamma' \in \mathcal{FP}(\Gamma)$, if $\forall k \in \omega \forall P \in \mathbb{M}_{k,p}(\Gamma') \forall M \in \omega\ \exists T \in \text{ext}(P) \exists x \in L_M (T \Vdash \text{“}\check{x} \leq \dot{X}\text{”})$. Also $\dot{X}$ is *symmetric below* $p \in \mathbb{C}(\Gamma)$ if $\forall \Gamma' \in \mathcal{FP}(\Gamma)$ $\dot{X}$ is Γ'-symmetric below p.

Lemma 1. *Let $\Gamma \in [\omega_2]^\omega$, ϕ a formula in the $\mathbb{C}(\Gamma)$-language of forcing such that $\forall \Gamma' \in \mathcal{FP}(\Gamma) \forall k \in \omega \forall P \in \mathbb{M}_k(\Gamma') \forall M \in \omega \exists T \in \exp(P) \exists x \in L_M$ such that $\phi(T, x)$. Then there is a $\mathbb{C}(\Gamma)$-symmetric name $\dot{X}$ for a pure condition such that $\forall \Gamma' \in \mathcal{FP}(\Gamma) \forall k \in \omega \forall P \in \mathbb{M}_k(\Gamma') \forall M \in \omega \exists T \in \exp(P) \exists x \in L_M$ for which $\phi(T, x)$ holds and $T \Vdash \text{“}\check{x} \in \dot{X}\text{”}$.*

Proof. Let $\{\Gamma_n\}_{n\in\omega}$ enumerate all finite ordered partitions of Γ, for every $n \in \omega$ let $\{P_{n,m}\}_{m\in\omega}$ enumerate $\mathbb{M}(\Gamma_n)$ and let $\tau : \omega \to \omega \times \omega$ such that $\forall (n, m) \in \omega \times \omega |\tau^{-1}(n, m)| = \omega$. Now for every $i \in \omega$ let $P_i = P_{\tau(i)}$. Then $\{P_i\}_{i\in\omega}$ is an enumeration of $\mathbb{M}(\Gamma)$ such that each matrix $P_{n,m}$ appears cofinally often. Let $i \in \omega$, $P_i = P_{n,m}$ for some n, m. By hypothesis there is $T_i \in \mathcal{T}(\Gamma_n)$ extending P_i and $x_i \in L_i$ such that $\phi(T_i, x_i)$. Let $\mathcal{A}_i = \{a_{is}\}_{s\in\omega}$ be a maximal antichain in $\mathbb{C}(\Gamma) - \mathbb{C}(\Gamma)^+(\{t\}_{t\in maxT_i})$ such that $\forall s \in \omega \exists x_{is} \in L_i(\phi(a_{is}, x_{is}))$. Let $\dot{X} = \cup_{i\in\omega}(\{\langle \check{x}_i, t\rangle : t \in \max T_i\} \cup \{\langle \check{x}_{is}, a_{is}\rangle\}_{s\in\omega})$. □

Remark 1. Whenever a name $\dot{X}$ is constructed by the method of Lemma 1, we say that $\dot{X}$ is obtained by diagonalization of $\mathbb{M}(\Gamma)$ with respect to $\phi(T, x)$. If C is a countable centered family of symmetric names for pure conditions, then there is a name $\dot{X} = \langle \dot{X}(i) : i \in \omega\rangle$ such that $\forall P \in \mathbb{M}(\Gamma) \forall M \in \omega \exists T \in \text{ext}(P) \exists x \in L_M$ such that $T \Vdash \check{x} \in \dot{X}$, $\forall m \in \omega \dot{X}_m = \langle \dot{X}(i) : i \geq m\rangle$ is symmetric and $\Vdash C \subseteq Q(\{\dot{X}_m\}_{m\in\omega})$. Such names are called *strongly symmetric*. Since all names constructed by diagonalization of $\mathbb{M}(\Gamma)$ are strongly symmetric, for every $\mathbb{C}(\Gamma)$ symmetric name $\dot{X}$ there is a strongly symmetric name $\dot{X}'$ such that $\Vdash \dot{X}' \leq \dot{X}$.

Lemma 2. *If $\dot{Y}$ is $\mathbb{C}(\Gamma)$ symmetric below e, then there is a $\mathbb{C}(\Gamma)$ symmetric name Y^*_e such that $e \Vdash Y^*_e \leq \dot{Y}$.*

Proof. Fix a maximal antichain $E = \{e_i\}_{i\in\omega}$ in $\mathbb{C}(\Gamma)$ such that $e_0 = e$. For every $i \in \omega$ let Φ_i be an isomorphism from $\mathbb{C}(\Gamma)^+(e_i)$ onto $\mathbb{C}(\Gamma)^+(e_0)$ such that $\forall \gamma \in \Gamma$ $\Phi_i''\mathbb{C}(\{\gamma\}) \subseteq \mathbb{C}(\{\gamma\})$.

Let $\Gamma' = \{\Gamma_j\}_{j\in n} \in \mathcal{FP}(\Gamma)$, $P \in \mathbb{M}_k(\Gamma')$, $M \in \omega$. Then $\forall i \in \omega$, $p_i = \cup_{j\in n} p^j_i \in \mathbb{C}(\Gamma)$ and so $\exists s(i)$ such that $p_i \not\perp e_{s(i)}$ with common extension q_i. Then $\forall j \in n$ let $q^j_i = q_i \restriction \Gamma_j \times \omega$. Thus $P_E = Q = (q^j_i)$ is a componentwise extension of P. Then $\forall i, j$, $\hat{q}^j_i = \Phi_{s(i)}(q^j_i) = \Phi_{s(i)}(q_i \restriction \Gamma_j \times \omega) = \Phi_{s(i)}(q_i) \restriction \Gamma_j \times \omega \leq e_0 \restriction \Gamma_j \times \omega$. Therefore $\hat{Q} = (\hat{q}^j_i)$ is a matrix below e. Since $\dot{Y}$ is symmetric below e, $\exists \hat{T} \in \text{ext}(\hat{Q}) \exists x \in L_M$ such that $\hat{T} \Vdash \check{x} \leq \dot{Y}$. If $\hat{t} : {}^{\leq n}k \to \cup_{j\in n}\mathbb{C}(\Gamma_j)$ induces $\hat{T}$, define $t : {}^{\leq n}k \to \cup_{j\in n}\mathbb{C}(\Gamma_j)$ as follows: $\forall j \in n \forall a \in {}^{j+1}k$, $a = (b, i)$, $i \in k$ let $t(a) = \Phi^{-1}_{s(i)}(\hat{t}(a))$. Then since $\hat{t}(a) \leq \Phi_{s(i)}(q^j_i)$, we have $t(a) \leq q^j_i$. Thus if T is induced by t, then $T \in$

$\mathrm{ext}(P_E) \subseteq \mathrm{ext}(P)$. Let $I : \mathrm{ext}(\hat{P}_E) \to \mathrm{ext}(P_E)$, $I(\hat{T}) = T$. Similarly define $J : \mathrm{ext}(P_E) \to \mathrm{ext}(\hat{T}_E)$ where if T is induced by t, then $\forall j \in n \forall a \in {}^{j+1}k$, $a = (b, i)$, $i \in k$ let $\hat{t}(a) = \Phi_{s(i)}(t(a))$ and let $J(T) = \hat{T}$ be the tree induced by $\hat{t}$. Then $\forall T \in \mathrm{ext}(P_E)(J \circ I(T) = T)$ and $\forall R \in \mathrm{ext}(\hat{P}_E)(I \circ J(R) = R)$.

The above construction did not depend on the choice of Γ'. Therefore $\forall \Gamma' \in \mathcal{FP}(\Gamma) \forall k \in \omega \forall P \in \mathbb{M}_k(\Gamma') \forall M \in \omega \exists T \in \mathrm{ext}(P) \exists x \in L_M$ such that $\hat{T} \Vdash \check{x} \le \dot{Y}$. To obtain Y_e^* diagonalize $\mathbb{M}(\Gamma)$ with respect to $\phi(T, x)$ where $\phi(T, x)$ holds iff $\hat{T}$ is defined and $\hat{T} \Vdash \check{x} \le \dot{Y}$. If $t \le e$ and $\langle t, \check{x} \rangle \in Y_e^*$, then $\hat{t} = t \Vdash \check{x} \le \dot{Y}$. Therefore $e \Vdash Y_e^* \le \dot{Y}$. □

Lemma 3. *Let G be a $\mathbb{C}(\Gamma)$-generic filter, $X \in [\omega]^\omega \cap V[G]$. If $\forall \Gamma' \in \mathcal{FP}(\Gamma)$ X has a Γ'-symmetric name, then X has a symmetric name.*

Proof. Proceed by the method of Lemma 1. At stage i of the construction if $P_i = P_{m,n} \in \mathbb{M}_k(\Gamma_m)$ for some partition Γ_m, use the Γ_m symmetry of a name for X to obtain $T_i \in \mathrm{ext}(P_i)$ and $x \in L_i$ such that $T_i \Vdash \check{x}_i \le \dot{X}$. □

3. An ultrafilter of symmetric names

Definition 10. Let $\Gamma' = \{\Gamma_j\}_{j \in n} \in \mathcal{FP}(\Gamma)$, $\phi : {}^{\le n}\omega_1 \to \cup_{j \in n} {}^{\Gamma_j \times \omega}\omega$ such that $\forall j \in n \forall u \in {}^{j+1}\omega_1 (\phi(u) \in {}^{\Gamma_j \times \omega}\omega)$. Then ϕ induces a tree $\Phi = \{\Phi(u)\}_{u \in {}^{\le n}\omega}$ where $\Phi(u) = (\Phi(v), \phi(u))$ where $u = (v, i)$, $i \in k$ and $\Phi(u) \le_\Phi \Phi(v)$ if $u \restriction |v| = v$. Let $\mathbf{\Phi}(\Gamma')$ be the set of all trees induced by some injective $\phi : {}^{\le n}k \to \cup_{j \in n} {}^{\Gamma_j \times \omega}\omega$. Again, capital letters will denote trees while the corresponding small letters will denote the inducing functions.

Consider ${}^{\Gamma \times \omega}\omega$ as the Tychonoff product of Γ copies of the Baire space ${}^\omega\omega$. Then for every basic open neighborhood U of ${}^{\Gamma \times \omega}\omega$, there is $p \in \mathbb{C}(\Gamma)$ such that $U = [p]_\Gamma = \{f \in {}^{\Gamma \times \omega}\omega : f \restriction \mathrm{dom}(p) = p\}$. If $\Gamma' = \{\Gamma_j\}_{j \in n} \in \mathcal{FP}(\Gamma)$, consider $\prod_{j=0}^n {}^{\Gamma_j \times \omega}\omega$ as a Tychonoff product of ${}^{\Gamma_j \times \omega}\omega$. Then every basic open neighborhood is of the form $\prod_{j=0}^n [p_j]_{\Gamma_j}$ where $p \in \mathbb{C}(\Gamma)$, $p_j = p \restriction \Gamma_j \times \omega$.

Definition 11. $\Phi \in \mathbf{\Phi}(\{\Gamma_j\}_{j \in n})$ is nowhere meager (denoted nwm), if $\forall j \in n \forall u \in {}^j\omega_1$ $\{\phi(u, \alpha)\}_{\alpha \in \omega_1}$ is a nowhere meager subset of ${}^{\Gamma_j \times \omega}\omega$.

Definition 12. An injective mapping $\psi : {}^{\le n}k \to {}^{\le n}\omega_1$ such that $|\psi(a)| = |a|$, $a \subseteq b \to \psi(a) \subseteq \psi(b)$ is called a tree embedding.

Lemma 4. *Let $n \ge 2$. For every ordered partition $\{\Gamma_j\}_{j \in n}$, for every nwm tree $\Phi \in \mathbf{\Phi}(\{\Gamma_j\}_{j \in n-1})$ and every $R : {}^{n-1}\omega_1 \times \mathbb{C}(\Gamma_{n-1}) \to \{0, 1\}$ either $(I)_n$ or $(II)_n$ holds, where:*

$(I)_n$ $\exists p = (p_i) \in \mathbb{M}_1(\{\Gamma_j\}_{j \in n})$ s.t. $\forall k \in \omega \forall P = (p_i^j) \in \mathbb{M}_k(\{\Gamma_j\}_{j \in n-1})$ below $p \restriction n-1$ there is a tree embedding $\psi : {}^{\le n-1}k \to {}^{\le n-1}\omega_1$ such that $\forall j \in n-1 \forall a \in {}^{j+1}k$ if $a = (b, i)$, $i \in k$, then $\phi \circ \psi(a) \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{n-1}k$, $R(\psi(a), p_{n-1}) = 1$.

$(II)_n$ $\forall k \in \omega \forall P = (p_i^j) \in \mathbb{M}_k(\{\Gamma_j\}_{j \in n-1})$ there is a tree embedding $\psi : {}^{\le n-1}k \to {}^{\le n-1}\omega_1$ such that $\forall j \in n-1 \forall a \in {}^{j+1}k$ if $a = (b, i)$, $i \in k$, then $\phi \circ \psi(a) \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{n-1}k \forall p \in \mathbb{C}(\Gamma_{n-1})$ $R(\psi(a), p) = 0$.

Proof. The statement is proved by induction on n. Let $n = 2$, let $\{\Gamma_j\}_{j \in 2}$ be a finite ordered partition, let $\Phi \in \mathbf{\Phi}(\Gamma_0)$ be a nwm tree (that is $\{\phi(\alpha)\}_{\alpha \in \omega_1}$

is a nwm subset of ${}^{\Gamma_0\times\omega}\omega$), $R^{\{0\}}\omega_1 \times \mathbb{C}(\Gamma_1) \to \{0,1\}$. If there is $p \in \mathbb{C}(\Gamma_1)$ such that $B_p = \{\phi(\alpha) : R(\alpha, p) = 1\}$ is not meager, then there is $q \in \mathbb{C}(\Gamma_0)$ such that $B_p \cap [q]_{\Gamma_0}$ is everywhere non-meager. Let $P = (p_i) \in \mathbb{M}_k(\Gamma_0)$ below q. Then $\forall i \in k \exists \phi(\alpha_i) \in [p_i]_{\Gamma_0} \cap B_p$ and so $\forall i \in k R(\alpha_i, p) = 1$. Take $\psi : k \to \omega_1$ where $\psi(i) = \alpha_i$. Then $(\mathrm{I})_2$ holds with witness (q, p).

Assume the statement holds for some $n \geq 2$. Let $\{\Gamma_j\}_{j\in n+1}$ be a finite ordered partition, $\Phi \in \mathbf{\Phi}(\{\Gamma_j\}_{j\in n})$ nwm tree, $R : {}^n\omega_1 \times \mathbb{C}(\Gamma_n) \to \{0,1\}$. Now, for every $\alpha \in \omega_1$, let $\Phi_\alpha \in T(\{\Gamma_j\}_{j=1}^n)$ be a nwm tree induced by $\phi_\alpha : \cup_{j=1}^{n-1}{}^{\{1,\dots,j\}}\omega_1 \to \cup_{j=1}^{n-1}{}^{\Gamma_j\times\omega}\omega$ where $\phi_\alpha(u) = \phi(\langle\alpha, u\rangle)$ and let $R_\alpha : {}^{\{1,\dots,n\}}\omega_1 \times \mathbb{C}(\Gamma_n) \to \{0,1\}$ where $R_\alpha(u, p) = R(\langle\alpha, u\rangle, p)$. Then for every $\alpha \in \omega_1$, by the inductive hypothesis applied to $\{\Gamma_j\}_{j=1}^n$, Φ_α, R_α either $(\mathrm{I})_n$ or $(\mathrm{II})_n$ holds. To specify the dependence on α, we say that $(\mathrm{I})_{n,\alpha}$ or $(\mathrm{II})_{n,\alpha}$ holds. For completeness of notation we state explicitly $(\mathrm{I})_{n,\alpha}$ and $(\mathrm{II})_{n,\alpha}$. If $(\mathrm{I})_{n,\alpha}$ holds with witness $p^\alpha = (p_i^\alpha)_{i=1}^n \in \mathbb{M}_1(\{\Gamma_j\}_{j=1}^n)$ then for every $k \in \omega$, every $P = (p_i^j)_{i\in k} \in \mathbb{M}_k(\{\Gamma_j\}_{j=1}^{n-1})$ below $(p_i^\alpha)_{i=1}^{n-1}$, there is a tree embedding $\psi_\alpha : \cup_{j=1}^{n-1}{}^{\{1,\dots,j\}}k \to \cup_{j=1}^{n-1}{}^{\{1,\dots,j\}}\omega_1$ such that $\forall a \in {}^{\{1,\dots,j\}}k$, $1 \leq j \leq n-1$, $a = (b, i)$, $i \in k$, $\phi_\alpha \circ \psi_\alpha \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{\{1,\dots,n-1\}}k$ $R_\alpha(\psi_\alpha(a), p_n^\alpha) = 1$. If $(\mathrm{II})_{n,\alpha}$, then for all $k \in \omega$, $P = (p_i^j) \in \mathbb{M}_k(\{\Gamma_j\}_{j=1}^{n-1})$ there is a tree embedding $\psi_\alpha : \cup_{j=1}^{n-1}{}^{\{1,\dots,j\}}k \to \cup_{j=1}^{n-1}{}^{\{1,\dots,j\}}\omega_1$ such that $\forall a \in {}^{\{1,\dots,j\}}k$, $1 \leq j \leq n-1$, $a = (b, i)$, $i \in k$, $\phi_\alpha \circ \psi_\alpha \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{\{1,\dots,n-1\}}k \forall p \in \mathbb{C}(\Gamma_n)$ $R_\alpha(\psi_\alpha(a), p) = 0$.

If $\mathcal{C}_0 = \{\phi(\alpha) : (\mathrm{I})_{n,\alpha}\}$ is non-meager in ${}^{\Gamma_0\times\omega}\omega$, then $\exists \mathcal{C}_1 \subseteq \mathcal{C}_0$ which is non-meager and such that $\forall \phi(\alpha) \in \mathcal{C}_1$ $(\mathrm{I})_{n,\alpha}$ holds with the same witness $(p_i)_{i=1}^n \in \mathbb{M}_1(\{\Gamma_j\}_{j=1}^n)$. Since $\mathcal{C}_1$ is non-meager, $\exists p_0 \in \mathbb{C}(\Gamma_0)$ such that $\mathcal{C}_1 \cap [p_0]_{\Gamma_0}$ is everywhere non-meager in $[p_0]_{\Gamma_0}$. It will be shown that $(\mathrm{I})_{n+1}$ holds with witness $(p_i)_{i=0}^n$. Let $k \in \omega$ and let $P = (p_i^j) \in \mathbb{M}_k(\{\Gamma_j\}_{j\in n})$ be a matrix below $(p_i)_{i\in n}$. Then $\forall i \in k \exists \alpha_i \in \omega_1 \phi(\alpha_i) \in [p_i^0] \cap \mathcal{C}_1$. Then $\psi : {}^{\leq n}k \to {}^{\leq n}\omega_1$ where $\psi(\langle i, a\rangle) = \alpha_i{}^\frown\psi_{\alpha_i}(a)$ is a tree embedding and $\forall j \in n \forall a \in {}^{j+1}k$, $a = (s, b, i)$, $s, i \in k$, $\phi \circ \psi(a) = \phi(\alpha_s{}^\frown\psi_{\alpha_s}(b, i)) = \phi_{\alpha_s} \circ \psi_{\alpha_s}(b, i) \in [p_i^j]_{\Gamma_j}$, as well as $\forall a \in {}^n k$, $a = (s, b)$, $s \in k$, $R(\psi(a), p_n) = R(\alpha_s{}^\frown\psi_{\alpha_s}(b), p_n) = R_{\alpha_s}(\psi_{\alpha_s}(b), p_n) = 1$. Otherwise $\mathcal{C}_0' = \{\phi(\alpha)\}_{\alpha\in\omega_1} \setminus \mathcal{C}_0 = \{\phi(\alpha) : (\mathrm{II})_{n,\alpha}\}$ is everywhere non-meager. Let $k \in \omega$, $P = (p_i^j) \in \mathbb{M}_k(\{\Gamma_j\}_{j\in n})$. Then $\forall i \in k \exists \alpha_i \in \omega_1 \phi(\alpha_i) \in \mathcal{C}_0' \cap [p_i^0]_{\Gamma_0}$. Then $\psi : {}^{\leq n}k \to {}^{\leq n}\omega_1$ where $\psi(i, \alpha) = \alpha_i{}^\frown\psi_{\alpha_i}(a)$ $(i \in k)$ is a tree embedding and $\forall j \in n \forall a \in {}^{j+1}k$, $a = (s, b, i)$, $s, i \in k$, $\phi \circ \psi(a) = \phi(\alpha_s{}^\frown\psi_{\alpha_s}(b, i)) = (\phi_{\alpha_s} \circ \psi_{\alpha_s})(b, i) \in [p_i^j]_{\Gamma_j}$, as well as $\forall a \in {}^n k$, $a = (s, b)$ $(s \in k)$ $\forall p \in \mathbb{C}(\Gamma_n)$, $R(\psi(a), p) = R(\alpha_s{}^\frown\psi_{\alpha_s}(b), p) = R_{\alpha_s}(\psi_{\alpha_s}(b), p) = 0$. $\square$

In the following $\mathcal{M}$ denotes a countable transitive model of sufficiently large portion of ZFC.

Definition 13. A tree $\Phi \in \mathbf{\Phi}(\Gamma')$ is Cohen generic over $\mathcal{M}$, if $\forall j \in n \forall u \in {}^{j+1}\omega_1$ where $u = (v, \alpha)$, $\alpha \in \omega_1$, $\phi(u)$ is $\mathbb{C}(\Gamma_j)$-generic over $\mathcal{M}[\Phi(v)]$ (thus $\phi(u)$ is a Γ_j-sequence of Cohen generic reals). Whenever the tree Φ is clear from context we will write $\mathcal{M}[u]$ for $\mathcal{M}[\Phi(u)]$.

Lemma 5. *Let $\Gamma' \in \mathcal{FP}(\Gamma)$, $\dot{X}$ a Γ'-symmetric name for a pure condition, $\Vdash \dot{X} = \dot{Y} \cup \dot{Z}$. Then $\forall p \in \mathbb{C}(\Gamma) \exists q \leq p$ such that $\dot{Y}$ is Γ'-symmetric below q, or $\dot{Z}$ is Γ' symmetric below q.*

Proof. Suppose $|\Gamma'| = 1$, i.e. $\Gamma' = \{\Gamma\}$. Note that $\dot{X}$ is $\{\Gamma\}$-symmetric below p iff for every finite tuple $(p_i)_{i\in k} \subseteq \mathbb{C}(\Gamma)^+(p)$ and every $M \in \omega$, there are $(q_i)_{i\in k}$, $x \in L_M$ such that $\forall i \in k(q_i \leq p_i)$ and $q_i \Vdash \check{x} \leq \dot{X}$. For every $p \in \mathbb{C}(\Gamma)$ let $\mathrm{hull}_p(\dot{X}) = \{x \in \mathrm{LM} : \exists q \leq p(q \Vdash \check{x} \leq \dot{X})\}$. Then $\dot{X}$ is $\{\Gamma\}$-symmetric below p iff for every finite tuple $(p_i)_{i\in k} \subseteq \mathbb{C}(\Gamma)^+(p)$ and $n \in \omega$, the set $\cap_{i\in k}\mathrm{hull}_{p_i}(\dot{X})$ meets L_n. Let $p \in \mathbb{C}(\Gamma)$ be a counterexample to the claim of the Lemma. Since $\dot{Y}$ is not $\{\Gamma\}$-symmetric below p, there are a tuple $(p_i)_{i\in k} \subseteq \mathbb{C}(\Gamma)^+(p)$ and $m \in \omega$ such that $(\cap_{p_i}\mathrm{hull}(\dot{Y})) \cap L_m = \emptyset$. For every $i \in k$ there are a finite tuple $(q_{ij})_{j\in n_i} \subseteq \mathbb{C}(\Gamma)^+(p_i)$ and $m_i \in \omega$ such that $(\cap_{j\in n_i}\mathrm{hull}(\dot{Z})) \cap L_{m_i} = \emptyset$. Consider $\{q_{ij}\}_{i\in k, j\in n_i}$. Since $\dot{X}$ is $\{\Gamma\}$-symmetric below p, for all i, j there are $t_{ij} \leq q_{ij}$ and $x \in L_M$ where $M > \{m, \max_{i\in k} m_i\}$ such that $t_{ij} \Vdash \check{x} \in \dot{X}$. Since $\Vdash \dot{X} = \dot{Y} \cup \dot{Z}$, for every i, j there is a further extension $r_{ij} \leq t_{ij}$ such that $r_{ij} \Vdash \check{x} \in \dot{Y}$ or $r_{ij} \Vdash \check{x} \in \dot{Z}$. If $\exists i \in k \forall j \in n_i(r_{ij} \Vdash \check{x} \in \dot{Z})$, we reach a contradiction since $x \in L_{m_i}$. Otherwise $\forall i \in k \exists j_i \in n_i(r_{ij_i} \Vdash \check{x} \in \dot{Y})$. But $r_{ij_i} \leq p_i$ and so $x \in \cap_{i\in k}\mathrm{hull}_{p_i}(\dot{Y})$ which is a contradiction since $x \in L_m$.

Let $|\Gamma'| \geq 2$, $\Gamma' = \{\Gamma_j\}_{j\in n}$, $\Phi \in \mathbf{\Phi}(\{\Gamma_j\}_{j\in n-1})$ a nowhere meager tree of Cohen generics over $\mathcal{M}$. For $u \in {}^{n-1}\omega_1, p \in \mathbb{C}(\Gamma_{n-1})$ let $E(u,p) = \{x \in \mathrm{LM} : \mathcal{M}[u] \vDash (\exists q \leq p)q \Vdash \check{x} \in \dot{X}[u]\}$. Then $\mathcal{E}_{n-1} = \{\cap_{i,j}^{k,\ell} E(u_i, p_j) : \{u_i\}_{i\in k} \subseteq {}^{n-1}\omega_1, \{p_j\}_{j\in\ell} \subseteq \mathbb{C}(\Gamma_{n-1}), k, \ell \in \omega\} \subseteq [\mathrm{LM}]$ is centered. Let $U \subseteq [\mathrm{LM}]$ be such that $\mathcal{E}_{n-1} \subseteq U$ and $\forall X \in [LM]$ either $X \in U$ or $\exists Y \in U(Y \cap X \notin [\mathrm{LM}])$. For $u \in {}^{n-1}\omega_1$, $p \in \mathbb{C}(\Gamma_{n-1})$ let $D(u,p) = \{x \in \mathrm{LM} : \mathcal{M}[u] \vDash p \Vdash_{\mathbb{C}(\Gamma_{n-1})} \check{x} \in (\dot{X}^c \cup \dot{Y})[u]\}$ and for $v \in {}^{n-2}\omega_1$, $p \in \mathbb{C}(\Gamma_{n-1})$ let $B(v,p) = \{\phi(v^\frown\alpha) : D(v^\frown\alpha, p) \in U\}$. Let $R : {}^{n-1}\omega_1 \times \mathbb{C}(\Gamma_{n-1}) \to \{0,1\}$ where $R(u,p) = 1$ if $D(u,p) \in U$ and $R(u,p) = 0$ otherwise. By Lemma 4 $(\mathrm{I})_n$ or $(\mathrm{II})_n$ holds.

If $(\mathrm{I})_n$ holds with witness $p = (p_i)_{i\in n} \in \mathbb{M}_1(\Gamma')$, let $P = (p_i^j) \in \mathbb{M}_k(\Gamma')$ below p and $M \in \omega$. Then there is a tree embedding $\psi : {}^{\leq n-1}k \to {}^{\leq n-1}\omega_1$ such that $\forall j \in n-1 \forall a \in {}^{j+1}k$ where $a = (b,i)$, $i \in k$ $\phi \circ \psi(a) \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{n-1}k$ $D(\psi(a), p_{n-1}) \in U$. Since $\forall a \in {}^{n-1}k$ $E(\psi(a), p_i^{n-1}) \in U$, also $A = (\bigcap E(\psi(a), p_i^{n-1}) \cap (\bigcap D(\psi(a), p_{n-1}) \in U$. Then $\exists x \in L_M \cap A$ and so $\forall a \in {}^{n-1}k$, $M[\psi(a)] \vDash (\exists p_{a,i} \leq p_i^{n-1})p_{a,i} \Vdash \check{x} \in \dot{X}[\psi(a)]$ and $\mathcal{M}[\psi(a)] \vDash p_{n-1} \Vdash \check{x} \in (\dot{X}^c \cup \dot{Y})[\psi(a)]$. Then since $\forall i(p_i^{n-1} \leq p_{n-1})$ we obtain that for all $a \in {}^{n-1}k$ $M[\psi(a)] \vDash p_{a,i} \Vdash$ "$\check{x} \in \dot{X}[\psi(a)]$ and $\check{x} \in (\dot{X}^c \cup \dot{Y})[\psi(a)]$". Therefore $M[\psi(a)] \vDash p_{a,i} \Vdash \check{x} \in \dot{Y}[\psi(a)]$. In finitely many steps obtain $T \in \mathrm{ext}(P)(T \Vdash$ "$\check{x} \in \dot{Y}$)". Otherwise $(\mathrm{II})_n$ holds. Let $k \in \omega$, $P = (p_i^j) \in \mathbb{M}_k(\Gamma')$, $M \in \omega$. Then there is a tree embedding $\psi : {}^{\leq n-1}k \to {}^{\leq n-1}\omega_1$ such that $\forall j \in n \forall a \in {}^{j+1}k$ where $a = (b,i)$, $i \in k$, $\phi \circ \psi(a) \in [p_i^j]_{\Gamma_j}$ and $\forall a \in {}^{n-1}k \forall p \in \mathbb{C}(\Gamma_{n-1})$ $D(\psi(a), p) \notin U$. Then $\exists x \in L_M$ such that $x \notin \cup_{a\in {}^{n-1}k, i\in k} D(\psi(a), p_i^{n-1})$ and so $\forall a \in {}^{n-1}k$ $\mathcal{M}[\psi(a)] \vDash p_i^{n-1} \not\Vdash$ "$\check{x} \in \dot{X}^c[\psi(a)] \cup \dot{Y}[\psi(a)]$". Therefore $\forall a \exists p_{a,i} \leq p_i^{n-1}$ such that $\mathcal{M}[\psi(a)] \vDash p_{a,i} \Vdash$ "$\check{x} \in \dot{Z}[\psi(a)]$". In finitely many steps obtain $T \in \mathrm{ext}(P)(T \Vdash$ "$\check{x} \in \dot{Z}$"). □

Lemma 6. *If $\dot{X}$ is a $\mathbb{C}(\Gamma)$ symmetric name for a pure condition, $\dot{A}$ is a name for an infinite subset of ω, then there is a $\mathbb{C}(\Gamma)$-symmetric name $\dot{Y}$ such that $\dot{Y} \leq \dot{X}$ and $\forall i \in \omega \Vdash$ int$(\dot{Y}(i)) \subseteq \dot{A}$ or int$(\dot{Y}(i)) \subseteq \dot{A}^c$.*

Proof. Diagonalize $\mathbb{M}(\Gamma)$ with respect to $\phi(T,x)$ where $\phi(T,x)$ holds iff $\forall t \in \max T$ $t \Vdash$ "$\check{x} \leq \dot{X}, \mathrm{int}(x) \subseteq \dot{A}$" or $t \Vdash$ "$\check{x} \leq \dot{X}, \mathrm{int}(x) \subseteq \dot{A}^c$". □

Lemma 7. *Let $\dot{X}$ be a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition, $\dot{A}$ a $\mathbb{C}(\Gamma)$-name for a set of integers, G a $\mathbb{C}(\Gamma)$-generic filter. Then in $V[G]$ there is a pure condition X^* with a symmetric name which extends $\dot{X}[G]$ and such that int$(X^*) \subseteq \dot{A}[G]$ or int$(X^*) \subseteq \dot{A}^c[G]$.*

Proof. Passing to a name for an extension if necessary, by Lemma 6 we can assume that $\forall P \in \mathbb{M}(\Gamma) \forall M \in \omega \exists T \in \mathrm{ext}(P) \exists x \in L_M$ such that $T \Vdash \check{x} \in \dot{X}$ and for all i, $\Vdash$ "$\mathrm{int}(\dot{X}(i)) \subseteq \dot{A}$ or $\mathrm{int}(\dot{X}(i)) \subseteq \dot{A}^c$". Then there are $\mathbb{C}(\Gamma)$-names $\dot{Y}$, $\dot{Z}$ such that $\Vdash \dot{Y} = \langle \dot{X}(i) : \mathrm{int}(\dot{X}(i) \subseteq \dot{A} \rangle$ and $\Vdash \dot{Z} = \langle \dot{X}(i) : \mathrm{int}(\dot{X}(i) \subseteq \dot{A}^c \rangle$. By Lemma 5 $\forall \Gamma' \in \mathcal{FP}(\Gamma) \forall p \in \mathbb{C}(\Gamma) \exists q \leq p$ such that $\dot{Y}$ is Γ' symmetric below p, or $\dot{Z}$ is Γ'-symmetric below p. For every $\Gamma' \in \mathcal{FP}(\Gamma)$ let $E(\Gamma')$ be a maximal antichain in $\mathbb{C}(\Gamma)$ such that $\forall e \in E(\Gamma')$ either there is no $t \leq e$ such that $\dot{Y}$ is Γ'-symmetric below t and $\dot{Z}$ is Γ'-symmetric below e, or $\dot{Y}$ is Γ'-symmetric below e. For every Γ' let $\{e(\Gamma')\} = G \cap E(\Gamma')$. If $\forall \Gamma' \in \mathcal{FP}(\Gamma)$, $\dot{Y}$ is Γ'-symmetric below $e(\Gamma')$, then by Lemmas 2 and 3, $\dot{Y}[G]$ has a symmetric name. Otherwise there is Γ' such that $\forall t \leq e(\Gamma')$ $\dot{Y}$ is not Γ'-symmetric below t and so by the choice of $E(\Gamma')$, $\dot{Z}$ is Γ'-symmetric below $e(\Gamma')$. Let $\Gamma'' \in \mathcal{FP}(\Gamma)$ be distinct from Γ' and $\Gamma_0 \in \mathcal{FP}(\Gamma)$ such that $\forall D \in \Gamma_0$ either $D \in \Gamma'$ or $D \in \Gamma''$. If $\dot{Y}$ is Γ_0-symmetric below $e(\Gamma_0)$, then $\dot{Y}$ is Γ'-symmetric below t, where $t \in G$ is a common extension of $e(\Gamma_0)$ and $e(\Gamma')$ which is a contradiction. Then $\dot{Z}[G]$ has a symmetric name. □

4. Unboundedness

Definition 14. Let $\Gamma \in [\omega_2]^\omega$, $\Gamma' = \{\Gamma_j\}_{j \in n}$ a finite ordered partition of Γ, $k \in \omega$. Let $\{\Gamma_a : a \in {}^{\leq n}k\}$ be a family of pairwise disjoint sets of ordinals such that $\forall j \leq n \forall a \in {}^j k$ $\Gamma_a \cong \Gamma_{j-1}$ with an isomorphism i_a such that $a <_{lex} b \to \sup \Gamma_a < \min \Gamma_b$. Let $\tilde{\Gamma} = \cup\{\Gamma_a : a \in {}^{\leq n}k\}$. Then $\mathbb{C}(\tilde{\Gamma})$ is said to be a Cohen tree defined by Γ, Γ' and k. For every $a \in {}^n k$ and $\mathbb{C}(\tilde{\Gamma})$-generic filter G, let $G^a = G \cap \prod_{i \in n} \mathbb{C}(\Gamma_{a|i})$.

Lemma 8. *Let $\dot{X}$ be a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition, $\Gamma \in [\omega_2]^\omega$, $\Gamma' = \{\Gamma_j\}_{j \in n} \in \mathcal{FP}(\Gamma)$, $k \in \omega$, $\tilde{\Gamma}$ a Cohen tree defined by Γ, Γ', $k \in \omega$, $A \in [\omega]^\omega \cap V$ and G a $\mathbb{C}(\tilde{\Gamma})$-generic filter. Then in $V[G]$ there is a pure condition $\tilde{X}$ with strongly $\mathbb{C}(\tilde{\Gamma})$-symmetric name such that $\forall a \in {}^n k$ $\tilde{X} \leq \dot{X}[G^a]$ and int$(\tilde{X}) \subseteq A$ or int$(\tilde{X}) \subseteq A^c$.*

Proof. For every $a \in {}^n k$ let $\Gamma^a = \cup_{j \in n} \Gamma_{a|j}$ and $I_a : \Gamma^a \cong \Gamma$ where $I_a \restriction \Gamma_{a|j} = i_{a|j}$. If $\tilde{\Gamma}' \in \mathcal{FP}(\tilde{\Gamma})$ $P \in \mathbb{M}(\tilde{\Gamma}, \tilde{\Gamma}')$ and $M \in \omega$, then there is a tree of extensions $T \in \mathrm{ext}(P)$ in $\mathcal{T}(\tilde{\Gamma}, \tilde{\Gamma}')$ and $x \in L_M$ such that $\forall t \in \max T$ $t \restriction \Gamma^a \Vdash \check{x} \leq I_a(\dot{X})$, and $\mathrm{int}(x) \subseteq A$ or $\mathrm{int}(x) \subseteq A^c$ (for such T, x we will say that $\phi(T,x)$ holds). Diagonalizing $\mathbb{M}(\tilde{\Gamma})$ obtain a $\mathbb{C}(\tilde{\Gamma})$-symmetric name $\tilde{X}$ such that $\forall P \in \mathbb{M}(\tilde{\Gamma}) \forall M \in \omega$ there are $T \in \mathrm{ext}(P)$, $x \in L_M$ such that

$\phi(T, x)$ and $T \Vdash \check{x} \in \tilde{X}$. Repeating the proof of Lemma 7 one can show that if $\tilde{Y}$, $\tilde{Z}$ are $\mathbb{C}(\tilde{\Gamma})$-names such that $\Vdash \tilde{Y} = \langle \tilde{X}(i) : \mathrm{int}(\tilde{X}(i)) \subseteq \check{A}\rangle$, $\Vdash \tilde{Z} = \langle \tilde{X}(i) : \mathrm{int}(\tilde{X}(i)) \subseteq \check{A}^c\rangle$, then $\tilde{Y}[G]$ or $\tilde{Z}[G]$ has a symmetric name. □

The following sufficient condition for an induced logarithmic measure to take arbitrarily high values can be found in [1]

Lemma 9. *Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family and let h be the logarithmic measure induced by P. Then if $\forall n \in \omega$ and every partition $\omega = A_0 \cup \cdots \cup A_{n-1}$, $\exists j \in n$ such that A_j contains a positive set, then $\forall k \in \omega \forall n \in \omega$ and partition $\omega = A_0 \cup \cdots \cup A_{n-1}$, $\exists j \in n$ such that A_j contains a set of h measure greater or equal k.*

Definition 15. A $\mathbb{C}(\Gamma) * Q(C)$-name for a real $\dot{f}$, where C is a centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions is *good*, if for every centered family C' of $\mathbb{C}(\omega_2)$-symmetric names for pure conditions, $\dot{f}$ is a $\mathbb{C}(\omega_2) * Q(C')$-name for a real. For every $i \in \omega$, let $\mathcal{A}_i(\dot{f})$ be a maximal antichain in $\mathbb{C}(\Gamma) * Q(C)$ of conditions deciding $\dot{f}(i)$.

Lemma 10. *Let $\dot{X} = \langle \dot{X}(i)\rangle_{i\in\omega}$ be a strongly symmetric $\mathbb{C}(\Gamma)$-name, $P \in \mathbb{M}(\Gamma, \Gamma')$, $\dot{f}$ a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real, where $C = \{\dot{X}_m\}_{m\in\omega}$, $\dot{X}_m = \langle \dot{X}(i)\rangle_{i\geq m}$. Then the logarithmic measure induced by the family $\mathcal{P}_k(\dot{X}, \dot{f}(i), P)$ of all $x \in [\omega]^{<\omega}$ such that there is a tree of extensions T of P which has the property that for every $a \in {}^n k$*

(1) $T(a) \Vdash (\check{x} \subseteq int(\dot{X}) \wedge (\exists l \in \omega(x \cap int(X(l))$ is $\dot{X}(l)$ -positive$))$

(2) $\exists N \in \omega \forall v \subseteq k \exists w^a_v \subseteq x \exists A_{va} \in \mathcal{A}_i(\dot{f})(T(a), (v \cup w^a_v, \dot{X}_N)) \leq A_{va}$

takes arbitrarily high values. T is said to witness that $x \in \mathcal{P}_k(\dot{X}, \dot{f}, P)$.

Proof. Let $\tilde{\Gamma}$ be a Cohen tree on Γ, Γ', k. Let G be $\mathbb{C}(\tilde{\Gamma})$-symmetric and $\omega = A_0 \cup \cdots \cup A_{M-1}$ a finite partition of ω. Then by Lemma 8, there is a pure condition with a $\mathbb{C}(\tilde{\Gamma})$-symmetric name $\tilde{X}$ such that $\forall a \in {}^n k$ $\tilde{X}[G] \leq \dot{X}[G^a]$ and for some $j_0 \in M$ $\mathrm{int}(\tilde{X}[G]) \subseteq A_{j_0}$. Then in particular $\tilde{C} = \{\tilde{X}_m[G]\}_{m\in\omega}$ where $\tilde{X}_m = \langle \tilde{X}(i) : i \geq m\rangle$ extends all of $C_a = \{X_m[G^a]\}_{m\in\omega}$, $a \in {}^n k$.

Let $v \subseteq k$, $a \in {}^n k$. Since $\dot{f}_a = \dot{f}/G^a$ is $Q(\tilde{C})$-name for a real, there is $\dot{R}_{av}$ a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition, $u_{av} \subseteq w$ and $q_{av} \in G^a$ such that $A_{av} = (q_{av}, (u_{av}, \dot{R}_{av})) \in \mathcal{A}_i(\dot{f})$ such that $(u_{av}, \dot{R}_{av}[G^a])$ and $(v, \tilde{X}[G])$ are compatible with common extension $(v \cup w_{av}, \tilde{T}[G])$. Since $\dot{R}_{av}$ belongs to $Q(C)$ there is N_{av} such that $\Vdash \dot{R}_{av} \leq \dot{X}_{N_{av}}$. Then there is $t_{av} \in G^a$ extending q_{av} and p^a such that $(t_{av}, (v \cup w_a, \dot{X}_{N_{av}})) \leq A_{av}$. In finitely many steps find $x \in [\mathrm{int}(\tilde{X})]^{<\omega}$ such that for all $v \subseteq k$, $a \in {}^n k$ there are $w_{av} \subseteq x$, $N_{av} \in \omega$, $t_{av} \in G^a$ such that $(t_{av}, (v \cup w_{av}, \dot{X}_{N_{av}})) \leq A_{av}$ and such that for some $\ell \in \omega$, $x \cap \mathrm{int}(\tilde{X}(\ell))[G]$ is $\tilde{X}(\ell)$-positive. Since $\tilde{X}[G] \leq \dot{X}[G^a]$ (for all $a \in {}^n k$) we have $x \subseteq \mathrm{int}(\dot{X}[G^a])$ and furthermore $\forall a \in {}^n k \exists \ell_a \in \omega$ such that $x \cap \mathrm{int}(\dot{X}(\ell_a))[G^a]$ is a positive subset of $\dot{X}(\ell_a)[G^a]$. Then $\forall a \in {}^n k \exists r_a \in G^a$ extending p^a and $\{t_{av}\}_{v\subseteq k}$ such that $r_a \Vdash (\check{x} \subseteq \mathrm{int}(\dot{X})$ and $x \cap \mathrm{int}(\dot{X}(\ell_a))$ is $\dot{X}(\ell_a)$-positive$)$. Furthermore for all $v \subseteq k$, $a \in {}^n k$ we have $(p^a, (v \cup w_{av}, \dot{X}_{N_{av}})) \leq A_{av}$. Let $N = \max_{a\in {}^n k, v\subseteq k} N_{av}$. Then for all $v \subseteq k, a \in {}^n k$, $(r^a, (v \cup w_{av}, \dot{X}_N)) \leq A_{av}$. From $\{r^a\}_{a\in {}^n k}$ one can obtain a tree of extensions of the given matrix, the maximal nodes of which have the desired properties. By Lemma 9 and $x \subseteq A_{j_0}$, the induced logarithmic measure takes arbitrarily high values. $\square$

Corollary 1. *Let $\dot{X} = \langle \dot{X}(i) \rangle_{i \in \omega}$ be a strongly $\mathbb{C}(\Gamma)$-symmetric name for a pure condition, $\dot{f}$ a good $Q(C)$-name for a real. Then there is a strongly symmetric name $\dot{Y} = \langle \dot{Y}(i) : i \in \omega \rangle$ for a pure condition such that $\forall m \in \omega$, $\dot{Y}_m = \langle \dot{Y}(i) : i \geq m \rangle \leq \dot{X}_m$ and $\forall i \in \omega \forall v \subseteq i \forall p \in \mathbb{C}(\Gamma)\forall$ and every $s \in [\omega]^{<\omega}$ such that $p \Vdash$ "$\check{s} \subseteq \dot{Y}(i)$ is $\dot{Y}(i)$-positive" there are $w_v \subseteq s$, $A \in \mathcal{A}_i(\dot{f})$ such that $(p, (v \cup w_v, \dot{Y})) \leq A$.*

Proof. Proceed by the method of Lemma 1. At stage i of the construction apply Lemma 10, to obtain $T_i \in \mathrm{ext}(P_i)$ and $x_i \in L_i$ such that T_i witnesses that $x_i \in \mathcal{P}_i(\dot{X}_i, \dot{f}(i), P_i)$. $\square$

Lemma 11. *Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, $\Gamma \in [\omega_2]^\omega$, $\dot{f}$ a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real, $\delta \in \omega_1 \backslash \Gamma$, $\dot{h} = \cup \dot{G}_\delta$, where $\dot{G}_\delta$ is the canonical name for the $\mathbb{C}(\{\delta\})$-generic filter. Then $\exists C'$ countable centered family of $\mathbb{C}(\Gamma \cup \{\delta\})$-symmetric names for pure conditions extending C such that $\forall C''$ of $\mathbb{C}(\omega_2)$-symmetric names extending C', $\Vdash_{\mathbb{C}(\omega_2) * Q(C'')}$ "$\dot{h} \not\leq^* \dot{f}$".*

Proof. By Corollary 1, we can assume that $C = \{\dot{Y}_m\}_{m \in \omega}$ where $\dot{Y}_m = \langle \dot{Y}(i) : i \geq m \rangle$, $\dot{Y} = \dot{Y}_0$ has the property that $\forall i \in \omega \forall v \subseteq i \forall p \in \mathbb{C}(\Gamma)$ and $s \in [\omega]^{<\omega}$ such that $p \Vdash$ "$\check{s} \subseteq \dot{Y}(i)$ is $\dot{Y}(i)$-positive" there are $w_v \subseteq s$ and $A \in A_i(\dot{f})$ such that $(p, (v \cup w_v, \dot{Y})) \leq A$. Let $\dot{g}$ be a $\mathbb{C}(\Gamma)$-name for a function in ${}^\omega\omega$ such that $\forall p \in \mathbb{C}(\Gamma) \forall i \in \omega$, $p \Vdash \dot{g}(i) = \check{k}$ if and only if k is maximal such that there are $v \subseteq i, w \in [\omega]^{<\omega}, A \in \mathcal{A}_i(\dot{f})$ such that $p \Vdash$ "$\check{w} \subseteq \dot{Y}(i)$", $(p, (v \cup w, \dot{Y})) \leq A$ and $A \Vdash$ "$\dot{f}(i) = \check{k}$". Let $\dot{J}$ be a $\mathbb{C}(\Gamma \cup \{\delta\})$-name such that $\Vdash \dot{J} = \langle i : \dot{g}(i) < \dot{h}(i) \rangle$ and $\forall m \in \omega$, let $\dot{Z}_m$ be a $\mathbb{C}(\Gamma \cup \{\delta\})$-name such that $\Vdash \dot{Z}_m = \langle \dot{Y}(i) : i > m$ and $i \in \dot{J} \rangle$.

Claim. For all $m \in \omega$ the name $\dot{Z}_m$ is $\mathbb{C}(\Gamma \cup \{\delta\})$-symmetric.

Proof. Let $P = (p_i^j) \in \mathbb{M}_k(\Gamma \cup \{\delta\}, \{\Gamma_j\}_{j \in n+1})$, $M \in \omega$ be given. Without loss of generality $\Gamma_n = \{\delta\}$. Then $Q = (p_i^j)_{i \in k, j \in n} \in \mathbb{M}_k(\Gamma, \{\Gamma_j\}_{j \in n})$. Pick $\ell \in \omega$, such that $\ell > m$ and $\ell > \max\{s : \mathrm{dom}(\delta, s) \in p_i^n, i \in k\}$. By the properties of $\dot{Y}$ there is $T \in \mathrm{ext}(Q)$, $x \in L_\ell$ such that $T \Vdash \check{x} = \dot{Y}(\ell)$. Successively on the lexicographic order on $\{a\}_{a \in {}^n k}$ extend the maximal nodes $\{T(a)\}_{a \in {}^n k}$ of T, to a tree $T' \in \mathrm{ext}(Q)$ consisting of Cohen conditions in $\mathbb{C}(\Gamma)$ such that $\forall a \in {}^n k \exists k_a \in \omega T'(a) \Vdash \dot{g}(\ell) = \check{k}_a$. Let $L > \max\{k_a\}_{a \in {}^n k}$ and $\forall i \in k$ let $q_i^n = p_i^n \cup \{\langle \langle \delta, \ell \rangle, \check{L} \rangle\}$. If T' is induced by $t' : {}^{\leq n}k \to \cup_{j \in n} \mathbb{C}(\Gamma_j)$, then $r : {}^{\leq n+1}k \to \cup_{j \in n+1} \mathbb{C}(\Gamma_j)$ where $\forall a \in {}^{\leq n}k$ $r(a) = t'(a)$ and $\forall a \in {}^{n+1}k, a = (b, i), i \in k$ $r(a) = q_i^n$ induces a tree $R \in \mathrm{ext}(P)$ such that $R \Vdash \dot{g}(\ell) < \dot{h}(\ell) \wedge \check{x} = \dot{Y}(\ell)$. That is $R \Vdash \check{\ell} \in \dot{J} \wedge \dot{Y}(\ell) = \check{x}$. Since $\ell > m$, $R \Vdash \check{x} \leq \dot{Z}_m$ and so $\dot{Z}_m$ is symmetric. $\square$

Let $C' = \{\dot{Z}_m\}_{m \in \omega}$, $\dot{Z} = \dot{Z}_0$ and let C'' be a centered family of $\mathbb{C}(\omega_2)$-symmetric names extending C'. It is sufficient to show that $\forall a \in [\omega]^{<\omega}$, $\forall k \in \omega$, $\Vdash_{\mathbb{C}(\omega_2)}$ "$(a, \dot{Z}) \Vdash_{Q(C'')}$ "$\exists i > k(\dot{f}(i) < \dot{h}(i)$"" since $\Vdash_{\mathbb{C}(\omega_2)}$ "$\{(a, \dot{Z}) : a \in [\omega]^{<\omega}\}$ is predense in $Q(C'')$". Let $a \in [\omega]^{<\omega}$, $k \in \omega$ and $(p, (b, \dot{R})) \in$

$\mathbb{C}(\omega_2) * Q(C'')$ such that $p \Vdash$ "$(b, \dot{R}) \leq (a, \dot{Z})$". Then $p \Vdash b \backslash a \subseteq \text{int}(\dot{Z})$ and $p \Vdash \dot{R} \leq \dot{Z}$. By definition of the extension relation there are $\ell > k$ such that $b \subseteq \ell$, $s \in [\omega]^{<\omega}$ and $\bar{p} \leq p$ such that $\bar{p} \Vdash$ "$\check{\ell} \in \dot{J}$ and $\check{s} = \text{int}(\dot{R}) \cap \text{int}(\dot{Z}(\ell))$ is $\dot{Z}(\ell)$- positive". By definition of $\dot{Z}(\ell)$ there is $w \subseteq s$ and $A \in \mathcal{A}_\ell(\dot{f})$ such that $(\bar{p}, (b \cup w, \dot{Y})) \leq A$ and so $(\bar{p}, (b \cup w, \dot{Z})) \leq A$ as well as $(\bar{p}, (b \cup w, R)) \leq A$. Note that $\bar{p} \Vdash \check{w} \subseteq \text{int}(\dot{R})$ and so $(\bar{p}, (b \cup w, R)) \leq (p, (b, \dot{R}))$. Furthermore $(\bar{p}, (b \cup w, \dot{R})) \Vdash$ "$\dot{f}(\ell) \leq \dot{g}(\ell) < \dot{h}(\ell)$". □

5. Countably closed and $\aleph_2$-c.c.

Definition 16. Let $\mathbb{P}$ be the partial order of all pairs $p = (\Gamma_p, C_p)$ where Γ is a countable subset of ω_2, C_p is a countable centered family of $\mathbb{C}(\Gamma_p)$-symmetric names for pure conditions with extension relation $p \leq q$ if $\Gamma_q \subseteq \Gamma_p$ and $\Vdash_{\mathbb{C}(\Gamma_p)} C_q \subseteq Q(C_p)$.

The partial order $\mathbb{P}$ has the $\aleph_2$-chain condition. Indeed, consider a model of CH and a subset $\{p_i : i \in I\}$ of $\mathbb{P}$ of size $\aleph_2$, $I \subseteq \omega_2$. By the Delta System Lemma there is $J \subseteq I$, $|J| = \aleph_2$ such that $\{\Gamma_i : i \in J\}$ form a delta system with root Δ where $\forall i \in I(\Gamma_i = \Gamma_{p_i})$. Furthermore J might be chosen so that for all $i < j$ in J there is an isomorphism $\alpha_{ij} : \Gamma_i \cong \Gamma_j$, such that $\alpha_{ij} \restriction \Delta$ is the identity and $C_j = C_{p_j} = \{\alpha_{ij}(\dot{X}) : \dot{X} \in C_{p_i}\}$. Suppose we have the proof of Lemma 12 below and let $\Gamma = \Gamma_i$, $\Theta = \Gamma_j$ for some $i < j$ from J, and $\alpha_{ij} = i$. Let $\Omega = \Gamma \cup \Theta$, $C = C_i \cup C_j \cup \{\tilde{X}_X\}_{X \in C_i}$ where for every $X \in C_i$, $\tilde{X}_X$ is the $\mathbb{C}(\Omega)$ symmetric name for a common extension of $\dot{X}$ and $i(\dot{X})$ constructed in Lemma 12. Suppose $\dot{R} \in C_i$, $\dot{Y} \in C_j$. Then $\dot{Y} = i(\dot{Z})$ for some $\dot{Z} \in C_i$. However C_i is centered, so there is $\dot{X} \in C_i$ which is a common extension of $\dot{R}$ and $\dot{Z}$. Then $\tilde{X}_X$ is a common extension of $\dot{R}$ and $\dot{Y}$. This implies that C is a centered family of $\mathbb{C}(\Omega)$ symmetric names for pure conditions and so $p = (\Omega, C)$ is a common extension of p_i, p_j. Thus it is sufficient to obtain Lemma 12. Note that this a particular case of Lemma 8.

Lemma 12. *Let Γ, Θ be countable subsets of ω_2, $\Delta = \Gamma \cap \Theta$ such that $\sup \Delta < \min \Gamma \backslash \Delta \leq \sup \Gamma \backslash \Delta < \min \Theta \backslash \Delta$ and let $i : \Gamma \cong \Theta$ be an isomorphism such that $i \restriction \Delta = id$. If $\dot{X}$ is a $\mathbb{C}(\Gamma)$ symmetric name for a pure condition, then there is a $\mathbb{C}(\Omega)$ symmetric name $\tilde{X}$ for a pure condition such that $\Vdash_{\mathbb{C}(\Omega)} \tilde{X} \leq \dot{X} \wedge \tilde{X} \leq i(\dot{X})$.*

Proof. Let $\Omega' \in \mathcal{FP}(\Omega)$. We can assume that $\Omega' = \Delta' \cup \Gamma' \cup \Theta'$ where $\Delta' \in \mathcal{FP}(\Delta)$, $\Gamma' \in \mathcal{FP}(\Gamma - \Delta)$, $\Theta' \in \mathcal{FP}(\Theta - \Delta)$. We can also assume that $\Delta' = \{\Gamma_i\}_{j \in n}$, $\Gamma' = \{\Gamma_j\}_{j \in [n, 2n)}$, $\Theta' = \{\Gamma_j\}_{j \in [2n, 3n)}$ and also that $\forall j \in [n, 2n) i(\Gamma_j) = \Gamma_{j+n}$. Let $P \in \mathbb{M}_k(\Omega, \Omega')$. Thus $P = (p_i^j)_{j \in 3n, i \in k}$. From P obtain a matrix $R \in \mathbb{M}_{2k}(\Gamma, \Delta' \cup \Gamma')$ as follows: if $(i, j) \in k \times 2n$ let $r_i^j = p_i^j$, if $(i, j) \in [k, 2k) \times n$ let $r_i^j = \emptyset$ and for $(i, j) \in k \times [n, 2n)$ let $r_{i+k}^j = i^{-1}(p_i^{j+n})$. By symmetry of $\dot{X}$ there is $x \in L_M$ and a tree of extensions $T = \{T(a) : a \text{ in } {}^{\leq 2n}2k\}$ of R such that $T \Vdash \check{x} \leq \dot{X}$. Having T obtain a tree of extensions $T' = \{T'(a) : a \text{ in } {}^{\leq 3n}k\}$ of P as follows. If $a \in {}^{\leq 2n}k$ let $T'(a) = T(a)$. If $a \in {}^{2n+m}k$ where $1 \leq m \leq n$ let $T'(a) = T(a) \restriction 2n \cup i(T(c))$ where $c = (a \restriction n)^\frown b$ and $b = \langle a(j) + k : j \in [2n, 2n+m) \rangle$. That is $T(c) \restriction n = T(a) \restriction n$ and since id $\restriction \Delta = $ id, $T(a) \restriction n = i(T(c)) \restriction n$. Note that $i(T(c)) \restriction [n, 2n) \in \mathbb{M}_{1\times n}(\Theta\backslash\Delta, \Theta')$. Then in particular the maximal nodes of T' belong to $\mathbb{M}_{1\times 3n}(\Omega, \Omega')$ and force "$\check{x} \leq \dot{X} \wedge \check{x} \leq i(\dot{X})$"

To obtain $\tilde{X}$, diagonalize $\mathbb{M}(\Omega)$ with respect to $\phi(T, x)$ where $\phi(T, x)$ holds iff $T \in \mathcal{T}(\Omega)$, $x \in \mathrm{LM}$ and $T \Vdash_{\mathbb{C}(\Omega)} \check{x} \leq \dot{X} \wedge \check{x} \leq i(\dot{X})$. □

The partial order $\mathbb{P}$ is countably closed and adds a centered family of $\mathbb{C}(\omega_2)$-symmetric names for pure conditions $C_H = \cup\{C_p : p \in H\}$ where H is $\mathbb{P}$-generic. By Lemma 7, forcing with $Q(C_H)$ over $V^{\mathbb{P}\times\mathbb{C}(\omega_2)}$ adds a real not split by $V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega = V^{\mathbb{C}(\omega_2)\times\mathbb{P}} \cap [\omega]^\omega$. By Lemma 11 any family of ω_1 Cohen reals remains unbounded in $V^{(\mathbb{C}(\omega_2)\times\mathbb{P})*Q(C_{\dot{H}})}$ where $\dot{H}$ is the canonical name $\mathbb{P}$ name for the generic filter.

Theorem 1. [CH] There is a countably closed, $\aleph_2$-cc forcing notion $\mathbb{P}$ such that in $V_1 = V^{\mathbb{C}(\omega_2)\times\mathbb{P}}$ there is a σ-centered poset Q which preserves the unboundedness of every family of ω_1 Cohen reals and adds a real not split by $V_1 \cap [\omega]^\omega$.

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